Uniform Convergence

Let X be a set and (Ω, d) a metric space and suppose $f, f_1, f_2, \dots$ are functions from X into Ω. The sequence $\{f_n\}$ converges uniformly to f (written as $f_n \rightrightarrows f$) if $\forall\, \epsilon > 0$, $\exists\, N = N(\epsilon)$ positive integer such that

$$d(f(x), f_n(x)) \leq \epsilon \quad \forall\, x \in X \text{ for } n \geq N$$

Hence $$\sup_{x \in X} \{ d(f(x), f_n(x)) \} \leq \epsilon \text{ for all } n \geq N$$

Equivalently, can be written as $\|f - f_n\|_\infty \leq \epsilon \;\forall\, n \geq N$.

Theorem Suppose that $f_n : (X, d) \to (\Omega, \rho)$ is continuous for each n and that $f_n \rightrightarrows f$, then f is continuous.

Remark Let $u_n : X \to \mathbb{C}$ be a sequence of complex valued function defined on X.

Define $f_n(x) = \sum_{k=1}^{n} u_k(x) = u_1(x) + u_2(x) + \dots + u_n(x)$

1. We say $f(x) = \sum_{n=1}^{\infty} u_n(x)$ iff $f(x) = \lim_{n\to\infty} f_n(x)$; $\forall\, x \in X$.
2. The series $\sum_{n=1}^{\infty} u_n$ is uniformly convergent to f iff

$$f_n \rightrightarrows f.$$

Weierstrass M-Test Let $u_n : X \to \mathbb{C}$ be a sequence of complex valued functions defined on X such that

$$|u_n(x)| \leq M_n, \;\forall\, x \in X \text{ and}$$

suppose that the constants M_n satisfy $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} u_n$ is uniformly convergent.

<u>Power Series</u> If $a_n \in \mathbb{C}$ for every $n=0,1,2,\dots$, then the series $\sum_{n=0}^{\infty} a_n$ converges to z iff $\forall\, \epsilon>0$, $\exists\, N \in \mathbb{Z}^+$ such that

$$\left| \sum_{n=0}^{m} a_n - z \right| < \epsilon \quad \forall\, m \geq N.$$

The series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges.

<u>Proposition</u> If $\sum_{n=0}^{\infty} a_n$ converges absolutely, then $\sum_{n=0}^{\infty} a_n$ converges.

Limit supremum and Limit infimum

<u>Def</u> Let $\{x_n\}$ be a real sequence. The limit supremum of $\{x_n\}$ is the extended real number

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k \geq n} x_k \right)$$

or, $$\limsup_{n\to\infty} x_n = \inf_{n \geq 1} \left(\sup_{k\geq n} x_k \right)$$

This limit exists as an extended real number.

<u>Notice that</u> if $s_n = \sup\{x_n, x_{n+1}, \dots\}$, that is

$s_1 = \sup\{x_1, x_2, \dots\}$

$s_2 = \sup\{x_2, x_3, \dots\}$

$\vdots$

Then $\{s_n\}$ is a decreasing sequence. We conclude that

1) if $\{s_n\}$ is bounded below then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n = s$

2) If $\{S_n\}$ is not bounded below then $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n = -\infty$

3) If $\limsup_{n\to\infty} x_n = +\infty$, then $S_n = +\infty$ for every $n \geq 1$.

This proves that $\limsup_{n\to\infty} x_n$ exists as an extended real number.

Remarks

1) Since $\{S_n\}$ is a decreasing sequence, then

$$\begin{aligned}\limsup_{n\to\infty} x_n &= \lim_{n\to\infty} (S_n) \\ &= \inf_{n\geq 1} (S_n) \\ &= \inf_{n\geq 1} \left(\sup_{k\geq n} x_n\right).\end{aligned}$$

2) $\limsup_{n\to\infty} x_n$ is the largest subsequential limit of $\{x_n\}$.

Def The limit infimum of $\{x_n\}$ is the extended real number

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k\right)$$

or, $$\liminf_{n\to\infty} x_n = \sup_{n\geq 1} \left(\inf_{k\geq n} x_k\right)$$

This limit exist as an extended real number. That is

$$\liminf_{n\to\infty} x_n = \begin{cases} +\infty & \text{if } (t_n) \text{ is not bouned above.} \\ t \text{ (finite real number)} & \text{if } \{t_n\} \text{ bouneded above.} \\ -\infty & \text{if } t_n = -\infty \ \forall\ n \geq 1. \end{cases}$$

Define $t_n = \inf\{x_n, x_{n+1}, \dots\}$ and argue as the same as for limit supremum.

Remarks

1) The ~~sequence~~ $\{t_n\}$ is an increasing sequence, then

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (t_n)$$
$$= \sup_{n\geq 1} (t_n)$$
$$= \sup_{n\geq 1} (\inf_{k\geq n} x_n).$$

2) $\liminf_{n\to\infty} x_n$ is the smallest subsequential limit of $\{x_n\}$.

3) $\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty} (-x_n)$

4) $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

5) If $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, then $\lim_{n\to\infty} (x_n)$ exists and

$$\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$$

Example Find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ of the following sequences

1) $\{x_n\} = (-1)^n$

Let $S_n = \sup\{x_n, x_{n+1}, \dots\}$; $n = 1, 2, \dots$

$S_1 = \sup\{-1, 1, -1, 1, \dots\} = 1$

$S_2 = \sup\{1, -1, 1, -1, \dots\} = 1$

$\vdots$

Then $\{S_n\} = \{1\}$, thus $\limsup_{n\to\infty} (-1)^n = \lim_{n\to\infty} (1) = 1$

similarly, we define

$t_n = \inf\{x_n, x_{n+1}, \dots\}$; $n = 1, 2, \dots$

$t_1 = \inf\{-1, 1, -1, 1, \dots\} = -1$

$t_2 = \inf\{1, -1, 1, -1, \dots\} = -1$

$\vdots$

Then $\{t_n\} = \{-1\}$, thus $\liminf_{n\to\infty} (-1)^n = \lim_{n\to\infty} (-1) = -1$

2) $\{x_n\} = \{1 + \frac{1}{n}\}$

Notice that $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup_{k \geq n} x_k) = \lim_{n\to\infty} \sup_{k\geq n} \left(1 + \frac{1}{k}\right)$

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$

$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf_{k\geq n} x_k) = \lim_{n\to\infty} \inf_{k \geq n} \left(1 + \frac{1}{k}\right) = \lim_{n\to\infty} (1) = 1.$

Def A power series about $a \in \mathbb{C}$ is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n ; \quad a_n \in \mathbb{C} \ \forall n = 0, 1, 2, \dots .$$

Example (Geometric Series) $\sum_{n=0}^{\infty} z^n$.

Notice that $1 + z + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}$, $z \neq 1$

If $|z| < 1$, then $\lim_{n\to\infty} |z|^n = 0$, then the geometric series $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$.

If $|z| > 1$, then $\lim_{n\to\infty} |z|^n = \infty$, then $\sum_{n=0}^{\infty} z^n$ diverges

If $|z| = 1$, then $\sum_{n=0}^{\infty} z^n$ diverges (Exercise).

Theorem For a given power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ define the number R, $0 \le R \le \infty$ by

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}},$$

then:

1. if $|z-a| < R$, the series converges absolutely;
2. if $|z-a| > R$, the terms of the series become unbounded and so the series diverges;
3. if $0 < r < R$ then the series converges uniformly on $\{z \in \mathbb{C} : |z-a| \le r\}$. Moreover, the number R is the only number having properties (1) and (2).

Def The number R is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$.

Remark If $\sum a_n (z-a)^n$ is a given power series with radius of convergence R, then

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

if this limit exists.

Example consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, by the remark we have that

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \lim_{n \to \infty} (n+1) = \infty .$$

Thus the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges at every complex number and the convergence is uniform on each compact subset of $\mathbb{C}$.

__Def__ $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$; exponential series or function.

__Proposition__ Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two absolutely convergent series and put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} .$$

Then $\sum_{n=0}^{\infty} c_n$ is absolutely convergent with sum $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$.

__Proposition__ Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ and $\sum_{n=0}^{\infty} b_n(z-a)^n$ be power series with radius of convergence $\geq r > 0$. Put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} ;$$

then both power series $\sum_{n=0}^{\infty} (a_n+b_n)(z-a)^n$ and $\sum_{n=0}^{\infty} c_n(z-a)^n$ have radius of convergence $\geq r$; and

1. $\sum_{n=0}^{\infty} (a_n+b_n)(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=0}^{\infty} b_n(z-a)^n.$

2. $\sum_{n=0}^{\infty} c_n(z-a)^n = \left(\sum_{n=0}^{\infty} a_n(z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(z-a)^n\right).$