Logarithm Function

For $w \in \mathbb{C} \setminus \{0\}$ we define $\log w$ be such that $w = e^z$ when $z = \log w$. Since $e^z \neq 0$ for all $z \in \mathbb{C}$, the $\log(0)$ cannot be defined.

Suppose $e^z = w$ and $w \neq 0$, and let $z = x + iy$.
Then we have the following equality

$$e^x e^{iy} = |w| (\cos\theta + i \sin\theta), \quad \text{where } \theta = \arg(w).$$

Therefore,

$$\begin{cases} |w| = e^x, \\ \arg(w) = y + 2\pi k, \text{ for some } k \in \mathbb{Z}. \end{cases}$$

Hence, we conclude that the following set

$$\{ \log_{\mathbb{R}} |w| + i(\arg w + 2\pi k) : k \text{ is any integer} \}$$

is the solution set for $e^z = w$.

Notation: $\log_{\mathbb{R}}$ is the usual logarithm function for positive real numbers.

Example Find the values of the following expressions

$\log(-1)$, $\log(1 + i\sqrt{3})$.

$$\log(-1) = i\pi(2k+1), \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\log(1 + i\sqrt{3}) = \log_{\mathbb{R}} 2 + i\left(\frac{\pi}{3} + 2\pi k\right), \quad k = 0, \pm 1, \pm 2, \ldots$$

Def If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function such that $e^{f(z)} = z$ for all $z \in G$ then f is a branch of the logarithm. (Notice that $0 \notin G$).

Remarks (G: open and connected in $\mathbb{C}$ be such that $0 \notin G$)

1. Let f be a given branch of the logarithm in G, and let k be a given integer. Then $g(z) = f(z) + 2\pi k i$ is a branch of the logarithm in G.

Proof clearly we see that g is continuous function in G and

$$e^{g(z)} = e^{f(z)+2\pi k i} = e^{f(z)} = z \quad \text{for all } z \in G.$$ Then g is a branch of the logarithm.

2. Let f and g be two branches of the logarithm in G, then for any $z \in G$ there exists an integer $k(z) = k$ such that

$$f(z) = g(z) + 2\pi k(z)\, i \quad (k(z) \text{ means } k \text{ depends on } z \in G)$$

Proof Let $z_1 \in G$, then $z_1 \neq 0$.

Notice that: $1 = \frac{z_1}{z_1} = \frac{e^{f(z_1)}}{e^{g(z_1)}} = e^{f(z_1) - g(z_1)}$

$\Leftrightarrow \exists\, k_1 \in \mathbb{Z}$ such that $f(z_1) - g(z_1) = 2\pi k_1 i$

$\Leftrightarrow f(z_1) = g(z_1) + 2\pi k_1 i$

3. In item (2) does the same $k \in \mathbb{Z}$ work for each $z \in G$?

Answer: Yes!

Proof Define $h : G \longrightarrow \mathbb{C}$ by the following form

$h(z) = \frac{1}{2\pi i}(f(z) - g(z))$, then clearly we see that h is continuous function on G. Furthermore, $h(G) \subset \mathbb{Z}$.

But G is connected set in $\mathbb{C}$, then $h(G)$ is connected set. Therefore, there exists a unique integer $k \in \mathbb{Z}$ such that

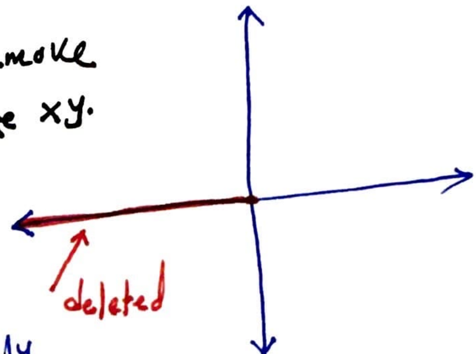
$$f(z) = g(z) + 2\pi k i \quad \text{for all } z \in G.$$

Proposition If $G \subset \mathbb{C}$ is open connected set, and f is branch of $\log z$ on G, then the totality of branches of $\log z$ are the functions: $f(z) + 2\pi k i$, $k = 0, \pm 1, \pm 2, \ldots$

Principal branch of the logarithm

To build one branch of the logarithm in some open connected set G, let us define G by the following form

Let $G = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$; that is, remove the non-positive real axis from the plane xy.



Then the set G is open connected subset of $\mathbb{C}$.

Therefore, for any $z \in G$, z can be uniquely represented by the form: $z = |z| e^{i\theta}$ where $-\pi < \theta < \pi$.

For $\theta \in (-\pi, \pi)$, define

$$f : G \longrightarrow \mathbb{C} \text{ by the form}$$

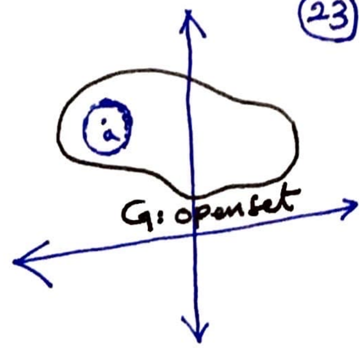
$$f(re^{i\theta}) = \log r + i\theta$$

Then f is continuous on G [Exercise].

Clearly we see that $e^{f(z)} = z$ for all $z \in G$. Then it follows that f is a branch of the logarithm on G.

Is f analytic on G? To answer this we first prove a general fact.

Proposition Let G and Ω be open subsets of $\mathbb{C}$. Suppose that $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ and $g(f(z)) = z$ for all z in G. If g is differentiable and $g'(z) \neq 0$, f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$.



Proof Fix $a \in G$ and let $h \in \mathbb{C}$ such that $h \neq 0$ and $a+h \in G$. Then we have the following

$$\begin{cases} a = g(f(a)), \\ a+h = g(f(a+h)). \end{cases}$$

Therefore we conclude that: $f(a) \neq f(a+h)$, and

$$1 = \frac{g(f(a+h)) - g(f(a))}{h}.$$

Thus,

$$1 = \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \cdot \frac{f(a+h) - f(a)}{h}$$

That is,
$$1 = \lim_{h \to 0} \left[\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \right] \cdot \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right].$$

From the continuity of f we have that:

$$f(a+h) \to f(a) \text{ as } h \to 0,$$

therefore, $\displaystyle\lim_{h \to 0} \left[\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \right] = g'(f(a)).$

That is, we conclude the following:

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) g'(f(a)) = 1$$

Therefore, $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{1}{g'(f(a))}.$

Thus $f'(a)$ exists and equals to $\dfrac{1}{g'(f(a))}.$

This gives f is differentiable on G and

$$f'(z) = \frac{1}{g'(f(z))}, \quad z \in G.$$

Corollary A branch of the logarithm function is analytic and its derivative is $\frac{1}{z}$.

Proof Let f be a branch of $\log z$ in G, then $e^{f(z)} = z$ for all $z \in G$. Let $g(z) = e^{z}$, for all $z \in \mathbb{C}$. Then $f(G) \subset \mathbb{C}$ and $g(f(z)) = z$ for all $z \in G$.

Since g is differentiable, then f is differentiable and

$$f'(z) = \frac{1}{g'(f(z))} = \frac{1}{z} \quad \text{for all } z \in G.$$

clearly f' is continuous on G since $0 \notin G$.

Therefore f is analytic in G.

Remarks

1. Principal branch of $\log z$ is defined in $G = \mathbb{C} - \{z : \mathrm{Re}(z) \le 0\}$

2. If we write $\log z$ as a function we will always take it to be the Principal brance of the logarithm unless otherwise stated.

principal branch   Real log $\arg(z) \in (-\pi, \pi)$

Fro example $\log(1+\sqrt{3}\, i) = \log 2 + \frac{\pi}{3} i$

3. If f is a branch of the logarithm on an open connected set G and if $b \in \mathbb{C}$ to be fixed, define

$$g(z) = e^{b f(z)}, \quad z \in G.$$

Then if $b \in \mathbb{Z}$ we see that $g(z) = z^b$ (Exercise).

In this manner we define a branch of z^b, $b \in \mathbb{C}$, for an open connected set on which there is a branch of $\log z$.

If we write $g(z) = z^b$ as a function we always understand

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that: $z^b = e^{b \log(z)}$, where $\log z$ is the principal branch of the logarithm.

Example $\underbrace{(1+i)^i}_{\text{Principal value}} = e^{-\pi/4}\left[\cos \log\sqrt{2} + i \sin \log\sqrt{2}\right]$

4. z^b is analytic because $\log z$ is analytic in $G = \mathbb{C} - \{\operatorname{Re} z \le 0\}$.

5. G is called region if G is open and connected