Analytic Functions

If G is an open set in $\mathbb{C}$ and $f: G \longrightarrow \mathbb{C}$ then f is differentiable at a point $a \in G$ if

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ exists};$$

the value of this limit is denoted by $f'(a)$ and is called the derivative of f at a.

Notice that if f is differentiable at each point of G, then we say f is differentiable on G.

<u>Remark</u> If f' is continuous on G, then f is called continuously differentiable on G.

The following proposition was surely predicted by the reader.

<u>Proposition</u> If $f: G \to \mathbb{C}$ is differentiable at a point $a \in G$ then f is continuous at a.

<u>Proof</u> We show that if $z \to a$, then $f(z) \to f(a)$.

$$\lim_{z \to a} |f(z)-f(a)| = \lim_{z \to a} \frac{|f(z)-f(a)|}{|z-a|} \lim_{z \to a} |z-a|$$

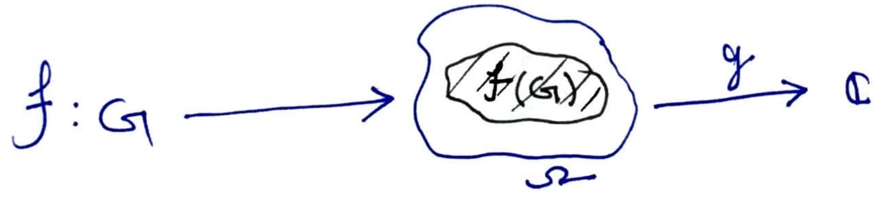
$$= |f'(a)| \cdot 0 = 0 \quad \blacksquare$$

Def A function $f: G \to \mathbb{C}$ is called analytic on G if f is continuously differentiable on G.

Remark If f, g are analytic on G then $f \pm g$, $f \cdot g$ are analytic on G, and $\frac{f}{g}$ is analytic if $g \neq 0$.

Theorem (Chain Rule)
Let f and g be analytic on G and Ω respectively and suppose $f(G) \subset \Omega$. Then $g \circ f$ is analytic on G and

$$(g \circ f)'(z) = g'(f(z)) \cdot f'(z), \quad \forall z \in G.$$



Remark The derivative of f is defined on an open set, but if we say f is analytic on a set A and A is not open, we mean that f is analytic on an open set containing A.

Remark The definition of differentiability of f in real analysis is different from the definition of Analytic functions in Complex Analysis. We will see that !! Every Analytic function is infinitely differentiable and furthermore, has a power series expansion about each point of its domain!!.

If we consider $G \subset \mathbb{C}$ and f is defined on G as a function of two variables as follows:

$$g(x,y) = f(x+iy) \; ; \; (x,y) \in G.$$

The requiring that g be Frechet differentiable will not ensure that f has derivative in our sense.

A function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is Frechet differentiable at $a \in G \subset \mathbb{R}^2$ if there is a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$\lim_{h \to 0} \frac{|g(a+h) - g(a) - T(h)|}{|h|} = 0.$$

The linear transformation T is denote $Df(a)$ and is called the Frechet derivative of g at a.

As we will see that $f(z) = |z|^2$ has derivative only at $z=0$ but $g(x,y) = f(x+iy) = x^2+y^2$ is Frechet differentiable.

Proposition (2.5) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence $R>0$. Then:

1. For each $k \geq 1$ the series

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (z-a)^{n-k} \quad \cdots \quad (*)$$

has radius of convergence R;

2. The function f is infinitely differentiable on $B(a;R)$, and, furthermore, $f^{(k)}(z)$ is given by the series $(*)$ for all $k \geq 1$ and $|z-a| < R$;

3. For $n \geq 0$; $a_n = \frac{1}{n!} f^{(n)}(a)$.

Corollary If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence R then $f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in $B(a;R)$.

Hence $f(z)=e^{z}=\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ is analytic in $\mathbb{C}$.

Proposition If G is open and connected and $f: G \rightarrow \mathbb{C}$ is differentiable with $f'(z)=0$ for all z in G, then f is constant.

Exponential Function

$f(z)=e^{z}=\sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $R=\infty$.

1. The differentiation of $f(z)$ is given by Proposition (2.5), that is $f'(z)=\sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = f(z)$.

   Therefore $\left(\frac{d}{dz} e^{z} = e^{z}\right)$. [The same as in real].

2. $e^{a+b} = e^{a} e^{b}$ for all $a, b \in \mathbb{C}$.

Proof First we prove that:

$$e^{z} e^{\tilde{a}-z} = e^{\tilde{a}} \quad \text{for fixed } \tilde{a} \in \mathbb{C} \text{ and all } z \in \mathbb{C}.$$

Let $g(z)=e^{z} e^{\tilde{a}-z}$, $z \in \mathbb{C}$.

Then $g'(z)=e^{z} e^{\tilde{a}-z} + (-e^{z} e^{\tilde{a}-z}) = 0 \quad \forall z \in \mathbb{C}$

Thus g is constant in $\mathbb{C}$, that is, $\exists\, w \in \mathbb{C}$ such that $g(z)=w$ for all $z \in \mathbb{C}$.

That is, $w=g(0)=e^{\tilde{a}}$ for all $z \in \mathbb{C}$ [use $e^{0}=1$].

Therefore, $g(z)=e^{\tilde{a}}$, $\forall z \in \mathbb{C}$.

Then we conclude that:

$$e^{z} e^{\tilde{a}-z} = e^{\tilde{a}} \; ; \forall \; z \in \mathbb{C}$$

Now let $z = a$ and $\tilde{a} = a+b$

Therefore, $\boxed{e^{a} e^{b} = e^{a+b}, \text{ for all } a, b \in \mathbb{C}}$.

3. $e^{-z} = \frac{1}{e^{z}}$ for all $z \in \mathbb{C}$

Notice that: $e^{z} e^{-z} = e^{0} = 1 \; ; \forall \; z \in \mathbb{C}$.

that is $e^{z} \neq 0$ for all $z \in \mathbb{C}$

Therefore, $e^{-z} = \frac{1}{e^{z}}, \; z \in \mathbb{C}$.

4. $\overline{(e^{z})} = e^{\bar{z}}, \; \forall \; z \in \mathbb{C}$.

From the power series of e^{z}, we have that

$$e^{z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{n},$$ all the coefficients are real.

Then $\overline{(e^{z})} = \sum_{n=0}^{\infty} \frac{1}{n!} (\bar{z})^{n} = e^{\bar{z}}$.

5. For $\theta \in \mathbb{R}$, we have that:

$|e^{i\theta}|^{2} = e^{i\theta} e^{-i\theta} = 1$. [We have used that $\overline{(e^{i\theta})} = e^{\overline{(i\theta)}} = e^{-i\theta}$]

\# More generally, $|e^{z}|^{2} = e^{z} e^{\bar{z}} = e^{2\mathrm{Re}(z)}$.

Thus $\boxed{|e^{z}| = e^{\mathrm{Re}(z)}}$.

6. We see that e^{z} has the same properties that the real function e^{x} has. Again by analogy with real power series we define the functions $\underline{\cos z}$ and $\underline{\sin z}$ by the power series:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} + \cdots$$

Each of the series has infinite radius of convergence and so $\cos z$ and $\sin z$ are analytic in $\mathbb{C}$. By using Proposition (2.5) we find that:

$$(\cos z)' = -\sin z \quad \text{and} \quad (\sin z)' = \cos z$$

Also we find that:

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \quad \text{and} \quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

This gives for all z in $\mathbb{C}$ that

$$\sin^2 z + \cos^2 z = 1 \quad \text{and}$$

$$e^{iz} = \cos z + i \sin z \quad \ldots \quad (**)$$

In particular, if we let $z = \theta \in \mathbb{R}$ in $(**)$ we get

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Hence, for $z \in \mathbb{C}$, we have that

$$z = |z| e^{i\theta}, \quad \text{where } \theta = \arg(z).$$

Since $e^{x+iy} = e^x e^{iy}$, we have

$$|e^z| = e^{\operatorname{Re} z} \quad \text{and} \quad \arg e^z = \operatorname{Im} z.$$

7. A function f is Periodic with Period c if $f(z+c)=f(z)$ for all $z \in \mathbb{C}$.

If c is a Period of e^z then $e^z = e^{z+c} = e^z e^c$, then we have $e^c = 1$.

Since $|e^c| = 1 = e^{Re(c)}$, $\Longrightarrow Re(c) = 0$.

Thus $c = i\theta$ for some $\theta \in \mathbb{R}$.

But $1 = e^c = e^{i\theta} = \cos\theta + i\sin\theta$

$\Longrightarrow \cos\theta + i\sin\theta = 1 \Longleftrightarrow \sin\theta = 0$

Therefore, $\cos\theta = 1 \Longleftrightarrow \theta = 2\pi k$, $k = 0, \pm 1, \pm 2, \ldots$

$\Longrightarrow c = 2\pi k i$

$\Longrightarrow$ Periods of e^z are multiple of $\underline{2\pi i}$.

That is, $e^{z+2\pi k i} = e^z$ for all $z \in \mathbb{C}$.

Thus if we divide the plane into infinitely many horizontal strips by the lines ($\text{Im} z = \pi(k-1)$) where k any integer, then the exponential function e^z behaves the same in each of these strips.