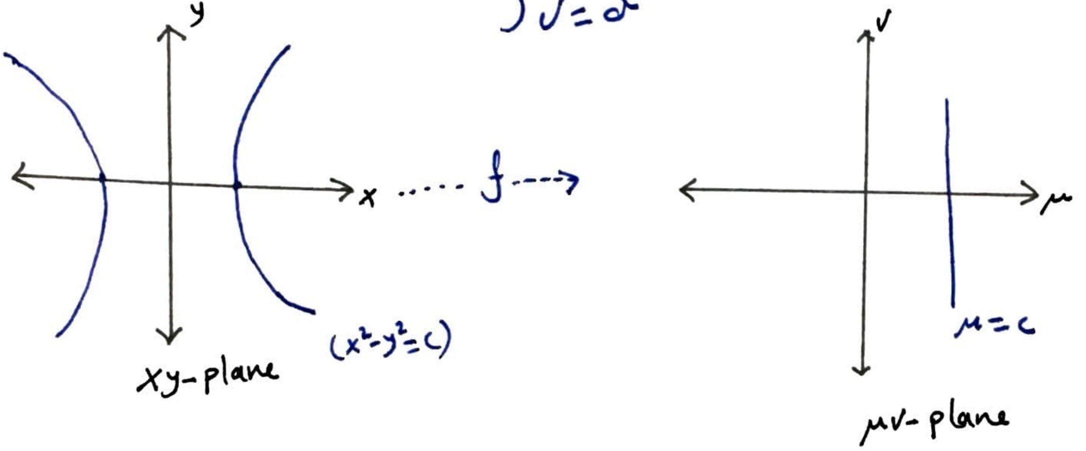
Analytic functions as mapping. Möbius transformations

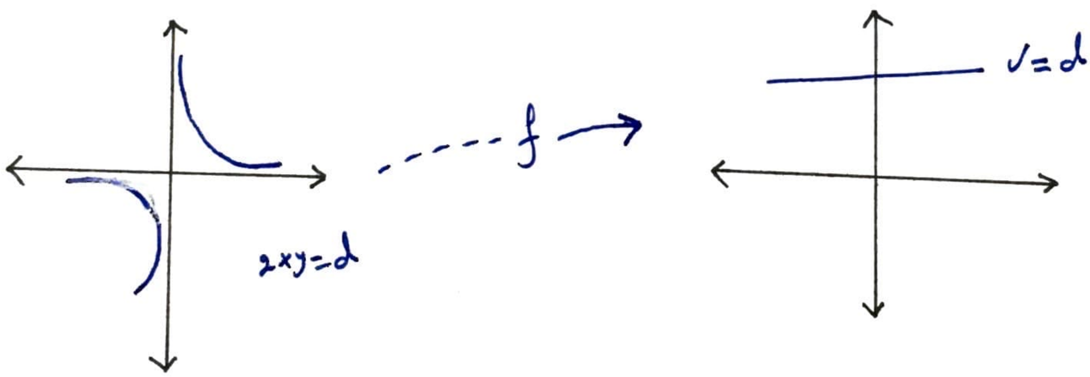
Consider the function defined by $f(z)=z^2$. If $z=x+iy$, then:

$$\begin{cases} f(z)=z^2=(x-y^2)+i(2xy) \\ \text{Assuming that: } f(z)=\mu+iv \\ \text{Then it follows that: } \mu=x^2-y^2 \text{ and } v=2xy. \end{cases}$$

Therefore, the hyperbolas $\begin{cases} x^2-y^2=c \\ 2xy=d \end{cases}$ in (xy-plane) are mapped by f

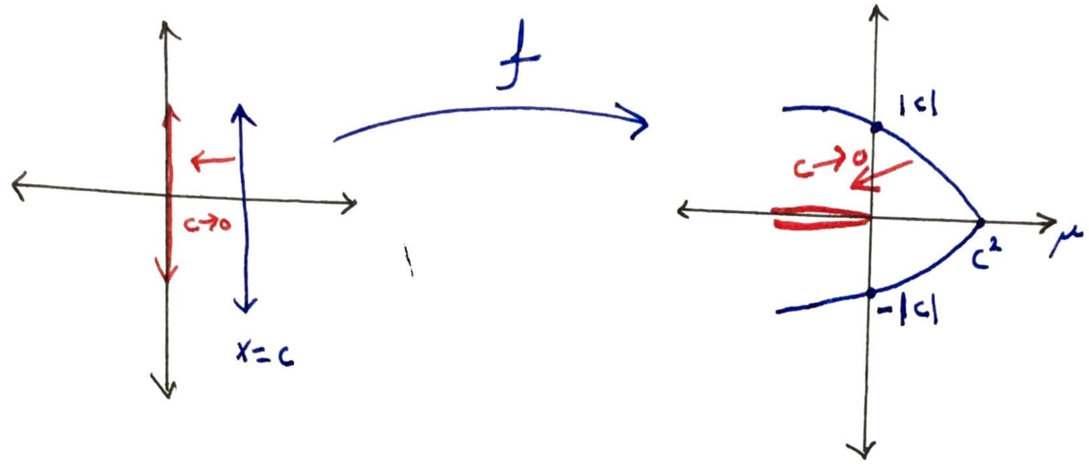
into the straight lines $\begin{cases} \mu=c \\ v=d \end{cases}$ in (μv-plane).





one interesting fact is that for c and d not zero, these hyperbolas intersects at right angle, just as their images do [Exercise: Think about Cauchy-Riemann]

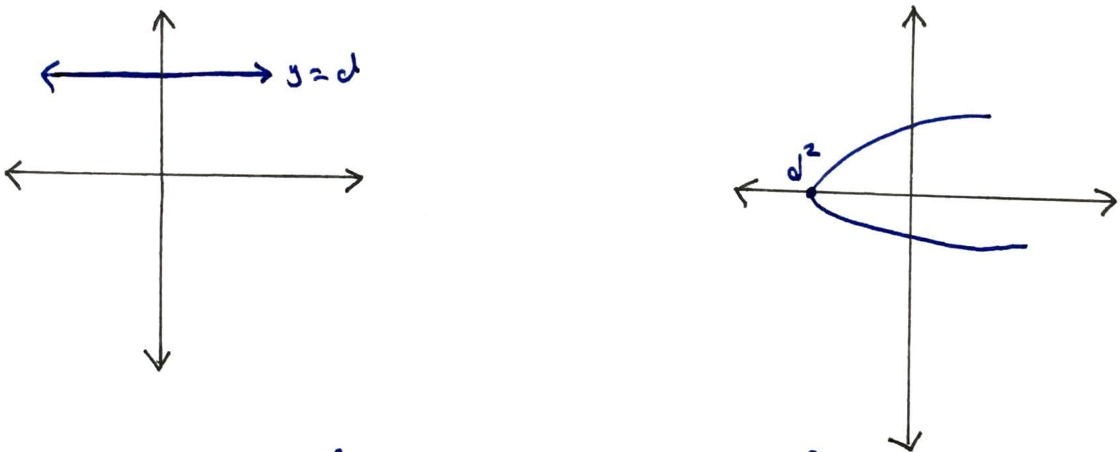
1. We examine the line $x=c$ for arbitrary y under the map $f(z)=z^2$.
We already have that $\mu=x^2-y^2$ and $v=2xy$ then for $x=c$, we obtain that: $\begin{cases} \mu=c^2-y^2 \\ v=2cy \end{cases} \Rightarrow \mu=c^2-\frac{1}{4c^2}v^2$ which is a parabola centered at $(c^2,0)$ pointing to the left.



2. We examine the line $y=d$ for arbitrary x under the map $f(z)=z^2$.

We have the following equations $\begin{cases} \mu = x^2-d^2 \\ \nu = 2dx \end{cases} \Rightarrow \mu = \frac{\nu^2}{4d} - d^2$

which is a parabola centered at $(-d^2, 0)$ pointing to the right.

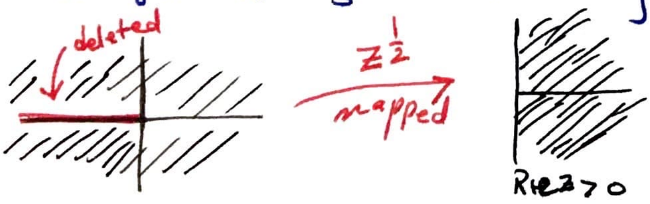


The parabolas $\mu = \frac{\nu^2}{4d} - d^2$ and $\mu = c^2 - \frac{1}{4c^2}\nu^2$ intersect at the points $(c^2-d^2, 2|cd|)$ and $(c^2-d^2, -2|cd|)$.

It is relevant to point out that as $c \to 0$ the parabola $\mu = c^2 - \frac{1}{4c^2}\nu^2$ gets closer and closer to the negative real axis. This corresponds to the fact that the function $z^{\frac{1}{2}}$ maps $G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$ onto the region $G' = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

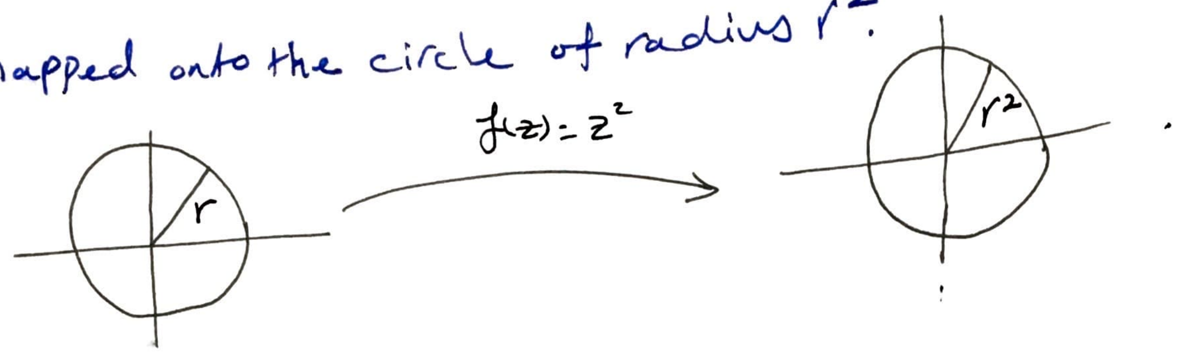
proof $z^{\frac{1}{2}} = e^{\frac{1}{2}\log(z)}$, $\arg(z) \in (-\pi, \pi)$

that is: $z^{\frac{1}{2}} = \frac{1}{2}|z|\, e^{i\frac{1}{2}\arg(z)}$, $\frac{1}{2}\arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$

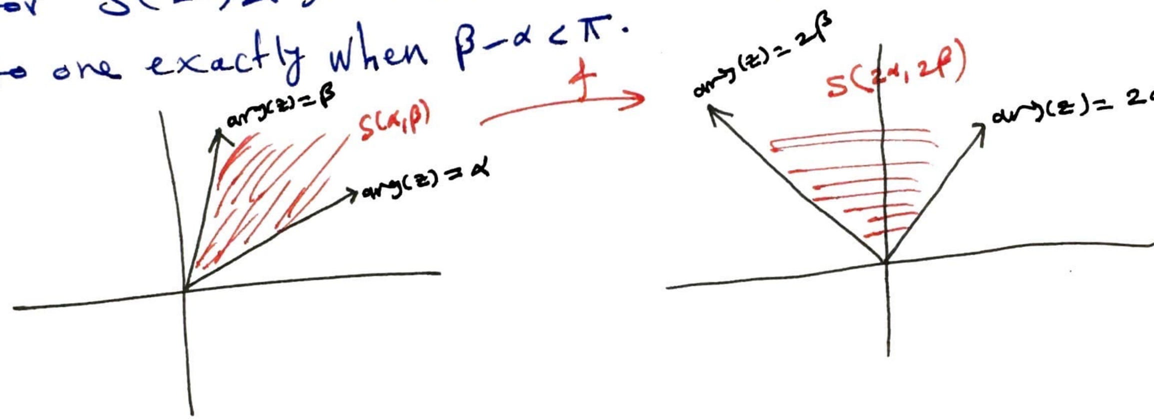


What happens to a circle of radius r about the origin?

If $z = re^{i\theta}$ then $f(z) = r^2 e^{2i\theta}$, thus the circle $z = re^{i\theta}$ mapped onto the circle of radius r^2.



Finally, what happens to the sector $S(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$, for $\alpha < \beta$? It is easily seen that the image of $S(\alpha,\beta)$ is the sector $S(2\alpha, 2\beta)$. The restriction of f to $S(\alpha,\beta)$ will be one to one exactly when $\beta - \alpha < \pi$.



In our study we proceed in order to answer the the following question:

Given two open connected sets G and Ω, is there an analytic function f defined on G such that $f(G) = \Omega$?

<u>Def</u> A path in a region $G \subset \mathbb{C}$ is a continuous function $\gamma : [a,b] \longrightarrow G$ for some interval $[a,b]$ in $\mathbb{R}$. If $\gamma'(t)$ exists for each $t \in [a,b]$ and $\gamma' : [a,b] \longrightarrow \mathbb{C}$ is continuous then γ is smooth path. Also γ is piecewise smooth if there is a partition of $[a,b]$, $a = t_0 < t_1 < \dots < t_n = b$, such that γ is smooth on each subinterval $[t_{j-1}, t_j]$, $j = 1, 2, \dots, n$.

Remarks

1. The continuous function $\gamma : [a,b] \to \mathbb{C}$ has derivative at $t \in (a,b)$ means that, the limit

$$\lim_{h \to 0} \frac{\gamma(t+h) \ \gamma(t)}{h}$$ exist, and in which case we denote $\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$.

2. The continuous function $\gamma : [a,b] \to \mathbb{C}$ has derivative at $t = a$ means that

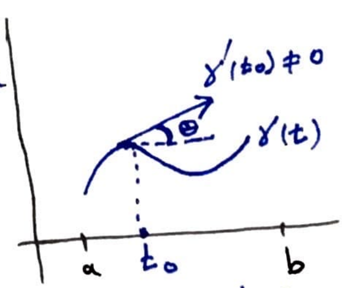
$$\lim_{h \to 0^+} \frac{\gamma(a+h) - \gamma(a)}{h} := \gamma'(a) \ (\text{Exist}) \ \left[\text{Right limit exist}\right]$$

3. The continuous function $\gamma : [a,b] \to \mathbb{C}$ has derivative at $t = b$ means that

$$\lim_{h \to 0^-} \frac{\gamma(b+h) - \gamma(b)}{h} := \gamma'(b) \ (\text{Exist}) \ \left[\text{Left limit exist}\right].$$

Conformal mapping

Suppose $\gamma : [a,b] \to G$ is a smooth Path and that for some t_0 in (a,b), $\gamma'(t_0) \neq 0$. Then γ has a tangent line at the Point $z_0 = \gamma(t_0)$. This line goes through the Point z_0 in the direction of the vector $\gamma'(t_0)$. The slope of this line is $\tan(\theta)$.

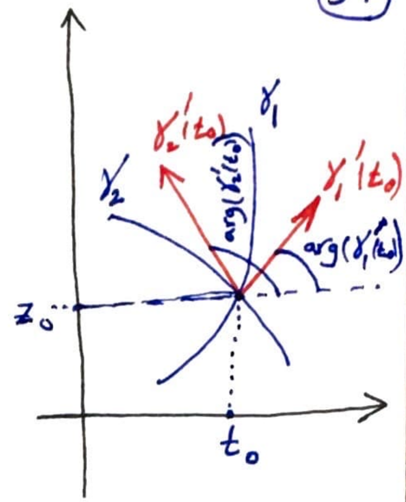


$\theta := \arg(\gamma'(t_0))$.

Now, let γ_1 and γ_2 be two smooth Paths intersect at the Point $t_0 \in (a,b)$. That is $\gamma_1(t_0) = \gamma_2(t_0) = z_0$.

Let $\gamma_1'(t_0) \neq 0$ and $\gamma_2'(t_0) \neq 0$, then define the angle between the paths γ_1 and γ_2 at z_0 to be the angle between their tangent lines, that is,

$$\text{Angle}(\gamma_1(t_0), \gamma_2(t_0)) = \text{Angle}(\gamma_1'(t_0), \gamma_2'(t_0))$$
$$= \arg \gamma_2'(t_0) - \arg \gamma_1'(t_0).$$



Now suppose that γ is a smooth path in G and $f: G \to \mathbb{C}$ is analytic. Then $\sigma = f \circ \gamma$ is also smooth path and

$$\sigma'(t) = f'(\gamma(t))\, \gamma'(t).$$

Let $z_0 = \gamma(t_0)$ and suppose that $\gamma'(t_0) \neq 0$ and $f'(z_0) \neq 0$, then $\sigma'(t_0) \neq 0$ and

$$\arg(\sigma'(t_0)) = \arg(f'(z_0)\, \gamma'(t_0)) = \arg(f'(z_0)) + \arg(\gamma'(t_0))$$

Therefore, $\arg(\sigma'(t_0)) - \arg(\gamma'(t_0)) = \arg(f'(z_0))$.

Let γ_1 and γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$, Let $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$. And suppose that $\gamma_1'(t_1) \neq \gamma_2'(t_2)$ (that is, γ_1 and γ_2 are not tangent to each other at z_0). Then it follows that:

$$\arg(\sigma_1'(t_1)) = \arg(f'(z_0)) + \arg(\gamma_1'(t_1)), \text{ and}$$
$$\arg(\sigma_2'(t_2)) = \arg(f'(z_0)) + \arg(\gamma_2'(t_2))$$

Therefore,

$$\arg(\sigma_2'(t_2)) - \arg(\sigma_1'(t_1)) = \arg(\gamma_2'(t_2)) - \arg(\gamma_1'(t_1))$$

This says that given any two paths through z_0, f maps these paths onto two paths through $w_0 = f(z_0)$ and, when $f'(z_0) \neq 0$, the angles between the curves are preserved both in magnitude and direction.

Theorem If $f: G \longrightarrow \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.

Def (conformal map)
A function $f: G \longrightarrow \mathbb{C}$ which has the angle preserving property and also has

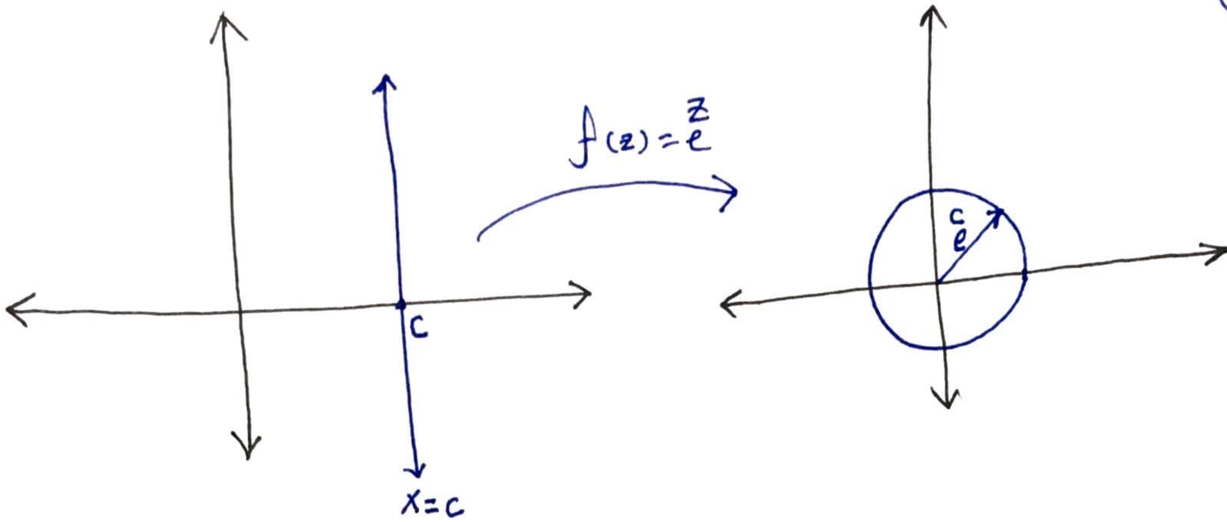
$$\lim_{z \to a} \frac{|f(z) - f(a)|}{|z-a|} := |f'(a)| \quad , a \in G$$

existing is called conformal map.

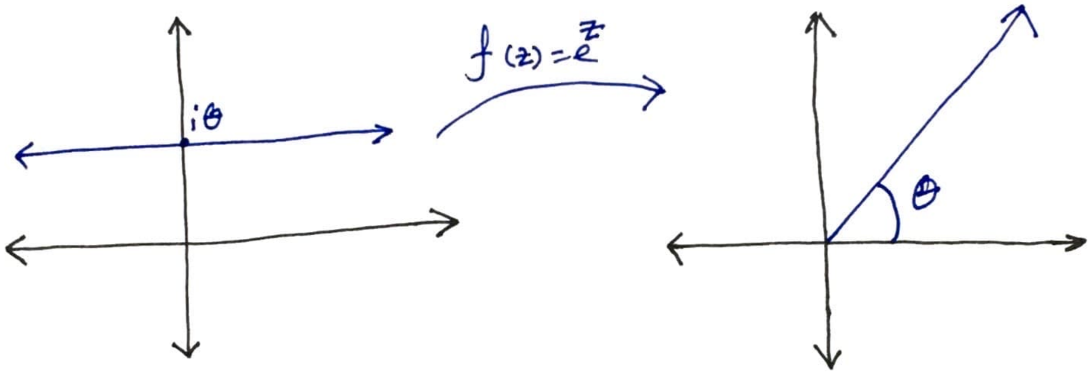
Remark If f is analytic and $f'(z) \neq 0$ for any z then f is conformal.

Example The exponential function $f(z) = e^z$ is conformal in $\mathbb{C}$.
(f is analytic and $f'(z) \neq 0 \; \forall z \in \mathbb{C}$).

1. The function $f(z) = e^z$ maps the line $z = c + iy$ into $e^c e^{iy}$
That is, f maps the vertical line $x = c$ into a circle of radius $r = e^c$ around the origin

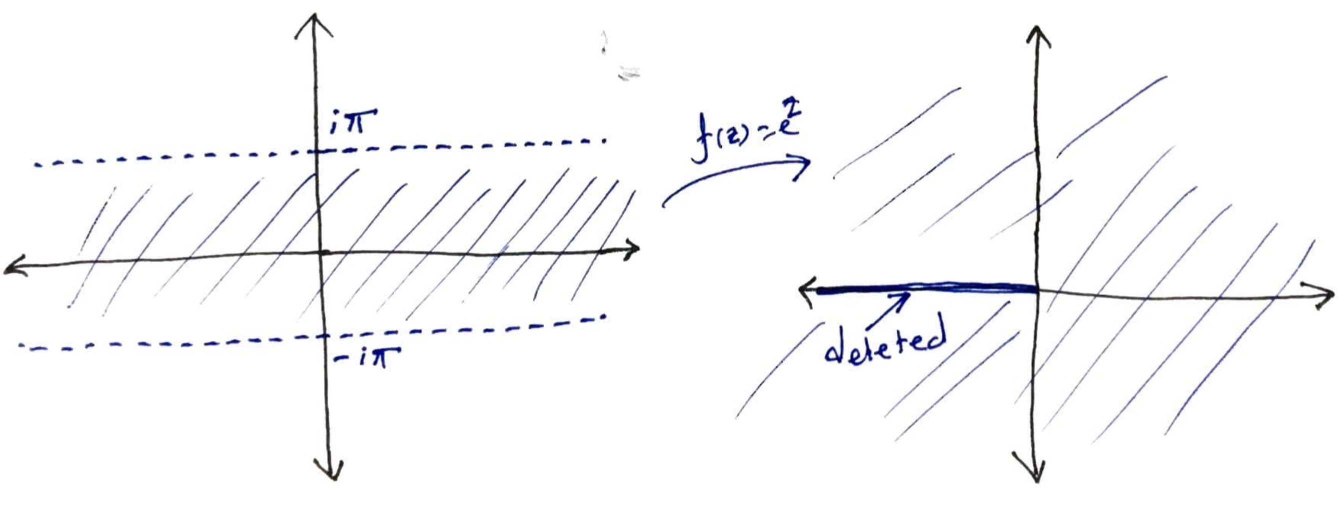


2. The function $f(z)=e^z$ maps the horizontal line $y=\theta$ into the infinite ray $\{re^{i\theta} : 0<r<\infty,\ \theta \text{ is fixed}\}$



3. The function $f(z)=e^z$ is one-to-one on any horizontal strip of width $< 2\pi$. Let $G=\{z\in\mathbb{C} : -\pi<\operatorname{Im} z<\pi\}$.

Then $f(G)=\Omega=\mathbb{C}-\{z\in\mathbb{R} : z\le 0\}$



4. The function $f(z) = e^{z}$ maps the vertical segment $\{z = c + iy, \ -\pi < y < \pi\}$ onto the circle $\{e^{c} e^{i\theta} : -\pi < \theta < \pi\}$

5. The function $f(z) = e^{z}$ maps the horizontal line $y = d, \ -\pi < d < \pi$ onto the ray making an angle d with the positive real axis.

6. The Principal branch of the logarithm maps the region $\Omega = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$ onto the strip $G = \{z : -\pi < \operatorname{Im} z < \pi\}$.