Unit One: Differential Equations

(Reference: Advanced Engineering Mathematics, by Dennis G. Zill, 6th edition, 2018.)

1.1 Basic Definitions and Concepts:

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation** (**DE**). The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule.

Ordinary Differential Equation (ODE) is a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable.

Partial Differential Equation (PDE) is an equation contains only partial derivatives of one or more functions of two or more independent variables.

The **order** of a differential equation is the order of the highest derivative appearing in the equation.

The **degree** of a differential equation is defined as the power to which the highest order derivative is raised.

Notation

The expressions $y', y'', y''', y^{(4)}, \ldots, y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth, . . ., *n*th derivatives of *y* with respect to the independent variable under consideration.

If the independent variable is time, usually denoted by *t*, primes are often replaced by dots. Thus, $\dot{y}, \ddot{y}$, and $\dddot{y}$ represent dy/dt, d^2y/dt^2, and d^3y/dt^3, respectively.

Examples: (1) $\frac{dy}{dx} + 6y = e^{-x}$

(2) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = 0$

(3) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}$

(4) $u_{xx} = u_{tt} - u_t$

(5) $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$

(6) $\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x$

We can express the *n*th-order ordinary differential equation in one dependent variable by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

or by the **normal form**

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

EXAMPLE Normal Form of an ODE

(a) $4x\frac{dy}{dx} + y = x$ is $\frac{dy}{dx} = \frac{x - y}{4x}$

(b) $y'' - y' + 6y = 0$ is $y'' = y' - 6y$

Linearity: An *n*th-order ordinary differential equation is said to be linear in the variable *y* if ***F*** is linear in $y, y', \ldots, y^{(n)}$.

A **nonlinear** ordinary differential equation is simply one that is not linear.

Standard and Differential Forms

Standard form for a first-order differential equation in the unknown function $y(x)$ is

$$y' = f(x,y)$$

while the ***differential form*** is

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

EXAMPLE Linear and Nonlinear Differential Equations

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y' + y = 0, \quad x^3 \frac{d^3y}{dx^3} + 3x\frac{dy}{dx} - 5y = e^x$$

are, in turn, examples of *linear* first-, second-, and third-order ordinary differential equations.

(b) The equations

$$\underbrace{(1 - y)}_{\text{nonlinear term: coefficient depends on } y}y' + 2y = e^x, \quad \frac{d^2y}{dx^2} + \underbrace{\sin y}_{\text{nonlinear term: nonlinear function of } y} = 0, \quad \frac{d^4y}{dx^4} + \underbrace{y^2}_{\text{nonlinear term: power not 1}} = 0,$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

A **solution** of a differential equation in the unknown function y and the independent variable x on the interval I is a function $y(x)$ that satisfies the differential equation identically for all x in I.

> ***Note***: A **particular** solution of a differential equation is any one solution.
> The **general** solution of a differential equation is the set of all solutions.

EXAMPLE Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}; \quad y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0; \quad y = xe^x$

SOLUTION One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x.

(a) From *left-hand side*: $\dfrac{dy}{dx} = 4 \cdot \dfrac{x^3}{16} = \dfrac{x^3}{4}$

right-hand side: $xy^{1/2} = x \cdot \left(\dfrac{x^4}{16}\right)^{1/2} = x \cdot \dfrac{x^2}{4} = \dfrac{x^3}{4},$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have for every real number x,

left-hand side: $y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0$
right-hand side: 0. ≡

Note, too, that each differential equation possesses the constant solution $y = 0$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution**.

Initial-Value and Boundary-Value Problems

A differential equation along with conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an ***initial-value problem*** (IVP). These conditions are ***initial conditions***. If the conditions are given at more than one value of the independent variable, the problem is a ***boundary-value problem*** (BVP) and the conditions are ***boundary conditions***.

Typically, initial value problems involve time dependent functions, while boundary value problems are spatial.

Example: The problem $y'' + 2y' = e^x; y(\pi) = 1, y'(\pi) = 2$ is an initial value problem, because the two subsidiary conditions are both given at $x = \pi$.

The problem $y'' + 2y' = e^x; y(0) = 1, y(1) = 1$ is a boundary-value problem, because the two subsidiary conditions are given at $x = 0$ and $x = 1$.

1.2 Solutions of the First Order D.Es

1.2.1 Separable D.Es

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables**. For example, the differential equations

$$\frac{dy}{dx} = x^2y^4e^{5x-3y} \quad \text{and} \quad \frac{dy}{dx} = y + \cos x$$

are separable and nonseparable, respectively. To see this, note that we can factor the first equation as

$$f(x, y) = x^2y^4e^{5x-3y} = (x^2e^{5x})(y^4e^{-3y})$$

but in the second there is no way writing $y + \cos x$ as a product of a function of x times a function of y.

EXAMPLE 1

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$|y| = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} \quad \leftarrow \text{laws of exponents}$$

$$= |1 + x|e^{c_1} \quad \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases}$$

and so

$$y = \pm e^{c_1}(1 + x).$$

Relabeling $\pm e^{c_1}$ by c then gives $y = c(1 + x)$.

EXAMPLE 2 Solution Curve

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $y(4) = -3$.

SOLUTION By rewriting the equation as $y\,dy = -x\,dx$ we get

$$\int y\,dy = -\int x\,dx \quad \text{and} \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by c^2. Now when $x = 4$, $y = -3$, so that $16 + 9 = 25 = c^2$. Thus $x^2 + y^2 = 25$.

EXAMPLE 3

Solve the initial-value problem

$$\cos x(e^{2y} - y)\frac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$$

SOLUTION Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\,dy = \frac{\sin 2x}{\cos x}\,dx.$$

$$\int (e^y - ye^{-y})\,dy = 2\int \sin x\,dx$$

yields

$$e^y + ye^{-y} + e^{-y} = -2\cos x + c.$$

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos x.$$

Exercises 2.2 (page 48): Solve exercises 1 to 27.

1.2.2 Exact Equations

The first order ODE

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **exact** if a function $f(x, y)$ exists such that the *total differential*

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x,y)dx + N(x,y)dy$$

or $M(x,y) = \partial f/\partial x$ and $N(x,y) = \partial f/\partial y$

It follows directly that if

$$M(x, y)dx + N(x, y)dy = 0$$

is exact, then the total differential

$$d\,[f(x, y)] = 0,$$

so the general solution of must be

$$f(x, y) = constant.$$

<u>Condition of Exactness:</u> $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Steps for Solving an Equation You Know to Be Exact:

1- Match the equation to the form

$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$$

to identify $\partial f/\partial x$ and $\partial f/\partial y$.

2- Integrate $\partial f/\partial x$ with respect to x, writing the constant of integration as $k(y)$.

3- Differentiate with respect to y and set the result equal to $\partial f/\partial y$ to find $k'(y)$.

4- Integrate to find $k(y)$ and determine $f(x,y)$.

5- Write the solution of the exact equation as $f(x,y) = C$.

EXAMPLE 1 Solving an Exact DE

Solve $2xy\,dx + (x^2 - 1)\,dy = 0.$

SOLUTION With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and so,

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2y + g(y).$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y.$

Hence, $f(x, y) = x^2y - y$, and so the solution of the equation is $x^2y - y = c$

Example (2): Show that the following equation is exact and find its general solution:
$\{3x^2 + 2y + 2\cosh(2x + 3y)\}dx + \{2x + 2y + 3\cosh(2x + 3y)\}dy = 0$

Solution:

$M(x, y) = 3x^2 + 2y + 2\cosh(2x + 3y)$,

and $N(x, y) = 2x + 2y + 3\cosh(2x + 3y)$,

then $M_y = 2 + 6\sinh(2x + 3y)$

and $N_x = 2 + 6\sinh(2x + 3y)$

so, as $M_y = N_x$ the equation is exact:

$$f(x,y) = \int M(x,y)dx = \int \{3x^2 + 2y + 2\cosh(2x + 3y)\}dx$$

$$= x^3 + 2xy + \sinh(2x + 3y) + k(y)$$

$$\frac{\partial f}{\partial y} = 2x + 3\cosh(2x + 3y) + k'(y) = N = 2x + 2y + 3\cosh(2x + 3y)$$

or $k'(y) = 2y$ and then

$$k(y) = \int 2y\,dy = y^2$$

so $f(x, y) = x^3 + 2xy + y^2 + sinh(2x + 3y)$

and the general solution is $x^3 + 2xy + y^2 + sinh(2x + 3y) = C$

(Do you know another method to find $k(y)$?)

To learn more about the exact equations, see examples 2 and 3 in pages 61 and 62.

1.2.3 Integrating Factors

It can be shown that every nonexact differential equation $M(x, y)dx + N(x, y)dy = 0$ can be made exact by multiplying both sides by a suitable factor called *integrating factor* $\mu(x,y)$.

- If $(M_y - N_x)/N$ is a function of x alone, then

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

- If $(N_x - M_y)/M$ is a function of y alone, then

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

As an example, the equation

$$2y\, dx + x\, dy = 0$$

is not exact, while the equation

$$2xy\, dx + x^2\, dy = 0$$

obtained by multiplying both sides by x, is exact.

EXAMPLE 4 A Nonexact DE Made Exact

The nonlinear first-order differential equation $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$ we find $M_y = x$ and $N_x = 4x$.

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20} \quad \text{depends on } x \text{ and } y.$$

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}. \quad \text{depends only on } y:$$

The integrating factor is then $e^{\int 3\,dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$.

and the resulting equation is $xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$.

(verify that the last equation is now exact, and the solution is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$).

Exercises 2.4 (page 64): Solve exercises 1 to 36.

1.2.4 Linear First Order Equations

A first-order linear differential equation has the form

$$y' + p(x)y = q(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

An integrating factor for equation (1) is

$$\mu(x) = e^{\int p(x)\,dx} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and the general solution of equation (1) is

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Note: When $q(x) = 0$, the linear equation (1) is said to be **homogeneous**; otherwise, it is nonhomogeneous.

Steps for solving a linear first order equation:

1- Put it in standard form, as in equation (1).

2- Find the integrating factor from equation (2).

3- Use equation (3) to find y.

Example 1: Solve $x\,y' - 3\,y = x^2$.

Solution: By dividing both sides on x, the equation can be written as $y' - (3/x)\,y = x$

So it is linear, with $p(x) = -3/x$ and $q(x)=x$.

$$\int p(x)dx = \int -\frac{3}{x}\,dx = -3\ ln\ x$$

$$\mu(x) = e^{\int p(x)dx} = e^{-3\ln x} = \frac{1}{x^3}$$

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)dx = \frac{1}{\frac{1}{x^3}}\int \frac{1}{x^3}\,x\,dx = x^3\left(\frac{1}{x} + C\right) = Cx^3 - x^2$$

The solution is $y = Cx^3 - x^2$.

***<u>Remark</u>*:**

Occasionally a first-order differential equation is not linear in one variable but is linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2 \quad \text{or} \quad \frac{dx}{dy} - x = y^2$$

is recognized as linear in the variable x. You should verify that the integrating factor

$$\mu(y) = e^{\int p(y)dy} = e^{\int(-1)dy} = e^{-y}$$

and integration by parts yield an implicit solution of the given equation:

$$x = -y^2 - 2y - 2 + ce^{y}.$$

Exercises 2.3 (page 57): Solve exercises 1 to 32.

1.2.5 **Homogeneous D.Es**

A function $f(x, y)$ is said to be homogeneous of degree n, if $f(tx, ty) = t^n f(x, y)$ for some real number n.

Examples

(a) If $f(x, y) = x^2 + 3xy + 4y^2$, then $f(tx, ty) = t^2(x^2 + 3xy + 4y^2) = t^2 f(x, y)$, so $f(x, y)$ is homogeneous of degree 2.

(b) If $f(x, y) = \ln|y| - \ln|x|$ for $(x, y) \neq (0, 0)$, then $f(x, y) = \ln|y/x|$, so $f(tx, ty) = f(x, y)$, showing that $f(x, y)$ is homogeneous of degree 0.

(c) If

$$f(x, y) = \frac{x^{3/2} + x^{1/2}y + 3y^{3/2}}{2x^{3/2} - xy^{1/2}}, \text{ then } f(tx, ty) = t^0 f(x, y),$$

showing that $f(x, y)$ is homogeneous of degree 0.

(d) If $f(x, y) = x^2 + 4y^2 + \sin(x/y)$, then $f(tx, ty) = t^2(x^2 + 4y^2) + \sin(x/y)$, so $f(x, y)$ is *not* homogeneous.

(e) If $f(x, y) = \tan(xy + 1)$, then $f(tx, ty) = \tan(t^2 xy + 1)$, so $f(x, y)$ is *not* homogeneous.

In addition, the first order ODE in differential form

$$P(x, y)dx + Q(x, y)dy = 0$$

is called homogeneous if P and Q are homogeneous functions of the same degree or, equivalently, if when written in the form

$$\frac{dy}{dx} = h(x, y)$$

the homogeneous function $h(x, y)$ can be written as $h(x, y) = F(y/x)$. We can change this equation into a separable equation by the substitution $y=vx$, then:

$$\frac{dy}{dx} = \frac{d}{dx}(vx) = v + x\frac{dv}{dx} = F(v)$$

which can be rearranged to give

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

Example 1: Show that the equation

$$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy}$$

is homogeneous and find the solution that satisfies the condition *y*(1)=1.

Solution:

$$\frac{dy}{dx} = -\frac{1+(\frac{y}{x})^2}{2(\frac{y}{x})}$$

$F(v) = -\frac{1+v^2}{2v}$ where $v=y/x$

$\frac{dx}{x} + \frac{dv}{v+\frac{1+v^2}{2v}} = 0$ or $\frac{dx}{x} + \frac{2v\,dv}{1+3v^2} = 0$

The solution of this equation is

$\ln|x| + \frac{1}{3}\ln(1 + 3v^2) = C$

or $x^3(1 + 3v^2) = \pm e^{3C} = C_1$

we substitute $v=y/x$ to find the corresponding *xy*-equation:

$x^3(1 + 3\frac{y^2}{x^2}) = C_1$

or $x^3 + 3xy^2 = C_1$

$(1^3 + 3(1)(1)^2) = C_1$ or $C_1 = 4$

The solution is $x^3 + 3xy^2 = 4$

1.2.6 <u>Bernoulli's Equation</u>

The Bernoulli equation is a nonlinear first order DE with the standard form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

1- When $n = 0$ the equation is First Order Linear DE.
2- When $n = 1$ the equation can be solved using Separation of Variables.
3- For other values of n the equation cannot be solved by separation of variables or linearity or homogeneity, but we can solve it by substituting

$$u = y^{1-n}$$

and turning it into a linear differential equation (and then solve that).

and thus, the Bernoulli equation becomes

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

(*Prove that !*)

Taking an integrating factor as

$$\mu(x) = e^{\int (1-n)\, P(x)\, dx}$$

then the general solution of Bernoulli equation is

$$u = \frac{1}{\mu(x)} \int (1-n)\, \mu(x)\, Q(x) dx$$

EXAMPLE 1 Solving a Bernoulli DE

Solve $x\dfrac{dy}{dx} + y = x^2y^2.$

SOLUTION

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2.$$

With $n = 2$, we next substitute $y = u^{-1}$, $P(x)=1/x$, and $Q(x)=x$ into equation

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

and simplify, the result is

$$\frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor is $\mu(x) = e^{-\int dx/x} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

$$u = \frac{1}{x^{-1}} \int (1-2)\, x^{-1}\, x\, dx = -x \int dx = -x(x+C)$$

$$u = -x^2 - Cx$$

But $u=y^{-1}$, then $y = -1/(x^2 + Cx)$

Exercises 2.5 (page 68): Solve exercises 1 to 22.

Additional Exercises: Solve the following DEs:

DE's	Answers
1. $\dfrac{dy}{dx} - \dfrac{1}{x}y = xy^2$	$\frac{1}{y} = -\frac{x^2}{3} + \frac{C}{x}$,
2. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2$	$\frac{1}{y} = x(C - \ln\, x)$,
3. $\dfrac{dy}{dx} + \dfrac{1}{3}y = e^x y^4$	$\frac{1}{y^3} = e^x(C - 3x)$,
4. $x\dfrac{dy}{dx} + y = xy^3$	$y^2 = \frac{1}{2x+Cx^2}$,
5. $\dfrac{dy}{dx} + \dfrac{2}{x}y = -x^2\cos x \cdot y^2$	$\frac{1}{y} = x^2(\sin x + C)$,
6. $2\dfrac{dy}{dx} + \tan x \cdot y = \dfrac{(4x+5)^2}{\cos x}y^3$	$\frac{1}{y^2} = \frac{-1}{12\cos x}(4x+5)^3 + \frac{C}{\cos x}$,
7. $x\dfrac{dy}{dx} + y = y^2x^2\ln\, x$	$\frac{1}{xy} = C + x(1 - \ln\, x)$,
8. $\dfrac{dy}{dx} = y\cot x + y^3 \mathrm{cosec} x$	$y^2 = \frac{\sin^2 x}{2\cos x + C}$.

1.3 Solutions of the Second-Order D.Es

1.3.1 Second-Order DE Reducible to First Order

A second order DE has the general form

$$F(x, y, y', y'') = 0$$

Equation above is called *reducible* second order DE if either the dependent variable *y* or the independent variable *x* is missing in it.

Case I : $F(x, y', y'') = 0$ (Dependent variable *y* missing)

The substitution $p = y' = \frac{dy}{dx}$, $y'' = \frac{dp}{dx}$ results in $F(x, p, p') = 0$.

Case II : $F(y, y', y'') = 0$ (Independent variable *x* missing)

The substitution $p = y' = \frac{dy}{dx}$, $y'' = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$ results in $F(y, p, p\frac{dp}{dy}) = 0$.

Example 1: Solve the equation $xy'' + 2y' = 6x$.

Solution: Let $p = y' = \frac{dy}{dx}$ and $y'' = \frac{dp}{dx}$, then

$$xp' + 2p = 6x$$

or $p' + 2xp = 6,$ which is a linear first order equation.

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln(x^2)} = x^2$$

$$p = \frac{1}{x^2}\int 6x^2\,dx = 2x + \frac{C_1}{x^2}$$

$$y = \int \left(2x + \frac{C_1}{x^2}\right) dx = x^2 - \frac{C_1}{x} + C_2$$

Example 2: Solve the equation $yy'' = (y')^2$.

Solution:

$$yp\frac{dv}{dy} = p^2.$$

$$\int \frac{dp}{p} = \int \frac{dy}{y} \implies \ln p = \ln y + \ln C_1 \implies p = C_1 y.$$

Since $p = \frac{dy}{dx}$, we have

$$\frac{dy}{dx} = C_1 y \implies \int \frac{dy}{y} = \int C_1 dx \implies \ln y = C_1 x + \ln C_2,$$

so that the general solution is $y(x) = C_2 e^{C_1 x}$.

Example 3: Solve the equation $y\,y''+(y')^2=0$

Solution:

$$y\,y''+(y')^2=0$$

$$p=y'\ ;\ p\frac{dp}{dy}=y''$$

$$y\,p\frac{dp}{dy}+p^2=0 \quad \Rightarrow \quad \frac{dp}{dy}=\frac{-1}{y}p$$

$$\frac{1}{p}dp=\frac{-1}{y}dy \quad \Rightarrow \quad \int\frac{1}{p}dp=\int\frac{-1}{y}dy \quad \Rightarrow \quad \ln p=-\ln y+C$$

$$p=e^C y^{-1} \quad \Rightarrow \quad p=\frac{C}{y} \quad \Rightarrow \quad C=y\,p$$

$$C=y\frac{dy}{dx} \quad \Rightarrow \quad y\,dy=C\,dx \quad \Rightarrow \quad \int y\,dy=\int C\,dx$$

$$\frac{y^2}{2}=C\,x+D \quad \Rightarrow \quad \boxed{y^2=C\,x+D}$$

Example 4: Solve the equation $\frac{d^2y}{d^2x}+y=0$.

Solution: Let $p=\frac{dy}{dx}$, $y''=p\frac{dp}{dy}$

$p\frac{dp}{dy}+y=0$ or $pdp+ydy=0$

$\frac{p^2}{2}+\frac{y^2}{2}=C$, let $C=\frac{C_1^2}{2}$

then $\frac{p^2}{2}+\frac{y^2}{2}=\frac{C_1^2}{2}\rightarrow \ p=\frac{dy}{dx}=\ \pm\sqrt{C_1-y^2}$

$$\frac{dy}{\pm\sqrt{C_1-y^2}}=\ \pm dx$$

$\sin^{-1}\frac{y}{C_1}=\ \pm(x+C_2)$ or $y=\ C_1\sin[\pm(x+C_2)]=\ \pm\,C_1\sin(x+C_2)$

$y=\ C_1\sin(x+C_2)$ (Since C_1 is arbitrary, there is no need for $\pm$ sign)

Exercises: Find general solutions of the following reducible second order differential equations.

a) $xy'' = y'$

b) $yy'' + (y')^2 = 0$

c) $xy'' + y' = 4x$

d) $y'' = (y')^2$

e) $x^2y'' + 3xy' = 2$

f) $yy'' + (y')^2 = yy'$

g) $y'' = (x + y')^2$

h) $y'' = 2y\,(y')^3$

i) $y^3y'' = 1$

j) $y'' = 2yy'$

k) $yy'' = 3\,(y')^2$

l) $y'' + 4y = 0$

1.3.2 Homogeneous Linear Equations with Constant Coefficients

A linear *n*th-order differential equation of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous**, whereas an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

with g(x) not identically zero is said to be **nonhomogeneous**.

If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation, then for any constants c_1 and c_2, the function $y(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution.

Differential Operator

Differentiation is often denoted by the capital letter D; that is

$$\frac{dy}{dx} = Dy$$

The symbol *D* is called a **differential operator**.

Examples: $D(\cos 4x) = -4\sin 4x$, $D(5x^3 - 6x^2) = 15x^2 - 12x$

The Characteristic Equation

Consider the special case of linear second order DE with constant coefficients:

$$a\frac{d^2y}{d^2x} + b\frac{dy}{dx} + cy = 0$$

If we try a solution of the form $y = e^{mx}$, then

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0$$

Since e^{mx} is never zero for real values of x, then

$$am^2 + bm + c = 0$$

This last equation is called the *characteristic equation*. There will be three forms of the general solution corresponding to the type of the roots m_1 and m_2.

Case I : Distinct Real Roots

If m_1 and m_2 are unequal real roots, the general solution is:

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

Case II : Repeated Real Roots

When m_1 and m_2 are equal real roots ($m_1 = m_2$,) the general solution is:

$$y = c_1e^{m_1x} + c_2xe^{m_1x}$$

Case III : Conjugate Complex Roots

If m_1 and m_2 are complex, or $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the general solution is

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

SOLUTION We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3)$, $m_1 = -\frac{1}{2}$, $m_2 = 3$.

$$y = c_1e^{-x/2} + c_2e^{3x}.$$

(b) $m^2 - 10m + 25 = (m - 5)^2$, $m_1 = m_2 = 5$.

$$y = c_1e^{5x} + c_2xe^{5x}.$$

(c) $m^2 + 4m + 7 = 0$, $m_1 = -2 + \sqrt{3}i$, $m_2 = -2 - \sqrt{3}i$. We have $\alpha = -2$, and $\beta = \sqrt{3}$.

$$y = e^{-2x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x).$$

EXAMPLE 2 An Initial-Value Problem

Solve the initial-value problem $4y'' + 4y' + 17y = 0,\ y(0) = -1,\ y'(0) = 2.$

SOLUTION $4m^2 + 4m + 17 = 0$

$$m_1 = -\tfrac{1}{2} + 2i \text{ and } m_2 = -\tfrac{1}{2} - 2i.$$

$$y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$$

Applying the condition $y(0) = -1$, $e^0(c_1 \cos 0 + c_2 \sin 0) = -1$

we see that $c_1 = -1$.

Differentiating $y = e^{-x/2}(-\cos 2x + c_2 \sin 2x)$ and then using $y'(0) = 2$ gives

$2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$.

Hence the solution is $y = e^{-x/2}(-\cos 2x + \frac{3}{4}\sin 2x)$

REMARKS:

(1) Characteristic equations are only defined for linear homogeneous differential equations with constant coefficients.

(2) The method of this section also works for homogeneous linear first-order differential equations $ay' + by = 0$ with constant coefficients.

Exercises 3.3 (page 125): Solve exercises 1 to 14, and exercises 29 to 34.

1.3.2 Non-Homogeneous Linear Equations with Constant Coefficients

The general solution $y = y(x)$ to the nonhomogeneous differential equation

$$ay'' + by' + cy = G(x), \qquad (1)$$

has the form $y = y_c + y_p$,

where the **complementary solution** $y_c = c_1y_1 + c_2y_2$ is the general solution to the associated homogeneous equation

$$ay'' + by' + cy = 0. \qquad (2)$$

and y_p is any **particular solution** to the nonhomogeneous equation (1).

1- Method of Undetermined Coefficients

The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	**And if**	**Then include this expression in the trial function for y_p.**
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx$, $\cos kx$	ki is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
a	0 is not a root of the auxiliary equation	A
$bx + c$	0 is not a root of the auxiliary equation	$Bx + C$
$dx^2 + ex + f$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
$dx^2 + ex + f$	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

EXAMPLE 1

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$

SOLUTION

Step 1 We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0.$

$$m^2 + 4m - 2 = 0$$

$$m_1 = -2 - \sqrt{6} \text{ and } m_2 = -2 + \sqrt{6}.$$

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}$$

Step 2 assume $y_p = Ax^2 + Bx + C.$

$$y_p' = 2Ax + B \text{ and } y_p'' = 2A$$

$$\begin{aligned} y_p'' + 4y_p' - 2y_p &= 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C \\ &= 2x^2 - 3x + 6. \end{aligned}$$

equal

$$\boxed{-2A}\, x^2 + \boxed{8A - 2B}\, x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6.$$

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6.$$

Solving this system of equations leads to the values $A = -1$, $B = -\frac{5}{2}$, and $C = -9$.

Thus $y_p = -x^2 - \dfrac{5}{2}x - 9.$

Step 3 The general solution is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

EXAMPLE 2

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

SOLUTION

$$y_p = A \cos 3x + B \sin 3x.$$

$$y_p'' - y_p' + y_p = (-8A - 3B) \cos 3x + (3A - 8B) \sin 3x = 2 \sin 3x$$

equal

$$\boxed{-8A - 3B} \cos 3x + \boxed{3A - 8B} \sin 3x = 0 \cos 3x + 2 \sin 3x.$$

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

$$A = \tfrac{6}{73} \text{ and } B = -\tfrac{16}{73}$$

$$y_p = \frac{6}{73} \cos 3x - \frac{16}{73} \sin 3x$$

EXAMPLE 3

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

SOLUTION

Step 1 First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is $y_c = c_1 e^{-x} + c_2 e^{3x}$.

Step 2 we also assume that the particular solution is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$.

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}.$$

$$-3A = 4, \qquad -2A - 3B = -5, \qquad -3C = 6, \qquad 2C - 3E = 0.$$

Solving, we find $A = -\frac{4}{3}$, $B = \frac{23}{9}$, $C = -2$, and $E = -\frac{4}{3}$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3 The general solution of the equation is

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}.$$

EXAMPLE 4 Find a particular solution of $y'' - 6y' + 9y = e^{3x}$.

Solution The equation $m^2 - 6m + 9 = (m-3)^2 = 0$

has $m = 3$ as a repeated root. Thus,

$$y_p = Ax^2e^{3x}$$

and we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$$

or $2Ae^{3x} = e^{3x}$.

Thus, $A = 1/2$, and the particular solution is $y_p = \frac{1}{2}x^2e^{3x}$.

EXAMPLE 5 Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the equation $r^2 - r = 0$.

Its roots are $r = 1$ and $r = 0$. Therefore, $y_c = c_1e^x + c_2$.

we choose y_p to be the sum $y_p = Axe^x + B\cos 2x + C\sin 2x$,

$$(Axe^x + 2Ae^x - 4B\cos 2x - 4C\sin 2x)$$

$$-(Axe^x + Ae^x - 2B\sin 2x + 2C\cos 2x) = 5e^x - \sin 2x$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x$$

$$A = 5, \qquad 4B + 2C = 0, \qquad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x$$

The general solution is $y = y_c + y_p = c_1e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$

Exercises 3.4

In Problems 1–26, solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3 \sin 2x$
14. $y'' - 4y = (x^2 - 3) \sin 2x$
15. $y'' + y = 2x \sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x \cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3 \sin x)$
19. $y'' + 2y' + y = \sin x + 3 \cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$

In Problems 27–32, solve the given initial-value problem.

27. $y'' + 4y = -2,\ \ y(\pi/8) = \frac{1}{2}, y'(\pi/8) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11,\ \ y(0) = 0, y'(0) = 0$
29. $5y'' + y' = -6x, y(0) = 0,\ \ y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x},\ \ y(0) = 2, y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x},\ \ y(0) = -3, y'(0) = 1$
32. $y'' - y = \cosh x,\ \ y(0) = 2, y'(0) = 12$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' + y = x^2 + 1,\ \ y(0) = 5, y(1) = 0$
38. $y'' - 2y' + 2y = 2x - 2,\ \ y(0) = 0, y(\pi) = \pi$
39. $y'' + 3y = 6x,\ \ y(0) = 0, y(1) + y'(1) = 0$
40. $y'' + 3y = 6x,\ \ y(0) + y'(0) = 0, y(1) = 0$

2- Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation

$$ay'' + by' + cy = G(x)$$

The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $u_1 = u_1(x)$ and $u_2 = u_2(x)$.

To use this method, follow the following steps:

1. Solve the associated homogeneous equation $ay'' + by' + cy = 0$ to find the functions y_1 and y_2.

2. Solve the equations (see page 137 for complete derivation):

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = G(x)$$

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ G(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & G(x) \end{vmatrix}.$$

to find the functions u_1' and u_2'.

3. Integrate u_1' and u_2' to find the functions $u_1 = u_1(x)$ and $u_2 = u_2(x)$.

4. Write down the particular solution as $y_P = u_1 y_1 + u_2 y_2$.

EXAMPLE 1

Solve $y'' - 4y' + 4y = (x + 1)e^{2x}$.

SOLUTION From equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ we have $y_c = c_1e^{2x} + c_2xe^{2x}$.

With the identifications $y_1 = e^{2x}$, $y_2 = xe^{2x}$ and $G(x) = (x + 1)e^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x + 1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x},$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x + 1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

and so $u_1' = -\dfrac{(x + 1)xe^{4x}}{e^{4x}} = -x^2 - x,\quad u_2' = \dfrac{(x + 1)e^{4x}}{e^{4x}} = x + 1$

Integrating u_1' and u_2 gives

$$u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2 \quad \text{and} \quad u_2 = \frac{1}{2}x^2 + x$$

Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and $$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

EXAMPLE 2

Solve $4y'' + 36y = \csc 3x$.

SOLUTION We first put the equation in the standard form by dividing by 4:

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Since the roots of the equation $m^2 + 9 = 0$ are $m_1 = 3i$ and $m_2 = -3i$,

$$y_c = c_1 \cos 3x + c_2 \sin 3x.$$

Using $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $G(x) = \frac{1}{4}\csc 3x$, we obtain

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3,$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4},$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Integrating $u_1' = \frac{W_1}{W} = -\frac{1}{12}$ and $u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$.

Thus $y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$.

The general solution is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

Exercises 3.5 (page: 140)

In Problems 1–18, solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec\theta \tan\theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 9y = \dfrac{9x}{e^{3x}}$

10. $4y'' - y = e^{x/2} + 3$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

13. $y'' + 3y' + 2y = \sin e^x$

14. $y'' - 2y' + y = e^t \arctan t$

15. $y'' + 2y' + y = e^{-t} \ln t$

16. $2y'' + y' = 6x$

17. $3y'' - 6y' + 6y = e^x \sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

1.4 Solutions of the Higher-Order D.Es ($n > 2$)

1.4.1 Higher-Order Homogeneous Linear DEs

The characteristic equation of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

is

$$m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

General Solution for *n*th-Order Equations:

The general solution of the nth-order DE is obtained directly from the roots of its characteristic equation, as in the following cases:

Case 1 If the roots $m_1, m_2, \ldots, m_n$ are all real and no two are equal, the solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case 2 If m is a real root appearing k times, the solution is

$$y = e^{mx}(c_1 + c_2 x + \cdots + c_k x^{k-1})$$

Case 3 If the complex roots are conjugate pairs of complex numbers:

$m_{1,2} = \alpha \pm i\beta$, $m_{1,2} = \gamma \pm i\delta, \ldots$

the solution is

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{\gamma x}(C_3 \cos \delta x + C_4 \sin \delta x) + \cdots$$

Case 4 If $m = \alpha \pm \beta i$ are complex conjugate roots each appears k times, the solution is

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + x e^{\alpha x}(C_3 \cos \beta x + C_4 \sin \beta x) + \cdots + x^{k-1} e^{\alpha x}(C_{2k-1} \cos \beta x + C_{2k} \sin \beta x)$$

EXAMPLE 1

Solve $y''' + 3y'' - 4y = 0$.

SOLUTION It should be apparent from inspection of $m^3 + 3m^2 - 4 = 0$ that one root is $m_1 = 1$ and so $m - 1$ is a factor of $m^3 + 3m^2 - 4$. By division we find

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

and so the other roots are $m_2 = m_3 = -2$. Thus the general solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$

EXAMPLE 2

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$ characteristic

SOLUTION The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$.

Thus the solution is (with α=0, β=1 and k=2)

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) + x e^{\alpha x}(C_3 \cos \beta x + C_4 \sin \beta x)$$

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$$

EXAMPLE 3

Find the general solution of the differential equation

$$y^{(4)} - y = 0.$$

Solution. The roots of the characteristic polynomial are $m = 1, \ -1, \ i, \ -i$. Thus, the general solution of the differential equation is

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

EXAMPLE 4

Solve the equation $y^{(4)} - y''' + 2y' = 0$

Solution: The characteristic equation is

$$m^4 - m^3 + 2m = 0$$

or

$$m(m^3 - m^2 + 2) = 0$$

$$m(m+1)(m^2 - 2m + 2) = 0$$

The characteristic equation has four distinct roots, two of which are complex:

$$m_1 = 0, \qquad m_2 = -1, \qquad m_{3,4} = 1 \pm i$$

The general solution is $y(x) = C_1 + C_2 e^{-x} + e^x (C_3 \cos x + C_4 \sin x)$

Exercises 3.3 (page:125)

In Problems 15–24, find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$

16. $y''' - y = 0$

17. $y''' - 5y'' + 3y' + 9y = 0$

18. $y''' + 3y'' - 4y' - 12y = 0$

19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$

20. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$

21. $y''' + 3y'' + 3y' + y = 0$

22. $y''' - 6y'' + 12y' - 8y = 0$

23. $y^{(4)} + y''' + y'' = 0$

24. $y^{(4)} - 2y'' + y = 0$

1.4.1 Higher-Order Non-Homogeneous Linear DEs

In this section, we give methods for obtaining a particular solution y_p once y_c is known.

1- Undetermined Coefficients for Higher-Order DEs

We have already seen how to solve a second-order linear nonhomogeneous DEs with constant coefficients. For higher-order nonhomogeneous differential equation, the exact same method will work.

EXAMPLE

Solve $y''' + y'' = e^x \cos x.$

SOLUTION From the characteristic equation $m^3 + m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = -1.$

Hence $y_c = c_1 + c_2x + c_3e^{-x}.$

With $g(x) = e^x \cos x$, we assume $y_p = Ae^x \cos x + Be^x \sin x.$

From $y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$

we get $-2A + 4B = 1, -4A - 2B = 0.$

This system gives $A = -\frac{1}{10}$ and $B = \frac{1}{5}.$

so that $y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$

The general solution is $y = y_c + y_p = c_1 + c_2x + c_3e^{-x} - \dfrac{1}{10}e^x \cos x + \dfrac{1}{5}e^x \sin x.$

Exercises 3.4 (page:135)

In Problems below, solve the given third-order DEs by undetermined coefficients.

21. $y''' - 6y'' = 3 - \cos x$

22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$

23. $y''' - 3y'' + 3y' - y = x - 4e^x$

24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$

25. $y^{(4)} + 2y'' + y = (x - 1)^2$

26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

2- Method of Variation of Parameters for Higher-Order DEs

This method will be illustrated here to find the particular solution y_p. For the nonhomogeneous second-order differential equation

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x)$$

$$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x)$$

where the u_k', $k = 1, 2, \ldots, n$, are determined by the n equations

$$\begin{aligned} y_1u_1' + \quad y_2u_2' + \cdots + \quad y_nu_n' &= 0 \\ y_1'u_1' + \quad y_2'u_2' + \cdots + \quad y_n'u_n' &= 0 \\ \vdots \qquad\qquad\qquad\qquad & \vdots \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' &= f(x) \end{aligned}$$

When $n = 3$, $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 are set of solutions of the associated homogeneous DE, and u_1, u_2, u_3 are determined from

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \text{ and } W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

Exercises 3.5 (page:140)

In Problems 29 and 30, solve the given third-order DEs by variation of parameters.

29. $y''' + y' = \tan t$ **30.** $y''' + 4y' = \sec 2x$

Solution of Problem 29:

For $y''' + y' = 0$ we have $m^3 + m = 0$

then $r(r^2 + 1) = 0$ and so $r = 0$ or $r = \pm i$

therefore $y_c = C_1 + C_2 \cos t + C_3 \sin t$

Furthermore, $y_1 = e^{0t} = 1$, $y_2 = e^{0t} \cos t = \cos t$, and $y_3 = e^{0t} \sin t = \sin t$

We have that: $y_p = u_1y_1 + u_2y_2 + u_3y_3$

$$y_p = u_1 + u_2\cos t + u_3\sin t$$

Thus we want to solve the following system of equations:

$$\begin{aligned} u_1'(1) + u_2'\cos t + u_3'\sin t &= 0 \\ u_1'(0) + u_2'(-\sin t) + u_3'(\cos t) &= 0 \\ u_1'(0) + u_2'(\cos t) + u_3'(\sin t) &= \tan t \end{aligned}$$

Now we have that:

$$W = \begin{bmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} = 1$$

$$u_1' = \frac{\begin{bmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ \tan t & -\cos t & -\sin t \end{bmatrix}}{1} = \tan t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \tan t$$

$$u_2' = \frac{\begin{bmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & \tan t & -\sin t \end{bmatrix}}{1} = -\begin{bmatrix} 1 & 0 & \sin t \\ 0 & \tan t & -\sin t \\ 0 & 0 & \cos t \end{bmatrix} = -\tan t\cos t = -\sin t$$

$$u_3' = \frac{\begin{bmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & \tan t \end{bmatrix}}{1} = \begin{bmatrix} -\sin t & 0 \\ -\cos t & \tan t \end{bmatrix} = -\sin t\tan t = -\frac{\sin^2 t}{\cos t}$$

We will now integrate u_1', u_2', and u_3' to get:

$$\int u_1'(t)\,dt = \int \tan t\,dt = \ln|\sec t| + C$$

$$\int u_2'(t)\,dt = \int -\sin t\,dt = \cos t + D$$

$$\begin{aligned} \int u_3'(t)\,dt &= \int -\frac{\sin^2 t}{\cos t}\,dt = \int \frac{\cos^2 t - 1}{\cos t}\,dt = \int (\cos t - \sec t)\,dt \\ &= \sin t + \ln|\sec t + \tan t| + E \end{aligned}$$

Thus we have that:

$$y(t) = (\ln|\sec t| + C)(1) + (\cos t + D)\cos t + (\sin t + \ln|\sec t + \tan t| + E)\sin t$$