Unit 4: MULTIPLE INTEGRALS

4.1 Double Integrals as Volumes

When *f*(*x*, *y*) is a positive function over a rectangular region *R* in the *xy*-plane, we may interpret the double integral of *f* over *R* as the volume of the 3-dimensional solid region over the *xy*-plane bounded below by *R* and above by the surface *z* = *f*(*x*, *y*), such that:

$$\text{Volume} = \iint_R f(x, y)\, dA,$$

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Example: Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region

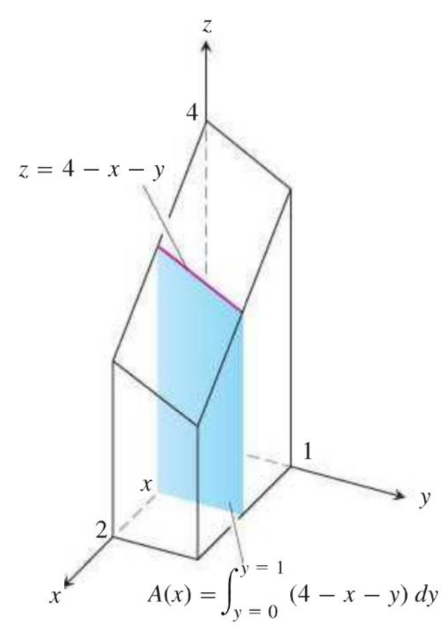
R: $0 \le x \le 2$; $0 \le y \le 1$, in the *xy*-plane.

Solution (1):

Applying the method of slicing, with slices perpendicular to the x-axis, the volume is

$$\int_{x=0}^{x=2} A(x)\, dx \quad \text{where } A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y.



$$\text{Volume} = \int_{x=0}^{x=2} A(x)\, dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\, dy \right) dx$$

$$= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx$$

$$= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 5.$$

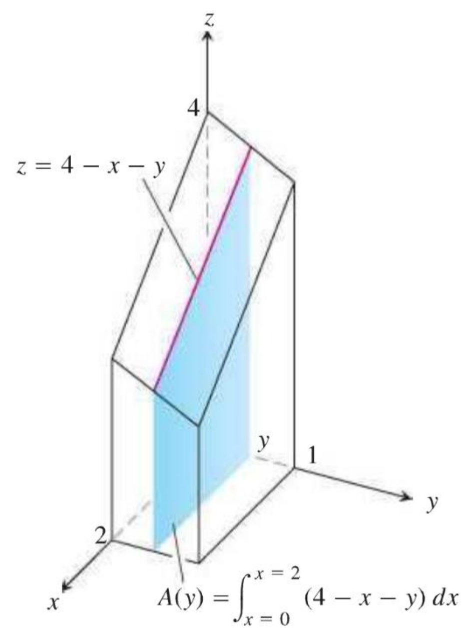
If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\, dy\, dx.$$

Solution (2):

By slicing with planes perpendicular to the *y*-axis, the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y.$$



The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\, dy = \int_{y=0}^{y=1} (6 - 2y)\, dy = \left[6y - y^2\right]_0^1 = 5$$

Again, we may give a formula for the volume as

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R:\ 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^1 \int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^1 \left[100x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^1 (200 - 16y)\, dy = \left[200y - 8y^2\right]_{-1}^1 = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2 \int_{-1}^1 (100 - 6x^2y)\, dy\, dx = \int_0^2 \left[100y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\, dx$$

$$= \int_0^2 200\, dx = 400. \quad \blacksquare$$

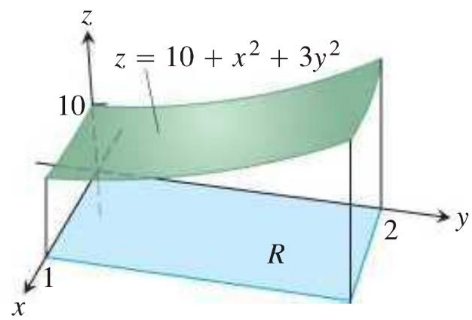
EXAMPLE 2 Find the volume of the region bounded above by the ellipitical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

$$= \int_0^1 \left[10y + x^2 y + y^3\right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \left[20x + \frac{2}{3}x^3 + 8x\right]_0^1 = \frac{86}{3}.$$



Exercises 15.1

Evaluating Iterated Integrals

In Exercises 1–12, evaluate the iterated integral.

1. $\int_1^2 \int_0^4 2xy\, dy\, dx$
2. $\int_0^2 \int_{-1}^1 (x - y)\, dy\, dx$
3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy$
4. $\int_0^1 \int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\, dy$
5. $\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$
6. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy)\, dy\, dx$
7. $\int_0^1 \int_0^1 \frac{y}{1 + xy}\, dx\, dy$
8. $\int_1^4 \int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\, dy$
9. $\int_0^{\ln 2} \int_1^{\ln 5} e^{2x+y}\, dy\, dx$
10. $\int_0^1 \int_1^2 xye^x\, dy\, dx$
11. $\int_{-1}^2 \int_0^{\pi/2} y \sin x\, dx\, dy$
12. $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\, dx\, dy$

Evaluating Double Integrals over Rectangles

In Exercises 13–20, evaluate the double integral over the given region R.

13. $\iint_R (6y^2 - 2x)\, dA, \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 2$
14. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA, \quad R: \ 0 \le x \le 4, \quad 1 \le y \le 2$
15. $\iint_R xy \cos y\, dA, \quad R: \ -1 \le x \le 1, \quad 0 \le y \le \pi$
16. $\iint_R y \sin (x + y)\, dA, \quad R: \ -\pi \le x \le 0, \quad 0 \le y \le \pi$
17. $\iint_R e^{x-y}\, dA, \quad R: \ 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$
18. $\iint_R xye^{xy^2}\, dA, \quad R: \ 0 \le x \le 2, \quad 0 \le y \le 1$
19. $\iint_R \frac{xy^3}{x^2 + 1}\, dA, \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 2$
20. $\iint_R \frac{y}{x^2 y^2 + 1}\, dA, \quad R: \ 0 \le x \le 1, \quad 0 \le y \le 1$

In Exercises 21 and 22, integrate f over the given region.

21. **Square** $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2, 1 \le y \le 2$
22. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

Volume Beneath a Surface

23. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.
24. Find the volume of the region bounded above by the ellipitical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: 0 \le x \le 2, 0 \le y \le 2$.
25. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.
26. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.
27. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.
28. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

4.2 Double Integrals over Bounded, Nonrectangular Regions

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x,y)$ to be $\iint_R f(x,y)\, dA$, as before. If R is a region bounded "above" and "below" by the curves $y = g_2(X)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(y \frac{\sin x}{x} \right]_{y=0}^{y=x} \Bigg) dx = \int_0^1 \sin x\, dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

we run into a problem because $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

A. Using Vertical Cross-sections:

1. *Sketch*. Sketch the region of integration and label the bounding curves (Figure 15.14a).

2. *Find the y-limits of integration*. Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure 15.14b).

3. *Find the x-limits of integration*. Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure 15.14c) is

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

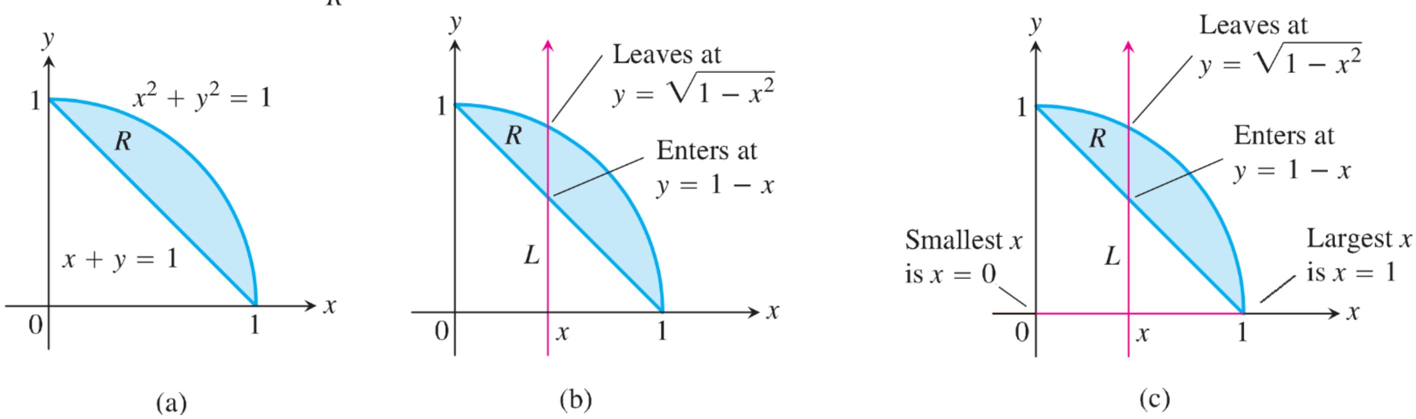
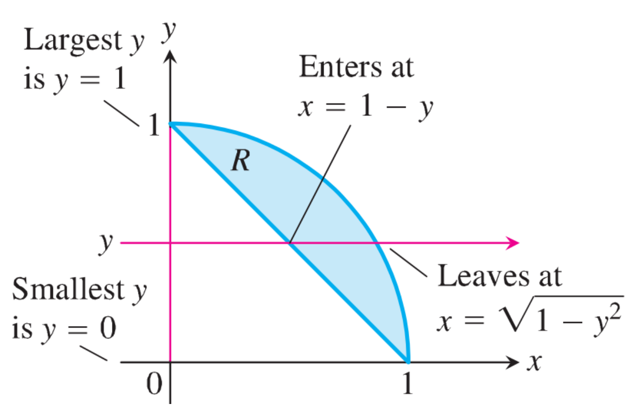


FIGURE 15.14 Finding the limits of integration.

B. Using Horizontal Cross-sections:

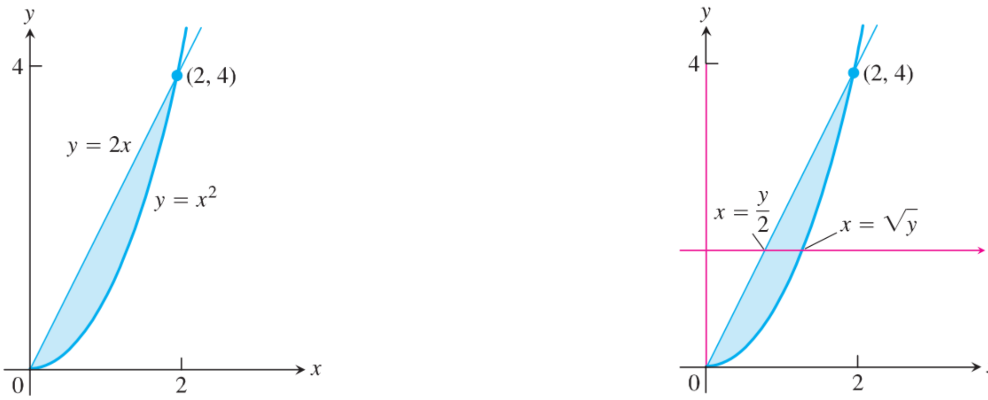
To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in steps 2 and 3.The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy$$



EXAMPLE 3 Sketch the region of integration for the integral $\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$ and write an equivalent integral with the order of integration reversed.

Solution $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy$. The common value of these integrals is 8.



EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

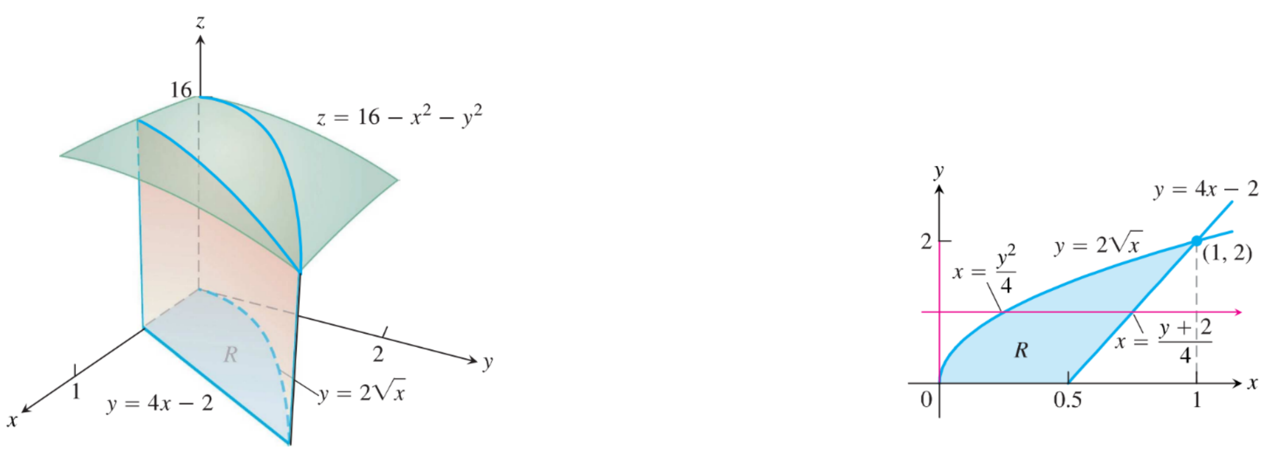
Solution $\iint_R (16 - x^2 - y^2)\, dA$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y + 2) - \frac{(y + 2)^3}{3 \cdot 64} - \frac{(y + 2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4$$

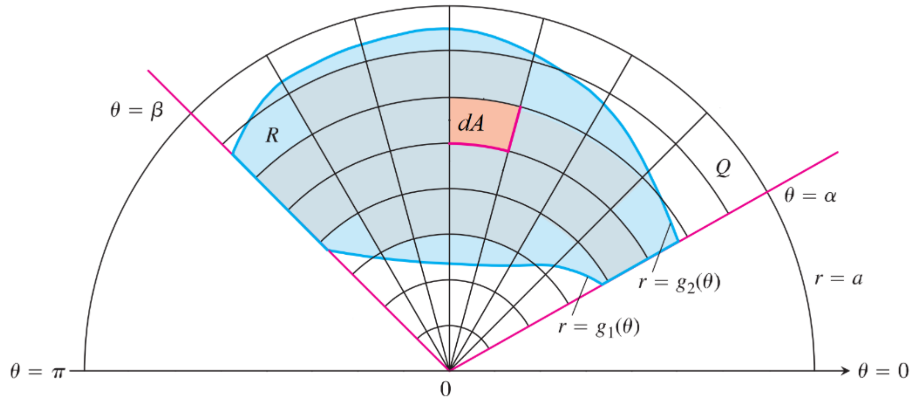


(a) The solid "wedgelike" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\, dy$.

(Exercises 15.2)

4.3 Double Integrals in Polar Form

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$.



A version of Fubini's Theorem says that
$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta$$

Finding Limits of Integration:

To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1. *Sketch*. Sketch the region and label the bounding curves (Figure 15.23a).
2. *Find the r-limits of integration*. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis (Figure 15.23b).
3. *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound R. These are the θ-limits of integration (Figure 15.23c). The polar iterated integral is

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r, \theta)\, r\, dr\, d\theta.$$

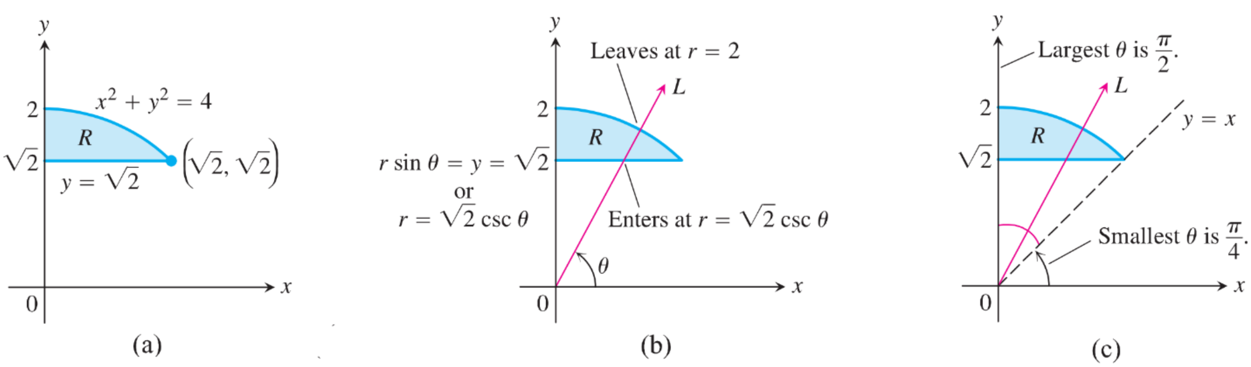


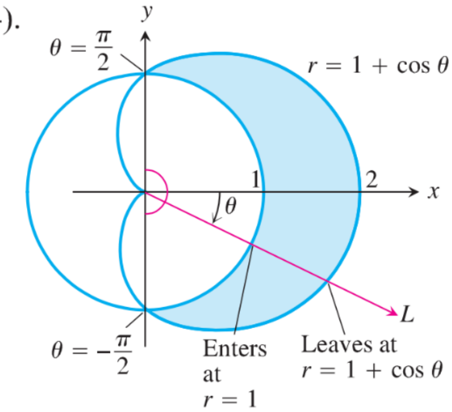
FIGURE 15.23 Finding the limits of integration in polar coordinates.

EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.24).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta.$$



If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

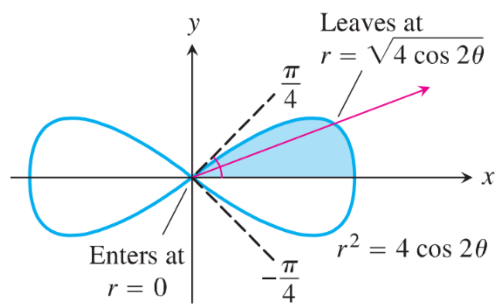
$$A = \iint_R r\, dr\, d\theta$$

EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$



Changing Cartesian Integrals into Polar Integrals

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

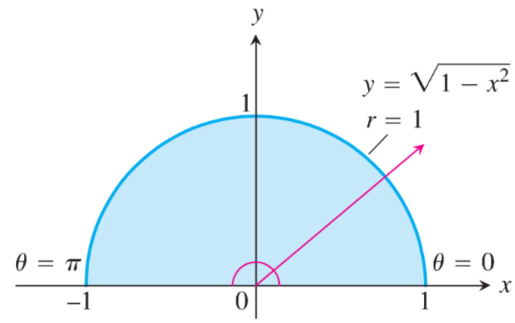
EXAMPLE 3 Evaluate $\displaystyle\iint_R e^{x^2+y^2}\, dy\, dx,$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

Solution

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$



EXAMPLE 4 Evaluate the integral $\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx.$

Solution Integration with respect to y gives

$$\int_0^1\left(x^2\sqrt{1-x^2}+\frac{(1-x^2)^{3/2}}{3}\right)dx,$$

an integral difficult to evaluate without tables.

$0 \le y \le \sqrt{1-x^2}$ and $0 \le x \le 1,$

Substituting the polar coordinates $x = r\cos\theta, y = r\sin\theta, 0 \le \theta \le \pi/2$ and $0 \le r \le 1,$ and replacing $dx\,dy$ by $r\,dr\,d\theta$ in the double integral, we get

$$\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^1 (r^2)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\frac{1}{4}\,d\theta = \frac{\pi}{8}.$$

EXAMPLE 5

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

Solution

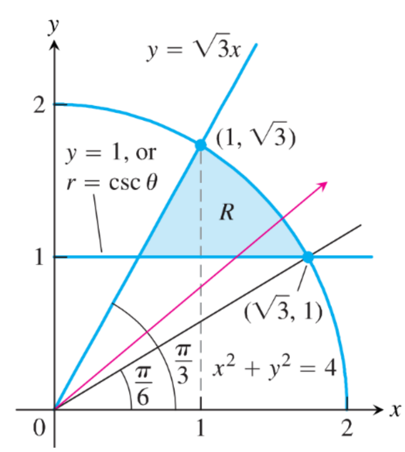
$$\iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4-\csc^2\theta\right]d\theta$$

$$= \frac{1}{2}\left[4\theta+\cot\theta\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{4\pi}{3}+\frac{1}{\sqrt{3}}\right)-\frac{1}{2}\left(\frac{4\pi}{6}+\sqrt{3}\right) = \frac{\pi-\sqrt{3}}{3}.$$



Exercises 15.4

Exercises 15.4

Area in Polar Coordinates

28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

4.4 Triple Integrals in Rectangular Coordinates

$$V = \iiint_D dz\,dy\,dx = \int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} dz\,dy\,dx$$

EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}(8 - 2x^2 - 4y^2)\,dy\,dx$$

$$= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx$$

$$= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx$$

$$= 8\pi\sqrt{2}.$$ After integration with the substitution $x = 2\sin u$