Def
Let $f:[a,b]\to\mathbb{R}$ be a bounded function, and $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$. Set $\Delta x_j = x_j - x_{j-1}$ for $j=1,2,\dots,n$.

1) The upper Riemann sum of f over P is the number

$$U(f,P)=\sum_{j=1}^{n}\Delta x_j \sup_{[x_{j-1},x_j]} f(x) .$$

2) The lower Riemann sum of f over P is the number

$$L(f,P)=\sum_{j=1}^{n}\Delta x_j \operatorname{Inf}_{[x_{j-1},x_j]} f(x)$$

Remark(1) Since we assumed that f is bounded, the numbers

$$\sup_{[x_{j-1},x_j]} f(x) \quad \text{and} \quad \inf_{[x_{j-1},x_j]} f(x) \quad ; \quad j=1,2,\dots,n$$

exist and are finite.

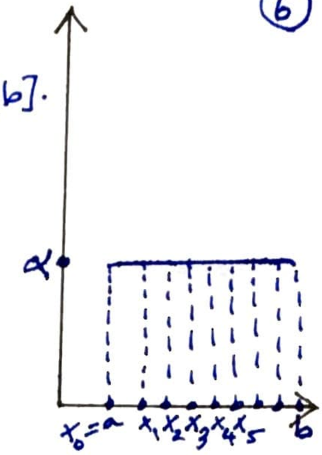
Remark(2) Notice that

$$\sum_{j=1}^{n}\Delta x_j = \sum_{j=1}^{n}(x_j - x_{j-1})$$

$$= (x_1-x_0)+(x_2-x_1)+\dots+(x_{n-1}-x_{n-2})+(x_n-x_{n-1})$$

$$= x_n - x_1 = b-a$$

Example Let $f(x)=\alpha$ be the constant function on $[a,b]$. Find $U(f,P)$ and $L(f,P)$, where P is any partion of $[a,b]$.



Let $P=\{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$.

Then, $U(f,P)=\sum_{j=1}^{n} \Delta x_j \sup_{[x_{j-1},x_j]} f(x)$

$$= \sum_{j=1}^{n} \Delta x_j (\alpha)$$

$$= \alpha \sum_{j=1}^{n} \Delta x_j = \alpha(b-a).$$

Similarly, we have:

$$L(f,P)=\sum_{j=1}^{n} \Delta x_j \inf_{[x_{j-1},x_j]} f(x) = \alpha(b-a).$$

Example

Let $f(x)=\begin{cases} 0, & \text{if } x \text{ is irrational,} \\ 1, & \text{if } x \text{ is rational.} \end{cases}$

Find $U(f,P)$ and $L(f,P)$ for any partition P of the interval $[a,b]$.

Notice that in each sub-interval $[x_{j-1}, x_j]$, there are infinitely many rational numbers and also there are infinitely many irrational numbers.

That is, in each sub-interval $[x_{j-1}, x_j]$, we have:

$$\sup_{[x_{j-1},x_j]} f(x) = 1 \text{ and } \inf_{[x_{j-1},x_j]} f(x) = 0 \text{ for } j=1,2,\dots,n.$$

Therefore, $U(f,P) = \sum_{j=1}^{n} \Delta x_j \sup_{[x_{j-1},x_j]} f(x) = \sum_{j=1}^{n} \Delta x_j (1) = b-a$

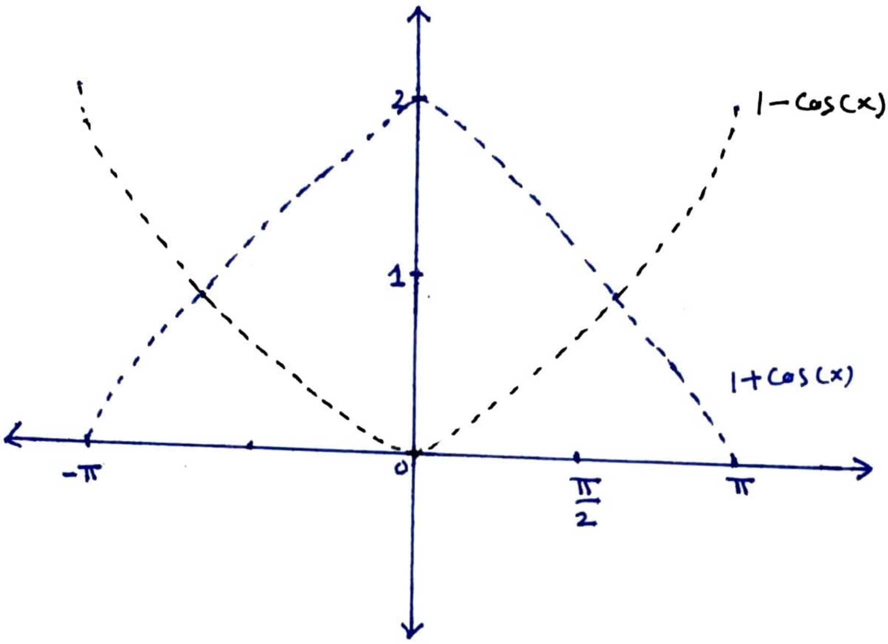
$$L(f,P) = \sum_{j=1}^{n} \Delta x_j \inf_{[x_{j-1},x_j]} f(x)$$

$$= \sum_{j=1}^{n} \Delta x_j (0) = 0$$

Example Let $P := -\pi < 0 < \frac{\pi}{2} < \pi$ be a partition of $[-\pi, \pi]$, and

$$f(x) = \begin{cases} 1+\cos(x), & \text{if } x \text{ is rational;} \\ 1-\cos(x), & \text{if } x \text{ is irrational.} \end{cases}$$

Find $U(f,P)$ and $L(f,P)$.



$$U(f,P) = \sum_{j=1} \Delta x_j \sup_{[x_{j-1},x_j]} f(x)$$

$$= (0-(-\pi)) \sup_{[-\pi,0]} f(x) + \left(\frac{\pi}{2}-0\right) \sup_{[0,\frac{\pi}{2}]} f(x) + \left(\pi-\frac{\pi}{2}\right) \sup_{[\frac{\pi}{2},\pi]} f$$

$$= \pi * (2) + \frac{\pi}{2} * (2) + \frac{\pi}{2} * (2)$$

$$= 2\pi + \pi + \pi = 4\pi$$

$$L(f,P) = \sum_{j=1}^{3} \Delta x_j \inf_{[x_{j-1},x_j]} f(x)$$

$$= (0-(-\pi)) \inf_{[-\pi,0]} f(x) + \left(\frac{\pi}{2}-0\right) \inf_{[0,\frac{\pi}{2}]} f(x) + \left(\pi-\frac{\pi}{2}\right) \inf_{[\frac{\pi}{2},\pi]} f(x)$$

$$= (\pi) * (0) + \left(\frac{\pi}{2}\right) * (0) + \left(\frac{\pi}{2}\right) * (0)$$

$$= 0$$

Remark $L(f,P) \le U(f,P)$ for all partitions P and all bounded functions f.

Proof clearly we see that $\inf_{[x_{j-1},x_j]} f(x) \le \sup_{[x_{j-1},x_j]} f(x)$; $j=1,2,\dots,n$

Then from the definitions of $U(f,P)$ and $L(f,P)$, we have $L(f,P) \le U(f,P)$ for any partition P of $[a,b]$ ∎

Theorem Let f be a bounded function on $[a,b]$, and Let P be any partition of $[a,b]$, and Q be a refinement of P, then

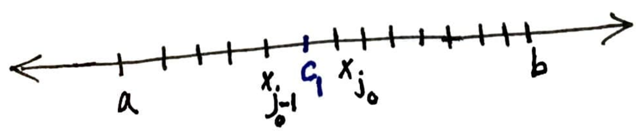
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Proof Let $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$. Since Q is finer than P, then Q can be obtained from P in a finite number of steps by adding one point at a time. That is, if

$$Q = P \cup \{c_1, c_2, \dots, c_m\}, \quad c_i \notin P \ \forall i=1,2,\dots,m$$

then it suffices to prove the inequalities above for the special case:

$$Q = P \cup \{c_1\}; \quad c_1 \notin P.$$



First, we prove the inequality $U(f,Q) \le U(f,P)$.

We assume that there is a unique index $j_0 \in \{1,2,\dots,n\}$ such that $x_{j_0-1} < c_1 < x_{j_0}$.

Therefore, notice that:

$$U(f,Q) = \sum_{\substack{j=1 \\ j\neq j_0,\, j\neq j_0-1}}^{n} \Delta x_j \sup_{[x_{j-1},x_j]} f(x) + (c_1 - x_{j_0-1}) \sup_{[x_{j_0-1},c_1]} f(x) + (x_{j_0} - c_1) \sup_{[c_1,x_{j_0}]} f(x)$$

$$U(f,P) = \sum_{j=1}^{n} \Delta x_j \sup_{[x_{j-1},x_j]} f(x)$$

$$= \sum_{\substack{j=1 \\ j\neq j_0,\, j\neq j_0-1}}^{n} \Delta x_j \sup_{[x_{j-1},x_j]} f(x) + (x_{j_0} - x_{j_0-1}) \sup_{[x_{j_0-1},x_{j_0}]} f(x)$$

Then, we see that:

$$U(f,Q) - U(f,P) = (c_1 - x_{j_0-1}) \sup_{[x_{j_0-1},c_1]} f(x) + (x_{j_0} - c_1) \sup_{[c_1,x_{j_0}]} f(x) - (x_{j_0} - x_{j_0-1}) \sup_{[x_{j_0-1},x_{j_0}]} f(x)$$

since $\sup_{[c_1,x_{j_0}]} f(x) \leq \sup_{[x_{j_0-1},x_{j_0}]} f(x)$ and $\sup_{[x_{j_0-1},c_1]} f(x) \leq \sup_{[x_{j_0-1},x_{j_0}]} f(x)$

Then, it follows that:

$$U(f,Q) - U(f,P) \leq (c_1 - x_{j_0-1}) \sup_{[x_{j_0-1},x_{j_0}]} f(x) + (x_{j_0} - c_1) \sup_{[x_{j_0-1},x_{j_0}]} f(x) - (x_{j_0} - x_{j_0-1}) \sup_{[x_{j_0-1},x_{j_0}]} f(x)$$

$$= \left[(c_1 - x_{j_0-1}) + (x_{j_0} - c_1) - (x_{j_0} - x_{j_0-1}) \right] \sup_{[x_{j_0-1},x_{j_0}]} f(x)$$

$$= 0$$

That is, $U(f,Q) - U(f,P) \leq 0$, which is $\boxed{U(f,Q) \leq U(f,P)}$.

From the Previous remark we have proved that:

$$\boxed{L(f,Q) \leq U(f,Q)}$$

To prove the last inequality $L(f,P) \leq L(f,Q)$, we use the same technique that we used in the Upper Riemann sum, so we leave it as an easy excercise for the students.