Unit one: Matrices

1.1 Basic Definitions:

A **matrix** is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called the **entries** (or sometimes the *elements*) of the matrix. For example,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix} \tag{1}$$

The first matrix has two **rows** (horizontal lines of entries) and three **columns** (vertical lines). The second and third matrices are square matrices. Matrices having just a single row or column are called vectors. Thus the fourth and last matrices are called **row** and **column** vectors, respectively.

We shall denote matrices by capital letters **A**, **B**, **C**, …, or **A** = $[a_{ij}]$

m* × *n is called the size of the matrix.

$$\mathbf{A} \equiv [a_{ij}] = \begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nm} \end{vmatrix}$$

If $m = n$, we call **A** an $n \times n$ **square matrix**. A matrix that is not square is called a **rectangular matrix**.

1.2 Equality of Matrices:

Two matrices **A** = $[a_{jk}]$ and **B** = $[b_{jk}]$ are equal (written **A** = **B**) if and only if they have the same size, and the corresponding elements are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**.

Example: **Equality of Matrices**

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

Then

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{matrix}$$

1.3 Addition of Matrices

The sum of two matrices **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ *of the same size* is written **A** + **B** and has the entries $a_{ij}+b_{ij}$ obtained by adding the corresponding entries of **A** and **B**. Matrices of different sizes cannot be added.

Example:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}, \quad \text{then} \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

1.4 Scalar Multiplication (Multiplication by a Number)

The product of any $m\times n$ matrix **A** = $[a_{ij}]$ and any **scalar** c (number c) is written c**A** and is the $m\times n$ matrix c**A** = $[ca_{jk}]$ obtained by multiplying each entry of **A** by c.

Example:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Rules for Matrix Addition and Scalar Multiplication:

(a) **A** + **B** = **B** + **A**

(b) (**A** + **B**) + **C** = **A** + (**B** + **C**)

(c) **A** + **0** = **A**

(d) **A** +(-**A**)= **0**

(a) c(**A** + **B**) = c**A** + c**B**

(b) $(c + k)$**A** = c**A** + k**A**

(c) $c(k$**A**) = (ck)**A**

(d) **IA** = **A**

1.4 Matrix multiplication

The product **C** = **AB** (in this order) of an $m\times n$ matrix **A** = $[a_{ij}]$ times an $r\times p$ matrix **B**=$[b_{ij}]$ is defined if and only if $r = n$ and is then the $m\times p$ matrix **C** = $[c_{ij}]$ with entries

$$c_{ij} = \sum_{l=1}^{n} a_{il}\, b_{lj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}\, b_{nj} \qquad i = 1,\cdots,m$$

$$j = 1,\cdots,p$$

$$\begin{array}{cccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m\times n] & [n\times p] & & [m\times p] \end{array}$$

Example:

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined. ■

Note: *Matrix Multiplication is Not Commutative,* ***AB*** *≠* ***BA in general****.*

Rules of Matrix Multiplication:

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

1.5 Transpose of a Matrix

Given any $m\times n$ matrix $\mathbf{A}=[a_{ij}]$, we define the transpose of **A**, denoted as $\mathbf{A}^{\mathrm{T}}$ and read as “**A**-transpose” as

$$\mathbf{A}^{\mathrm{T}} = [a_{ij}]^{\mathrm{T}} = [a_{ji}] = \begin{bmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{1n} & a_{2n} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

1.6 Types of Matrices:

Type of Matrix	Details	Example
Row Matrix	$\mathbf{A} = [a_{ij}]_{1\times n}$	[0 1 1 -2]
Column Matrix	$\mathbf{A} = [a_{ij}]_{m\times 1}$	$\begin{bmatrix} -1 \\ 2 \\ -4 \\ 5 \end{bmatrix}$
Zero or Null Matrix	$\mathbf{A} = [a_{ij}]_{m\times n}$ where, $a_{ij} = 0$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
Singleton Matrix	$\mathbf{A} = [a_{ij}]_{m\times n}$ where, $m = n = 1$	[2]
Horizontal Matrix	$[a_{ij}]_{m\times n}$ where $n > m$	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$
Vertical Matrix	$[a_{ij}]_{m\times n}$ where, $m > n$	$\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$
Square Matrix	$[a_{ij}]_{m\times n}$ where, $m = n$	$\begin{bmatrix} 4 & 7 \\ 9 & 13 \end{bmatrix}$
Diagonal Matrix	$\mathbf{A} = [a_{ij}]$, $a_{ij} = 0$, when $i \neq j$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$
Identity (Unit) Matrix (I)	$\mathbf{A} = [a_{ij}]_{m\times n}$ where, $a_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Triangular Matrix	Can be either upper triangular ($a_{ij} = 0$, when $i > j$) or lower triangular ($a_{ij} = 0$ when $i < j$)	$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ Upper Matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ Lower Matrix
Singular Matrix	$\|\mathbf{A}\| = 0$	
Non-Singular Matrix	$\|\mathbf{A}\| \neq 0$	
Symmetric Matrix	$\mathbf{A} = [a_{ij}]$ where, $a_{ij} = a_{ji}$ or $\mathbf{A}=\mathbf{A}^T$ (**A** is a square matrix)	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{pmatrix}$
Skew-Symmetric Matrix	$\mathbf{A} = [a_{ij}]$ where, $a_{ij} = -a_{ji}$ or $\mathbf{A} = -\mathbf{A}^T$ (**A** is a square matrix)	$\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

1.7 Determinants

The scalar quantity associated with a square matrix, is called "determinant". We denote the determinant of an $n\times n$ matrix $\mathbf{A}=[a_{ij}]$ as

$$\det \mathbf{A}=\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} = \sum_{1}^{n} a_{ij}\, C_{ij}$$

$$|\mathbf{A}|= a_{i1}\, A_{i1} + a_{i2}\, A_{i2} + \ldots + a_{in}\, A_{in} = \sum_{j=1}^{n} a_{ij}\, C_{ij} \qquad \text{(row definition)}$$

$$|\mathbf{A}|= a_{1j}\, A_{1j} + a_{2j}\, A_{2j} + \ldots + a_{nj}\, A_{nj} = \sum_{i=1}^{n} a_{ij}\, C_{ij} \qquad \text{(column definition)}$$

where C_{ij} is called the cofactor of the a_{ij} element and is defined as

$$C_{ij} \equiv (-1)^{i+j}\, M_{ij}$$

where M_{ij} is called the minor of a_{ij}, namely the determinant of $(n-1) \times (n-1)$ matrix that survives when the row and column containing a_{ij} are struck out.

Special Cases in Determinants:

1- Determinant of order 1: The determinant of a matrix of the first order (n=1) is the element itself.
   Example: If $\mathbf{A}=[-1]$, then $\det\mathbf{A}= -1$.

2- Determinant of order 2:

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = (a)(d) - (b)(c)$$

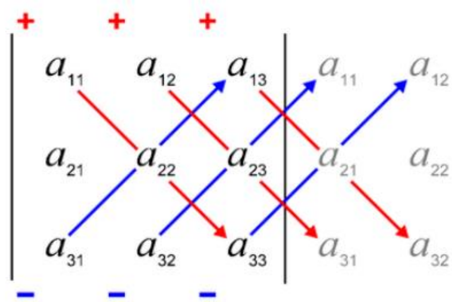
$$\det \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} = (3)(6) - (4)(5) = 18 - 20 = -2$$

3- Determinant of order 3:

(a) The Leibniz formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

(b) The rule of Sarrus:

det(**A**)= 

det(**A**)= $a_{11}\, a_{22}\, a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$

Notes:

i. The scheme of Sarrus for calculating the determinant of a 3 × 3 matrix does not carry over into higher dimensions.

ii. The Leibniz formula expresses the determinant of an $n \times n$-matrix in a manner which is consistent across higher dimensions.

4- If the matrix is upper or lower triangular matrix, the determinant of the matrix is the product of the elements on the main diagonal.

Some Properties of determinants:

1- If any row or column of a determinant det**A** only contains zero elements, then det**A**= 0.

2- If **A** is a square matrix with the transpose $\mathbf{A}^T$, then det**A** = det$\mathbf{A}^T$.

3- If each element of a row or column of a square matrix **A** is multiplied by a constant k, then the value of the determinant is $k$det**A**.

4- If two rows (or columns) of a square matrix are interchanged, the sign of the determinant is changed.

5- If any two rows or columns of a square matrix **A** are proportional, then det**A** = 0.

Example 1: Evaluate the determinant of the given matrix by cofactor expansion.

$$\begin{pmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{pmatrix}$$

Solution: $|\mathbf{A}|= \begin{vmatrix} 2 & 1 & -2 & 1 \\ 0 & 5 & 0 & 4 \\ 1 & 6 & 1 & 0 \\ 5 & -1 & 1 & 1 \end{vmatrix} = 5\begin{vmatrix} 2 & -2 & 1 \\ 1 & 1 & 0 \\ 5 & 1 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 & -2 \\ 1 & 6 & 1 \\ 5 & -1 & 1 \end{vmatrix} = 5(0) + 4(80) = 320$

Example 2: Explain the following results:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 3 \\ 4 & 10 & 14 & 6 \\ 3 & 4 & 2 & 7 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 6 & 6 \\ 4 & 7 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 9 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 2 & 1 & 2 & 3 \\ 6 & 7 & 6 & 9 \\ 0 & 6 & 0 & 0 \\ 1 & 2 & 1 & 4 \end{vmatrix} = 0$$

1.8 Matrix Partitioning:

The idea of matrix partitioning is that any matrix **A** (which is larger than 1×1) may be portioned into a number of smaller matrices called *blocks*.

Example: The matrix

$$A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$$

- Can also be written as the 2×3 **partitioned** (or **block**) **matrix**
$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$
- Whose entries are the *blocks* (or *submatrices*)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}$$

Example : Find AB, where

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ 0 & -4 & -2 & 7 & -1 \end{bmatrix}, B = \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ -1 & 3 \\ 5 & 2 \end{bmatrix}$$

Solution:

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix}$$

$$= \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right]$$

1.9 Inverse of a Matrix

If **A** is a non-singular square matrix, there is an existence of $n \times n$ matrix $\mathbf{A}^{-1}$, which is called the inverse matrix of **A**, such that it satisfies the property:

$$\mathbf{A}^{-1} = \frac{1}{\boldsymbol{A}} \quad \text{and } \mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where **I** is the Identity matrix. The inverse of a nonsingular $n \times n$ matrix $\mathbf{A}=[a_{ij}]$ is:

$$\mathbf{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \left[C_{ij}\right]^T = \frac{1}{|\boldsymbol{A}|} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} = \frac{\mathrm{adj}(\boldsymbol{A})}{|\boldsymbol{A}|}$$

where $[\boldsymbol{C}_{ij}]^{\mathrm{T}}$ is the transpose of cofactors matrix of **A**, adj(**A**) is the adjoint matrix of **A**.

Properties of the Inverse:

Let **A** and **B** be nonsingular matrices. Then

(i) $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

(ii) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(iii) $(\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$

Example: Find the inverse of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$.

SOLUTION Since $\det \mathbf{A} = 12$, we can find $\mathbf{A}^{-1}$ from (5). The cofactors corresponding to the entries in **A** are

$$C_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad C_{12} = -\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = 5 \qquad C_{13} = \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = -2 \quad C_{22} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2 \qquad C_{23} = -\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 6$$

$$C_{31} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad C_{32} = -\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = -2 \quad C_{33} = \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = 6.$$

From (5) we then obtain

$$\mathbf{A}^{-1} = \frac{1}{12}\begin{pmatrix} 1 & -2 & 2 \\ 5 & 2 & -2 \\ -3 & 6 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & -\frac{1}{6} & \frac{1}{6} \\ \frac{5}{12} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

1.10 Solution of linear simultaneous equations:

A system of m linear equations in n variables $x_1, x_2, \ldots, x_n$,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \qquad (1)$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

can be written completely as a matrix **AX=B**, where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

Let us suppose that $m = n$ in the equation above, so that the coefficient matrix **A** is $n \times n$. In particular, if **A** is nonsingular, then the system **AX = B** can be solved by:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Where **X** is variable matrix, **A** coefficients matrix and **B** constant matrix. (Note: if det **A**=0, the system of linear equations is called **inconsistent**, i.e. there is no unique solution).

<u>Example</u>: Use the inverse of the coefficient matrix to solve the system

$$\begin{aligned} 2x_1 + \qquad\quad x_3 &= 2 \\ -2x_1 + 3x_2 + 4x_3 &= 4 \\ -5x_1 + 5x_2 + 6x_3 &= -1 \end{aligned}$$

Solution:

$$\mathbf{A}^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}$$ (*Home work !*)

Thus

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 19 \\ 62 \\ -36 \end{pmatrix}$$

Consequently, $x_1 = 19$, $x_2 = 62$, and $x_3 = -36$.

In addition to the *inverse matrix* method shown above, there are two another widely used important methods:

1- Cramer's Rule

Let **A** be the coefficient matrix of system (1). If det **A** ≠ 0, then the solution of (1) is given by

$$x_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}}\ ,\ x_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}}\ ,\ \ldots,\ x_n = \frac{\det \mathbf{A}_n}{\det \mathbf{A}}$$

where A_k, k = 1, 2, . . ., *n*, is defined as:

$$\mathbf{A}_k = \begin{array}{c} \begin{matrix} \phantom{a_{11}} & \phantom{a_{12}} & & \phantom{a_{1\ k-1}} & k\text{th column} & \phantom{a_{1\ k+1}} & & \phantom{a_{1n}} \\ & & & & \downarrow & & & \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\ k-1} & b_1 & a_{1\ k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\ k-1} & b_2 & a_{2\ k+1} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\ k-1} & b_n & a_{n\ k+1} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

Example: Use Cramer's rule to solve the system

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= 3 \\ 5x_1 + 4x_2 - 2x_3 &= 1. \end{aligned}$$

SOLUTION The solution requires the evaluation of four determinants:

$$\det \mathbf{A} = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 13, \qquad \det \mathbf{A}_1 = \begin{vmatrix} 7 & 2 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = -39,$$

$$\det \mathbf{A}_2 = \begin{vmatrix} 3 & 7 & 1 \\ 1 & 3 & 3 \\ 5 & 1 & -2 \end{vmatrix} = 78, \qquad \det \mathbf{A}_3 = \begin{vmatrix} 3 & 2 & 7 \\ 1 & -1 & 3 \\ 5 & 4 & 1 \end{vmatrix} = 52.$$

Thus, (6) gives

$$x_1 = \frac{\det \mathbf{A}_1}{\det \mathbf{A}} = -3, \quad x_2 = \frac{\det \mathbf{A}_2}{\det \mathbf{A}} = 6, \quad x_3 = \frac{\det \mathbf{A}_3}{\det \mathbf{A}} = 4.$$

2- Gaussian Elimination Method:

There are three types of elementary row operations which may be performed on the rows of a matrix:

1- Swap the positions of two rows.
2- Multiply a row by a non-zero scalar.
3- Add to one row a scalar multiple of another.

The solution is then found by the process called **back-substitution**.

Example: Solve the following linear system using Gaussian elimination:

$$\begin{aligned} 2x_1 + 6x_2 + x_3 &= 7 \\ x_1 + 2x_2 - x_3 &= -1 \\ 5x_1 + 7x_2 - 4x_3 &= 9. \end{aligned}$$

Solution: Using row operations on the augmented matrix of the system, we obtain:

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right) \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

$$\overset{\substack{-2R_1+R_2 \\ -5R_1+R_3}}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right) \overset{\frac{1}{2}R_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$$\overset{3R_2+R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{11}{2} & \frac{55}{2} \end{array}\right) \overset{\frac{2}{11}R_3}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right).$$

Substituting $x_3 = 5$ into the second equation gives $x_2 = -3$. Substituting both these values back into the first equation finally yields $x_1 = 10$.

Home Works:

Exercises 8.2: Problems 1-20 (Only use Gaussian elimination).
Exercises 8.4: All Problems.
Exercises 8.5: 1-16.
Exercises 8.7: Problems 1-11.