# Unit 3 Vector-valued Functions

## 3.1 Curves in Space and Their Tangents

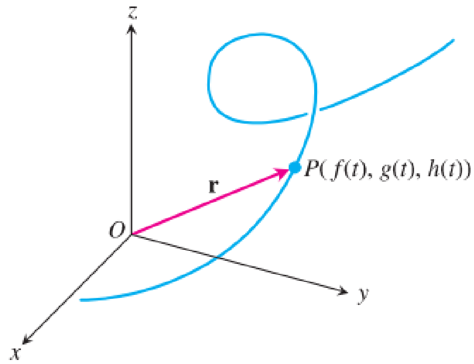
When a particle moves through space during a time interval I, its coordinates can be written as:

$$x = f(t), \qquad y = g(t), \qquad z = h(t), \qquad t \in I.$$

and in vector form as

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

which is the particle's **position vector**.




A **vector-valued function** or **vector function**, is a rule that assigns a vector in space.

EXAMPLE 1 Graph the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

Solution The vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

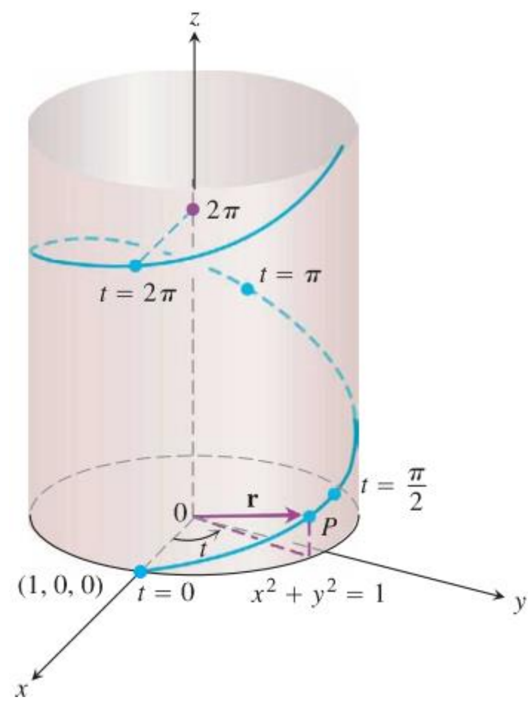
is defined for all real values of *t*. The curve traced by **r** winds around the circular cylinder $x^2 + y^2 = 1$.

The curve rises as the **k**-component $z = t$ increases.

The equations

$$x = \cos t, \qquad y = \sin t, \qquad z = t$$

parametrize the helix.




The upper half of the helix in Example 1.

DEFINITIONS: If **r** is the position vector of a particle moving along a smooth curve in space, then

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time $t$.

EXAMPLE 2

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\cos^2 t\,\mathbf{k}$, at $t=7\pi/4$.

**Solution** The velocity and acceleration vectors at time $t$ are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k}\\ &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k},\end{aligned}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},$$

and the speed is

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 5\,\mathbf{k}, \qquad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}, \qquad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

## Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of $t$, $\mathbf{C}$ a constant vector, $c$ any scalar, and $f$ any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$

7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

## Exercises 13.1

In Exercises 1-4, $\mathbf{r}(t)$ is the position of a particle in the *xy*-plane at time *t*. Find an equation in *x* and *y* whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of *t*.

**1.** $\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j}, \quad t = 1$

**2.** $\mathbf{r}(t) = \dfrac{t}{t+1}\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t = -1/2$

**3.** $\mathbf{r}(t) = e^t\mathbf{i} + \dfrac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$

**4.** $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3\sin 2t)\mathbf{j}, \quad t = 0$

In Exercises 19-22, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_o$.

**19.** $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

**20.** $\mathbf{r}(t) = t^2\mathbf{i} + (2t-1)\mathbf{j} + t^3\mathbf{k}, \quad t_0 = 2$

**21.** $\mathbf{r}(t) = \ln t\,\mathbf{i} + \dfrac{t-1}{t+2}\mathbf{j} + t\ln t\,\mathbf{k}, \quad t_0 = 1$

**22.** $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \dfrac{\pi}{2}$

## 3.2 Unit Tangent Vector ( T )

We already know the velocity vector v = $d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and that the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is a unit tangent vector.

EXAMPLE 1 Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

**Solution**

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9+4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9+4t^2}}\mathbf{k}.$$ ■

# 3.3 Arc Length Along a Space Curve

The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as $t$ increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

**EXAMPLE** A glider is soaring upward along the helix $\mathbf{r}(t) = a \cos t\mathbf{i} + a \sin t\mathbf{j} + ct\mathbf{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

**Solution** $\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$;

$$L = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2}\; dt = \sqrt{a^2 + c^2}\, t\,\Big|_0^{2\pi} = 2\pi\sqrt{a^2 + c^2}$$

## Exercises 13.3

### Finding Tangent Vectors and Lengths

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

1. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \le t \le \pi$
2. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \le t \le \pi$
3. $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \le t \le 8$
4. $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 3$
5. $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \le t \le \pi/2$
6. $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \le t \le 2$
7. $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$
8. $\mathbf{r}(t) = (t\sin t + \cos t)\mathbf{i} + (t\cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \le t \le 2$
9. Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance $26\pi$ units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length.

10. Find the point on the curve

$$\mathbf{r}(t) = (12\sin t)\mathbf{i} - (12\cos t)\mathbf{j} + 5t\mathbf{k}$$

at a distance $13\pi$ units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.

## 3.4 Unit Normal Vector ( N )

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

**EXAMPLE** Find $\mathbf{T}$ and $\mathbf{N}$ for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

**Solution** We first find $\mathbf{T}$:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}.$$

From this we find

$$\frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$

$$\left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

and

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

$$= -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}.$$

Notice that $\mathbf{T} \cdot \mathbf{N} = 0$, verifying that $\mathbf{N}$ is orthogonal to $\mathbf{T}$. Notice too, that for the circular motion here, $\mathbf{N}$ points from $\mathbf{r}(t)$ towards the circle's center at the origin.

**Exercises 13.4**

Find $\mathbf{T}$, $\mathbf{N}$, and $\kappa$ for the space curves in Exercises 9–16.

9. $\mathbf{r}(t) = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k}$
10. $\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} + 3\mathbf{k}$
11. $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + 2\mathbf{k}$
12. $\mathbf{r}(t) = (6\sin 2t)\mathbf{i} + (6\cos 2t)\mathbf{j} + 5t\mathbf{k}$
13. $\mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j}, \quad t > 0$
14. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 < t < \pi/2$
15. $\mathbf{r}(t) = t\mathbf{i} + (a\cosh(t/a))\mathbf{j}, \quad a > 0$
16. $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

## 3.5 Directional Derivatives

If $f(x,y)$ is a differentiable function, defined in a region R, and $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}$ is a unit vector, then:

1- The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.

2- The **Directional Derivative** is

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

Note: ( $D_{\mathbf{u}}f$ ) is The derivative of $f$ in the direction of **u**.

**EXAMPLE 1** Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

**Solution** The direction of **v** is the unit vector obtained by dividing **v** by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of $f$ are everywhere continuous and at (2, 0) are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$
$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of $f$ at (2, 0) is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The derivative of $f$ at (2, 0) in the direction of **v** is therefore

$$(D_{\mathbf{u}}f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u}$$
$$= (\mathbf{i} + 2\mathbf{j})\cdot\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

Properties of the directional derivative $D_{\mathbf{u}}f$ :

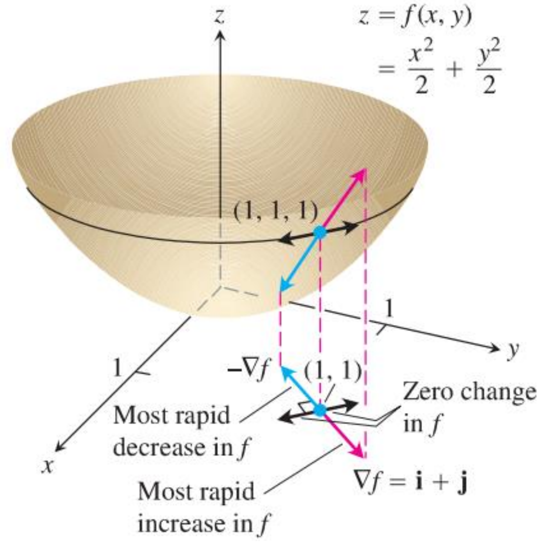
1- The function $f$ increases most rapidly (greatest ascent) in the direction of $\nabla f$. The derivative in this direction is $|\nabla f|$.
2- The function $f$ decreases most rapidly (greatest descent) in the direction of $-\nabla f$. The derivative in this direction is $-|\nabla f|$.
3- Any direction **u** orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in $f$.

**EXAMPLE 2** Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

**(a)** increases most rapidly at the point $(1, 1)$.

**(b)** decreases most rapidly at $(1, 1)$.

**(c)** What are the directions of zero change in $f$ at $(1, 1)$?




**Solution**

**(a)** The function increases most rapidly in the direction of $\nabla f$ at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(b)** The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(c)** The directions of zero change at $(1, 1)$ are the directions orthogonal to $\nabla f$:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

**EXAMPLE 3**

**(a)** Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

**(b)** In what directions does $f$ change most rapidly at $P_0$, and what are the rates of change in these directions?

**Solution** **(a)** The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of $f$ at $P_0$ are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy|_{(1,1,0)} = -2, \qquad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of $f$ at $P_0$ is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of $f$ at $P_0$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

**(b)** The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3.$$

## Exercises 14.5

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$
2. $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$
3. $g(x, y) = xy^2, \quad (2, -1)$
4. $g(x, y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}, \quad \left(\sqrt{2}, 1\right)$
5. $f(x, y) = \sqrt{2x + 3y}, \quad (-1, 2)$
6. $f(x, y) = \tan^{-1} \dfrac{\sqrt{x}}{y}, \quad (4, -2)$

In Exercises 7–10, find $\nabla f$ at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$
8. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz, \quad (1, 1, 1)$
9. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$
10. $f(x, y, z) = e^{x+y} \cos z + (y + 1) \sin^{-1} x, \quad (0, 0, \pi/6)$

In Exercises 11–18, find the derivative of the function at $P_0$ in the direction of $\mathbf{u}$.

11. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$
13. $g(x, y) = \dfrac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$
14. $h(x, y) = \tan^{-1}(y/x) + \sqrt{3} \sin^{-1}(xy/2), \quad P_0(1, 1), \mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$
15. $f(x, y, z) = xy + yz + zx, \quad P_0(1, -1, 2), \quad \mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
16. $f(x, y, z) = x^2 + 2y^2 - 3z^2, \quad P_0(1, 1, 1), \quad \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
17. $g(x, y, z) = 3e^x \cos yz, \quad P_0(0, 0, 0), \quad \mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
18. $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2), \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at $P_0$. Then find the derivatives of the functions in these directions.

**19.** $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$

**20.** $f(x, y) = x^2 y + e^{xy} \sin y, \quad P_0(1, 0)$

**21.** $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$

**22.** $g(x, y, z) = xe^y + z^2, \quad P_0(1, \ln 2, 1/2)$

**23.** $f(x, y, z) = \ln xy + \ln yz + \ln xz, \quad P_0(1, 1, 1)$

**24.** $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z, \quad P_0(1, 1, 0)$

**29.** Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f(1, -1)$ for which

**a.** $D_{\mathbf{u}} f(1, -1)$ is largest **b.** $D_{\mathbf{u}} f(1, -1)$ is smallest

**c.** $D_{\mathbf{u}} f(1, -1) = 0$ **d.** $D_{\mathbf{u}} f(1, -1) = 4$

**e.** $D_{\mathbf{u}} f(1, -1) = -3$

**30.** Let $f(x, y) = \dfrac{(x - y)}{(x + y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which

**a.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest **b.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest

**c.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$ **d.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = -2$

**e.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$

**31.** In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?

**32.** In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?

**33.** Is there a direction $\mathbf{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

# 3.6 Divergence and Curl of a Vector

A vector field $\mathbf{F}(x,y,z)$ is a function that assigns a vector to each point in its domain.

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

## Divergence

The divergence of a vector field $\mathbf{F}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

**EXAMPLE 1** The following vector fields represent the velocity of a gas flowing in space. Find the divergence of each vector field and interpret its physical meaning. Figure 16.67 displays the vector fields.

**(a)** Expansion: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

**(b)** Compression: $\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

**(c)** Rotation about $z$-axis: $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

**(d)** Shearing along horizontal planes: $\mathbf{F}(x, y, z) = z\mathbf{j}$

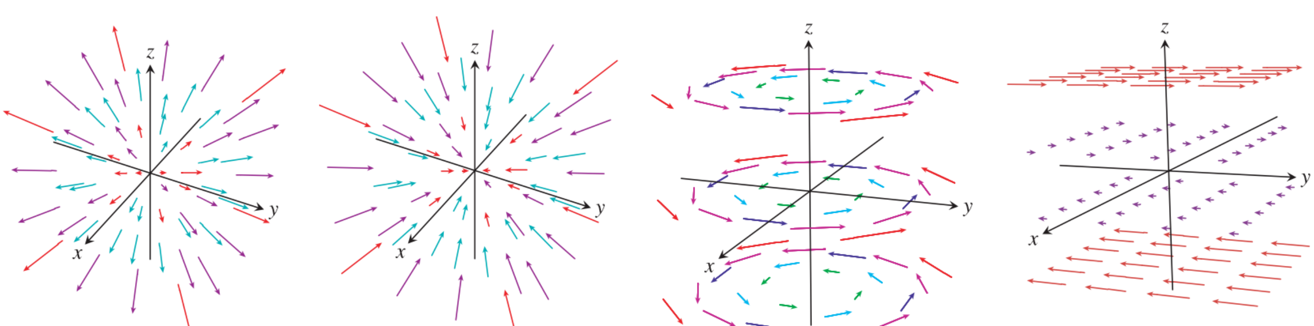
**Solution**

**(a)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$: The gas is undergoing uniform expansion at all points.

**(b)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) = -3$: The gas is undergoing uniform compression at all points.

**(c)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$: The gas is neither expanding nor compressing at any point.

**(d)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial y}(z) = 0$: Again, the divergence is zero at all points in the domain of the velocity field, so the gas is neither expanding nor compressing at any point. ■




(a) (b) (c) (d)

## Curl

The curl of a vector field **F**= $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the vector field

$$\text{curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

The curl of **F** is $\nabla \times \mathbf{F}$ :

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = \text{curl } \mathbf{F}$$

**EXAMPLE 1** Find the curl of $\mathbf{F} = (x^2 - z)\mathbf{i} + xe^z\mathbf{j} + xy\mathbf{k}$.

**Solution** We use Equation (3) and the determinant form, so

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - z & xe^z & xy \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xe^z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x^2 - z)\right)\mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial y}(x^2 - z)\right)\mathbf{k} \\
&= (x - xe^z)\mathbf{i} - (y + 1)\mathbf{j} + (e^z - 0)\mathbf{k} \\
&= x(1 - e^z)\mathbf{i} - (y + 1)\mathbf{j} + e^z\mathbf{k}
\end{aligned}$$

**EXAMPLE 2** If $\mathbf{F} = (x^2y^3 - z^4)\mathbf{i} + 4x^5y^2z\,\mathbf{j} - y^4z^6\mathbf{k}$, find **(a)** curl **F**, **(b)** div **F**, and **(c)** div(curl **F**).

**SOLUTION** **(a)**

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^3 - z^4 & 4x^5y^2z & -y^4z^6 \end{vmatrix}$$

$$= (-4y^3z^6 - 4x^5y^2)\mathbf{i} - 4z^3\mathbf{j} + (20x^4y^2z - 3x^2y^2)\mathbf{k}.$$

**(b)** $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(x^2y^3 - z^4) + \dfrac{\partial}{\partial y}(4x^5y^2z) + \dfrac{\partial}{\partial z}(-y^4z^6) = 2xy^3 + 8x^5yz - 6y^4z^5.$

**(c)**

$$\begin{aligned}
\text{div(curl } \mathbf{F}) &= \frac{\partial}{\partial x}(-4y^3z^6 - 4x^5y^2) + \frac{\partial}{\partial y}(-4z^3) + \frac{\partial}{\partial z}(20x^4y^2z - 3x^2y^2) \\
&= 0 - 20x^4y^2 + 0 + 20x^4y^2 = 0.
\end{aligned}$$

**Exercises 9.7**, page 514 (Advanced Engineering Mathematics- by Dennis G. Zill-6th ed.)

In Problems 7–16, find the curl and the divergence of the given vector field.

7. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$
8. $\mathbf{F}(x, y, z) = 10yz\,\mathbf{i} + 2x^2z\,\mathbf{j} + 6x^3\,\mathbf{k}$
9. $\mathbf{F}(x, y, z) = 4xy\,\mathbf{i} + (2x^2 + 2yz)\,\mathbf{j} + (3z^2 + y^2)\,\mathbf{k}$
10. $\mathbf{F}(x, y, z) = (x - y)^3\,\mathbf{i} + e^{-yz}\,\mathbf{j} + xye^{2y}\,\mathbf{k}$
12. $\mathbf{F}(x, y, z) = 5y^3\,\mathbf{i} + (\frac{1}{2}x^3y^2 - xy)\,\mathbf{j} - (x^3yz - xz)\,\mathbf{k}$
13. $\mathbf{F}(x, y, z) = xe^{-z}\,\mathbf{i} + 4yz^2\,\mathbf{j} + 3ye^{-z}\,\mathbf{k}$
14. $\mathbf{F}(x, y, z) = yz\ln x\,\mathbf{i} + (2x - 3yz)\,\mathbf{j} + xy^2z^3\,\mathbf{k}$
15. $\mathbf{F}(x, y, z) = xye^x\,\mathbf{i} - x^3yze^z\,\mathbf{j} + xy^2e^y\,\mathbf{k}$
16. $\mathbf{F}(x, y, z) = x^2\sin yz\,\mathbf{i} + z\cos xz^3\,\mathbf{j} + ye^{5xy}\,\mathbf{k}$

Solutions:

7. curl $\mathbf{F} = (x - y)\mathbf{i} + (x - y)\mathbf{j}$; div $\mathbf{F} = 2z$
8. curl $\mathbf{F} = -2x^2\mathbf{i} + (10y - 18x^2)\mathbf{j} + (4xz - 10z)\mathbf{k}$; div $\mathbf{F} = 0$
9. curl $\mathbf{F} = \mathbf{0}$; div $\mathbf{F} = 4y + 8z$
10. curl $\mathbf{F} = (xe^{2y} + ye^{-yz} + 2xye^{2y})\mathbf{i} - ye^{2y}\mathbf{j} + 3(x - y)^2\mathbf{k}$; div $\mathbf{F} = 3(x - y)^2 - ze^{-yz}$
11. curl $\mathbf{F} = (4y^3 - 6xz^2)\mathbf{i} + (2z^3 - 3x^2)\mathbf{k}$; div $\mathbf{F} = 6xy$
12. curl $\mathbf{F} = -x^3z\mathbf{i} + (3x^2yz - z)\mathbf{j} + \left(\frac{3}{2}x^2y^2 - y - 15y^2\right)\mathbf{k}$; div $\mathbf{F} = (x^3y - x) - (x^3y - x) = 0$
13. curl $\mathbf{F} = (3e^{-z} - 8yz)\mathbf{i} - xe^{-z}\mathbf{j}$; div $\mathbf{F} = e^{-z} + 4z^2 - 3ye^{-z}$
14. curl $\mathbf{F} = (2xyz^3 + 3y)\mathbf{i} + (y\ln x - y^2z^3)\mathbf{j} + (2 - z\ln x)\mathbf{k}$; div $\mathbf{F} = \dfrac{yz}{x} - 3z + 3xy^2z^2$
15. curl $\mathbf{F} = (xy^2e^y + 2xye^y + x^3ye^z + x^3yze^z)\mathbf{i} - y^2e^y\mathbf{j} + (-3x^2yze^z - xe^x)\mathbf{k}$; div $\mathbf{F} = xye^x + ye^x - x^3ze^z$
16. curl $\mathbf{F} = (5xye^{5xy} + e^{5xy} + 3xz^3\sin xz^3 - \cos xz^3)\mathbf{i} + (x^2y\cos yz - 5y^2e^{5xy})\mathbf{j}$
$+ (-z^4\sin xz^3 - x^2z\cos yz)\mathbf{k}$; div $\mathbf{F} = 2x\sin yz$