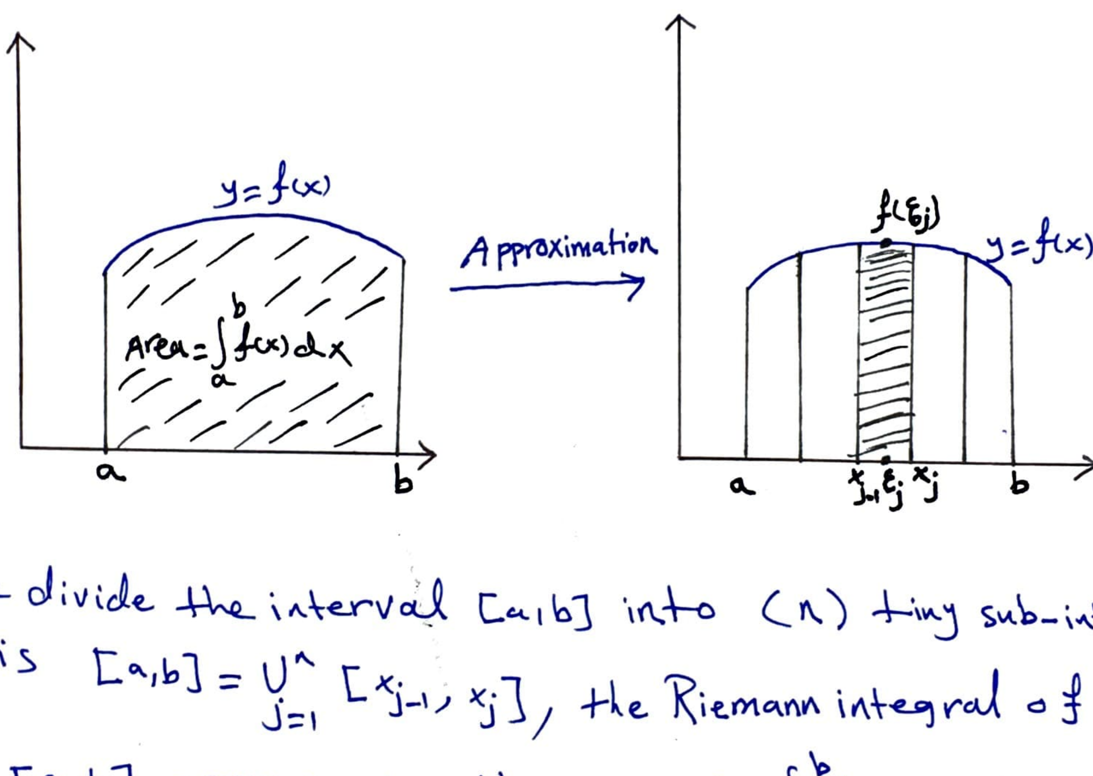
Chapter (1)

Riemann Integration

We have seen in our earlier study in Calculus (Math 101, Math 102, Math 201), the integral of a non-negative bounded real valued function f over an interval $[a,b]$, $\int_a^b f(x)\,dx$ (if it exists), is the area of the region bounded by the curves $y=f(x)$, $y=0$, $x=a$, and $x=b$.



If we divide the interval $[a,b]$ into (n) tiny sub-intervals that is $[a,b]=\bigcup_{j=1}^{n}[x_{j-1},x_j]$, the Riemann integral of f over $[a,b]$ approximates the area $A=\int_a^b f(x)\,dx$ by rectangles whose bases are $\Delta x_j=(x_j-x_{j-1})$ and whose heights are $f(\xi_j)$ for some $\xi_j\in[x_{j-1},x_j]$, that is,

$$\int_a^b f(x)\,dx=\lim_{\substack{\max(\Delta x_j)\to 0\\ 1\le j\le n}}\left[\sum_{j=1}^{n}\Delta x_j\, f(\xi_j)\right].$$

To make the idea of Riemann integral clear, we give the following definitions:

Definition

Let $a, b \in \mathbb{R}$ with $a<b$.

1. A Partition of the interval $[a,b]$ is a set of Points

$$P=\{x_0, x_1, \ldots, x_n\} \text{ such that}$$

$$a=x_0<x_1<x_2<\cdots<x_n=b.$$

In other words, $[a,b]=\bigcup_{j=1}^{n}[x_{j-1}, x_j]$

2. The norm of a Partition $P=\{x_0, x_1, \ldots, x_n\}$ is the number

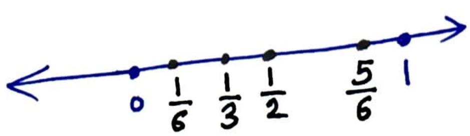
$$\|P\|=\max_{1\leq j\leq n}|x_j-x_{j-1}|=\max_{1\leq j\leq n}(x_j-x_{j-1}).$$

3. A refinement of a Partition $P=\{x_0, x_1, \ldots, x_n\}$ is a Partition Q of $[a,b]$ which satisfies $P\subseteq Q$. In which case we say that Q is finer than P.

Examples Let $[a,b]=[0,1]$ be the unit interval

1. $P=\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}, 1\}$ is a Partition of $[0,1]$

$$\|P\|=\max_{1\leq j\leq 5}(x_{j-1}-x_j)$$

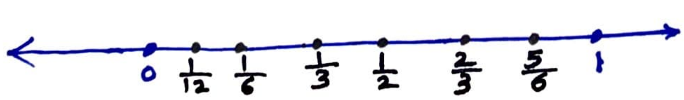


$$=\max\{(\tfrac{1}{6}-0), (\tfrac{1}{3}-\tfrac{1}{6}), (\tfrac{1}{2}-\tfrac{1}{3}), (\tfrac{5}{6}-\tfrac{1}{2}), (1-\tfrac{5}{6})\}$$

$$=\max\{\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{6}\}$$

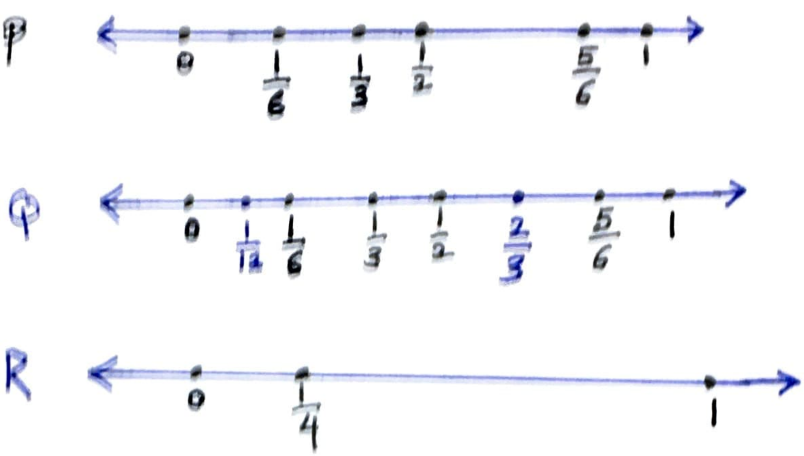
$$=\frac{1}{3}$$

2. $Q=\{0, \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$ is a Partion of $[0,1]$



$$\|Q\| = \max_{1\le j\le 7}(x_j - x_{j-1}) = \max\left\{\frac{1}{12}, \frac{1}{6}\right\} = \frac{1}{6}$$

Notice that $P \subseteq Q$, then Q is a refinement of P.



(3) $R = \{0, \frac{1}{4}, 1\}$ is a partition of $[0,1]$ and $\|R\| = \frac{3}{4}$

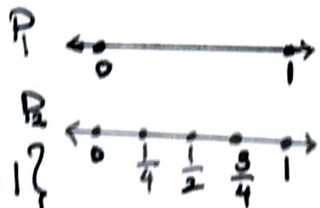
(4) The dyadic partition prove that for each $n \in \mathbb{N}$,

$P_n = \{\frac{j}{2^n},\ j = 0, 1, \dots, 2^n\}$ is a partition of the interval $[0,1]$,

and P_m is finer than P_n when $m > n$.

Notice that $n=1$, $P_1 = \{0, 1\}$

$n = 2$, $P_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$



Proof: Fix $n \in \mathbb{N}$, if $x_j = \frac{j}{2^n}$, then

$0 = x_0 < x_1 < \dots < x_{2^n} = 1$. Thus P_n is a partition of $[0,1]$.

Now, let $m > n$ and set $r = m - n$, $k = j2^r$.

Notice that: if $0 \leq j \leq 2^n$, then $0 \leq k \leq 2^n 2^r$

That is, $0 \leq k \leq 2^m$

We need to prove that:

$$P_n = \left\{ \frac{j}{2^n},\ 0 \leq j \leq 2^n \right\} \subseteq P_m = \left\{ \frac{k}{2^m},\ 0 \leq k \leq 2^m \right\}$$

Let $x_j \in P_n$ for some $0 \leq j \leq 2^n$

Then $x_j = \frac{j}{2^n}$, for some $0 \leq j \leq 2^n$

Therefore, x_j can be written as follows:

$$x_j = \frac{j}{2^n} = \frac{j}{2^m} \cdot \frac{1}{2^{n-m}} = \frac{j}{2^m} \frac{1}{2^{-r}} = \frac{j 2^r}{2^m} = \frac{k}{2^m}$$

Thus $x_j = \frac{k}{2^m}$ for some $0 \leq k \leq 2^m$

That is $x_j \in P_m$

Therefore, $P_n \subseteq P_m$ for $n < m$ ■

Remarks on the dyadic partition P_n

1. P_n has $(2^n + 1)$ elements
2. $\|P_n\| = \frac{1}{2^n}$ for any $n \in \mathbb{N}$.
3. $P_n := 0 < \frac{1}{2^n} < \frac{2}{2^n} < \frac{3}{2^n} < \cdots < \frac{2^n - 1}{2^n} < 1$

Remarks on Partitions

1. If P and Q are partitions of $[a,b]$, then $P \cup Q$ is finer than <u>both</u> P and Q.

2. If Q is a refinement of P, then $\|P\| \leq \|Q\|$.