Def (Mõbius transformation)

A mapping of the form $S(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$, is called a linear fractional transformation. If $ad-cb \neq 0$, then $S(z)$ is called Mõbius transformation.

Remarks

1. If S is a Mõbius transformation, then $S^{-1}(z) = \frac{dz-b}{-cz+a}$ satisfies $SS^{-1}(z) = z = S^{-1}S(z)$. That is S^{-1} is the inverse mapping of S.

2. If S and T are both Mõbius transformations, then the composition $S \circ T$ is also Mõbius transformation.

3. In our study, the only linear fractional transformations we will consider are Mõbius transformations.

4. If $S(z) = \frac{az+b}{cz+d}$, then we cannot have $a=c=0$ or $d=c=0$ because either case would contradict the assumption of the Mõbius transformation $ad-bc \neq 0$.

5. The Mõbius transformation $S(z) = \frac{az+b}{cz+d}$ may define on $\mathbb{C}_\infty$ as follows:
$S(\infty) = \frac{a}{c}$ and $S(-\frac{d}{c}) = \infty$.
Since S has inverse, then S maps $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$.

6. If $S(z) = z+a$ then S is called translation.
If $S(z) = rz$ with $r>0$ then S is called dilation.
If $S(z) = e^{i\theta} z$ then S is called rotation.
If $S(z) = \frac{1}{z}$ then S is called inversion.

Proposition If S is a Mõbius transformation then S is the composition of translations, dilations, rotations, and the inversion (of course, some of these may be missing.)

Def (Fixed Point)

Let $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Mõbius transformation. A Point z in $\mathbb{C}_\infty$ is called a fixed Point for S if it satisfies $S(z)=z$.

Remark If z is a fixed Point for $S(z)=\frac{az+b}{cz+d}$, then we conclude that z satisfies the following equation

$$cz^2+(d-a)z-b=0.$$

Hence, a Mõbius transformation can have at most two fixed Points unless $S(z)=z$ for all z. (Why?).

Let S be a Mõbius transformation and let a_1, a_2, a_3 be three distinct points in $\mathbb{C}_\infty$ be such that

$$\begin{cases} S(a_1)=\alpha \\ S(a_2)=\beta \\ S(a_3)=\gamma \end{cases}$$

Let T be another Mõbius transformation with this Property:

$$\begin{cases} T(a_1)=\alpha \\ T(a_2)=\beta \\ T(a_3)=\gamma \end{cases}$$

Then we would have the following:

$$\begin{cases} T^{-1}\circ S(a_1)=a_1 \\ T^{-1}\circ S(a_2)=a_2 \\ T^{-1}\circ S(a_3)=a_3 \end{cases}$$

That is, $T^{-1}\circ S$ has three fixed Point, then $T^{-1}\circ S(z)=z$ for all $z\in\mathbb{C}_\infty$. which means that $T^{-1}\circ S=I \Rightarrow T=S$.

Hence a Mõbius transformation is uniquely determined by its action on any three given Points in $\mathbb{C}_\infty$.

Let z_2, z_3, z_4 be points in $\mathbb{C}_\infty$. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \begin{cases} \left(\dfrac{z - z_3}{z - z_4}\right)\cdot\left(\dfrac{z_2 - z_4}{z_2 - z_3}\right) & \text{if } z_2, z_3, z_4 \in \mathbb{C}; \\[2ex] \dfrac{z - z_3}{z - z_4} & \text{if } z_2 = \infty; \\[2ex] \dfrac{z_2 - z_4}{z - z_4} & \text{if } z_3 = \infty; \\[2ex] \dfrac{z - z_3}{z_2 - z_3} & \text{if } z_4 = \infty. \end{cases}$$

In any case $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$ and S is the only transformation having this property.

<u>Def</u> If $z_1 \in \mathbb{C}_\infty$ then (z_1, z_2, z_3, z_4) is the image of z_1 under the unique Möbius transformation that takes z_2 to 1, z_3 to 0, and z_4 to ∞. That is, $(z_1, z_2, z_3, z_4) = \dfrac{z_1 - z_3}{z_1 - z_4}\cdot\dfrac{z_2 - z_4}{z_2 - z_3}$.

<u>Remark</u> $(z_1, z_2, z_3, z_4) :=$ is called the cross ratio of z_1, z_2, z_3 and z_4.

<u>Example</u>

1. $(z_2, z_2, z_3, z_4) = 1$
2. $(z, 1, 0, \infty) = z$ [Three fixed points].

<u>Remark</u> If M be any Möbius transformation and w_2, w_3, w_4 are points such that $Mw_2 = 1$, $Mw_3 = 0$, $Mw_4 = \infty$ then $Mz = (z, w_2, w_3, w_4)$.

<u>Proposition</u> If z_2, z_3, z_4 are distinct points and T is any Möbius transformation then $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any z_1.

Proof Let $Sz = (z, z_2, z_3, z_4)$ and let $M = S \circ T^{-1}$; then $M(Tz_2) = 1$, $M(Tz_3) = 0$, $M(Tz_4) = \infty$. Then by the previous remark, we conclude that:

$Mz = (z, Tz_2, Tz_3, Tz_4)$. That is, $ST^{-1}(z) = (z, Tz_2, Tz_3, Tz_4)$ for all $z \in \mathbb{C}_\infty$. Take $z = Tz_1$ to obtain that:

$$Sz_1 = (Tz_1, Tz_2, Tz_3, Tz_4).$$

Proposition If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and w_2, w_3, w_4 are also distinct points of $\mathbb{C}_\infty$, then there is one and only one Möbius transformation S such that $Sz_2 = w_2$, $Sz_3 = w_3$, $Sz_4 = w_4$.

Proof Let $Tz = (z, z_2, z_3, z_4)$ and $Mz = (z, z_2, z_3, z_4)$

$$\begin{tikzcd} z_i \arrow[rr, "M^{-1}\circ T"] \arrow[dr, "T"'] & & w_i \arrow[dl, "M"] \\ & 1, 0, \infty & \end{tikzcd}$$

Then, Define $S = M^{-1} \circ T$

Therefore, $Sz_2 = w_2$, $Sz_3 = w_3$, $Sz_4 = w_4$.

The uniqueness of S can be done as follows:

Let $Rz_i = w_i$, $i = 2, 3, 4$.

Then $R^{-1} \circ S$ is a Möbius transformation such that $R^{-1} \circ S(z_i) = z_i$ $i = 2, 3, 4$.

That is $R^{-1} \circ S$ has three fixed points.

Then we conclude that $R^{-1} \circ S = I$ which means $R = S$ ∎

Remark Recall that a circle in $\mathbb{C}_\infty$ passing through ∞ corresponds to a straight line in $\mathbb{C}$. So, the word circle means either a circle or a straight line when dealing with Möbius transformations.

Proposition Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Then the cross ration $(z_1, z_2, z_3, z_4) = \dfrac{z_1 - z_3}{z_1 - z_4} \cdot \dfrac{z_2 - z_4}{z_2 - z_3}$ is a real number if and only if all four points lie on a circle. (circle or a straight line).

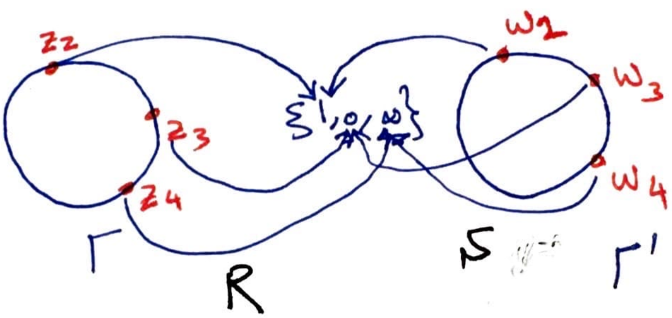
Theorem A Möbius transformation takes circles onto circles

Proof Let Γ be any circle in $\mathbb{C}_\infty$ and let S be any Möbius transformation. Let z_2, z_3, z_4 be three distinct points on Γ and put $w_j = Sz_j$ for $j = 2,3,4$. Then w_2, w_3, w_4 determine a circle Γ'. We only need to show that $S(\Gamma) = \Gamma'$.
Let $z \in \Gamma$, then the cross ration (z, z_2, z_3, z_4) is real by the Preceding proposition. But also we have that:

$$(z, z_2, z_3, z_4) = (Sz, w_2, w_3, w_4).$$

Therefore, (Sz, w_2, w_3, w_4) is real, which means that $Sz \in \Gamma'$. ∎

Now let Γ and Γ' be two circles in $\mathbb{C}_\infty$ and let $z_2, z_3, z_4 \in \Gamma$; $w_2, w_3, w_4 \in \Gamma'$.



Define:

$Rz = (z, z_2, z_3, z_4)$

$Sz = (z, w_2, w_3, w_4)$

Then $T = S^{-1} \circ R$ maps Γ onto Γ'.
Infact, $Tz_j = w_j$ for $j = 2,3,4$ and, as in the above proof it follows that $T(\Gamma) = \Gamma'$.

Proposition For any given circles Γ and Γ' in $\mathbb{C}_\infty$ there is a unique Möbius transformation T such that $T(\Gamma) = \Gamma'$.