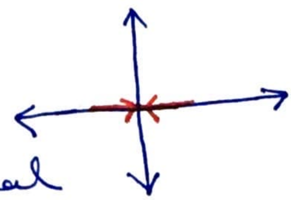
Cauchy – Riemann Equations

Let G be a region (open, connected) in the complex plane $\mathbb{C}$.
And let $f: G \longrightarrow \mathbb{C}$ be analytic function.
Assume that: $u(x,y) = \operatorname{Re} f(x+iy)$ and $v(x,y) = \operatorname{Im} f(x+iy)$.
Let us evaluate the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

in two different ways as follows:



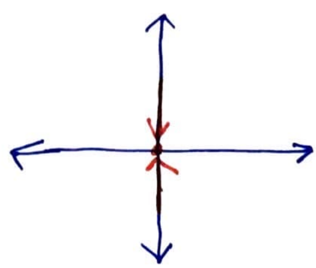
1. $h \to 0$ through the real axis

In this case h takes only the real values. In this case we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h}$$

$$\boxed{f'(z) = u_x(x,y) + i\, v_x(x,y)} \quad \text{---- } (1)$$



2. $h \to 0$ through the imaginary axis

In this case we write $h = ih^*$ ($h^* \in \mathbb{R}$).

$$f'(z) = \lim_{h^*\to 0} \frac{f(x+i(y+h^*)) - f(x+iy)}{ih^*}$$

$$= \lim_{h^*\to 0} \frac{u(x,y+h^*) - u(x,y)}{ih^*} + \lim_{h^*\to 0} \frac{v(x,y+h^*) - v(x,y)}{h^*}$$

$$\boxed{f'(z) = -i\, u_y(x,y) + v_y(x,y)} \quad \text{---- } (2)$$

Combining (1) and (2), it follows that:

Cauchy-Riemann Equations

$$\boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x}$$

Suppose that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ have continuous second partial derivatives. Differentiate the Cauchy-Riemann equation to obtain the following:

$$\begin{cases} u_{xx} = v_{yx} \quad \text{or} \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \\ u_{yy} = -v_{xy} \quad \text{or} \quad \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial y \partial x} \end{cases}$$

Therefore, we obtain that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad (u \text{ is harmonic in } G).$$

Similarly, we obtain that $\boxed{\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0}$ (v is harmonic in G).

Remarks

1. If $f = u + iv$ is analytic in G, then u and v satisfy the Cauchy-Riemann Equations
2. We will show later that !! the derivative of an analytic function $f'(z)$ is itself analytic, that is f' is analytic !! In which case, the assumption of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ have second partial derivatives always hold.

Cauchy-Riemann Equations $\longrightarrow$ Analytic function

Let G be a region in the plane and let u and v be functions defined on G with first order continuous partial derivatives. Assume that u and v satisfy Cauchy-Riemann in G. If $f = u + iv$ in G, then f is analytic in G

to see that: let $z \in G$ and $B(z, r) \subset G$

Let $h = s + it \in B(0, r)$, then it follows that $z + h \in B(z, r) \subset G$.

G

We prove that, the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \text{exists}$$

and its value $f'(z)$ is continuous at z.

Notice that:

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \left\{ \frac{u(x+s, y+t) - u(x,y)}{s+it} + i \frac{v(x+s, y+t) - v(x,y)}{s+it} \right\}$$

1) $$\lim_{s+it \to 0} \frac{u(x+s, y+t) - u(x,y)}{s+it} = \lim_{s+it} \frac{u(x+s, y+t) - u(x, y+t) + u(x, y+t) - u(x,y)}{s+it}$$

mean value theorem gives that: there exist s_1 and t_1 such that $|s_1| < |s|$ and $|t_1| < |t|$ and

$$\begin{cases} u(x+s, y+t) - u(x, y+t) = u_x(x+s_1, y+t)\, s \\ u(x, y+t) - u(x,y) = u_y(x+y+t_1)\, t \end{cases}$$

Therefore,

$$\lim_{s+it \to 0} \frac{u(x+s, y+t) - u(x,y)}{s+it} = \lim_{s+it \to 0} \frac{u_x(x+s_1, y+t)\, s + u_y(x, y+t_1)\, t}{s+it}$$

From the continuity of u_x and u_y in G, it follows that

$u_x(x+s_1, y+t) = u_x(x,y) + \epsilon_1$ and

$u_y(x, y+t_1) = u_y(x,y) + \epsilon_2$

where $\epsilon_1, \epsilon_2 \to 0$ as $s + it \to 0$

Therefore, we obtain that

$$\lim_{s+it \to 0} \frac{u(x+s, y+t) - u(x,y)}{s+it} = \lim_{s+it \to 0} \left[\frac{s}{s+it} \left(u_x(x,y) + \epsilon_1 \right) + \frac{t}{s+it} \left(u_y(x,y) + \epsilon_2 \right) \right]$$

Similarly, we obtain that

$$\lim_{s+it\to 0} \frac{V(x+s,y+t)-V(x,y)}{s+it} = \lim_{s+it\to 0}\left[\frac{s}{s+it}\left(V_x(x,y)+\epsilon_3\right)+\frac{t}{s+it}\left(V_y(x,y)+\epsilon_4\right)\right]$$

where $\epsilon_3, \epsilon_4 \to 0$ as $s+it \to 0$.

Then we conclude that:

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{s+it\to 0}\left[\frac{s}{s+it}\left(u_x(x,y)+\epsilon_1\right)+\frac{t}{s+it}\left(\overset{=-V_x}{u_y(x,y)}+\epsilon_2\right)\right]$$

$$+\, i\lim_{s+it\to 0}\left[\frac{s}{s+it}\left(V_x(x,y)+\epsilon_3\right)+\frac{t}{s+it}\left(\overset{=u_x}{V_y(x,y)}+\epsilon_4\right)\right]$$

From the assumption, we have that:

$$u_x = V_y \text{ and } u_y = -V_x$$

Therefore, we obtain that:

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{s+it\to 0}\left[\frac{s+it}{s+it}u_x(x,y)+(i)\left(\frac{s+it}{s+it}\right)V_x(x,y)\right]$$

$$+\lim_{s+it\to 0}\left[(\epsilon_1+i\epsilon_3)\left(\frac{s}{s+it}\right)+(\epsilon_2+i\epsilon_4)\left(\frac{t}{s+it}\right)\right]$$

since $\left|\frac{s}{s+it}\right|\le 1$ and $\left|\frac{t}{s+it}\right|\le 1$ and

$\epsilon_1+i\epsilon_3\to 0$, $\epsilon_2+i\epsilon_4\to 0$ as $s+it\to 0$,

it follows that:

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = u_x(x,y)+iV_x(x,y)$$

Therefore, $f'(z)=u_x(x,y)+iV_x(x,y)$, that is, f is differentiable

Since u_x and V_x are continuous, f' is continuous and f is analytic.

These results are summarized as follows.

Theorem Let u and v be real-valued functions on a region G and suppose that u and v have continuous partial derivatives. Then $f: G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.

Remark If f is analytic in G, then

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad \text{for all } z \in G.$$

Example Is the function $u(x,y) = \log(x^2+y^2)^{\frac{1}{2}}$ harmonic on $G = \mathbb{C} - \{0\}$

Answer: Yes

1. Direct calculation show that $u_{xx} + u_{yy} = 0$ in $G = \mathbb{C} - \{0\}$

2. It can be shown by observing that in a neighborhood of each point of G, u is the real part of an analytic function defined in that neighborhood. (Which function)?

Answer: $u(x,y) = \operatorname{Re} f(z) = \log(z)$ (locally). $\arg(z)$ can be defined locally in the region $(\theta, \theta + 2\pi)$ for a suitable choice of θ in the Region $\tilde{G} = \mathbb{C} - \{\text{ray from zero to infinity}\}$.

Harmonic Conjugate

Suppose G is a region in the plane and $u: G \to \mathbb{R}$ is harmonic function. Does there exist a harmonic function $v: G \to \mathbb{R}$ such that $f = u + iv$ is analytic in G? if such a function v exist it is called a harmonic conjugate of u.

Notice that: If v_1 and v_2 are two harmonic conjugates of u

then, $i(V_1 - V_2) = (u + iV_1) - (u + iV_2)$ is analytic on G and only takes on purely imaginary values.

Thus if $g(z) = i(V_1 - V_2)$ which is analytic in G and

$$\begin{cases} \tilde{u} = Re(g(z)) = 0 \\ \tilde{V} = Im(g(z)) = V_1 - V_2 \end{cases},$$

Satisfy Cauchy-Riemann Equations, that is,

$$\begin{cases} \tilde{u}_x = \tilde{V}_y \\ \tilde{u}_y = -\tilde{V}_x \end{cases} \Longrightarrow \tilde{V}_x = \tilde{V}_y = 0 \text{ in } G$$

Since G is connected, then $\tilde{V}(x,y) = V_1(x,y) - V_2'(x,y) = c$

Therefore, $g(z) = i(V_1 - V_2) = ic$

That is, Any two harmonic conjugates of a harmonic function differ by constant.

Defining the harmonic Conjugate in special regions

Harmonic Conjugate <u>Can not</u> be defined for general region. For example in the region $G = \mathbb{C} - \{0\}$ there is no harmonic conjugate for $u(z) = \log|z|$. If there was a harmonic Conjugate for $\log(z)$ in $G = \mathbb{C} - \{0\}$, then it would be possible to define an analytic branch of $\log(z)$ in G, and this can not be done as we have seen in exercise (21) in the textbook (Page 44).

However, there are some regions for which every harmonic function has a Conjugate. Particularly, if G = disk or $G = \mathbb{C}$. as we will see in the following theorem.

<u>Theorem</u> Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u: G \to \mathbb{R}$ is a harmonic function then u has a harmonic Conjugate.

Proof We use Leibniz's rule for differentiating under the integral sign. (Proposition 2.1 in Ch IV page 68).

Let $G = B(0;R)$, $0 < R \le \infty$.

Let $u : G \to \mathbb{R}$ be harmonic function.

Define $V(x,y) = \int_0^y u_x(x,t)\,dt + \varphi(x)$.

Then by Leibniz's rule, we find that:

$$V_x(x,y) = \int_0^y u_{xx}(x,t)\,dt + \varphi'(x).$$

$$= -\int_0^y u_{yy}(x,t)\,dt + \varphi'(x)$$

$$V_x(x,y) = -u_y(x,y) + u_y(x,0) + \varphi'(x)$$

But $V_x(x,y) = -u_y(x,y)$. Therefore, we obtain the following

$$u_y(x,0) + \varphi'(x) = 0.$$

That is, $\varphi'(x) = -u_y(x,0)$, integrating from 0 to x, having that

$$\varphi(x) - \varphi(0) = -\int_0^x u_y(s,0)\,ds.$$

Therefore, $\varphi(x) = -\int_0^x u_y(s,0)\,ds$ [up to constant].

The harmonic conjugate in G is:

$$V(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds. \quad ∎$$