Chapter (1)

The Complex number system

1. Introduction

The set of the complex numbers $\mathbb{C}$ is defined as a set of ordered pairs in the following form

$$\mathbb{C} = \{ (a,b) : a,b \in \mathbb{R} \},$$

where, $\mathbb{R}$ is the set of the real numbers.

The set $\mathbb{C}$ equipped with the following two binary operations

1. Addition $(a,b) + (c,d) = (a+c, b+d)$,

2. Multiplication $(a,b).(c,d) = (ac-bd, bc+ad)$.

Then we easily can verify that $(\mathbb{C},+,.)$ is a field.
That is, $(\mathbb{C},+,.)$ is a commutative ring with identity.

The field of real numbers $(\mathbb{R},+,.)$ can be embedded in the field of complex numbers $(\mathbb{C},+,.)$ by defining the following field monomorphism

$$\varphi : (\mathbb{R},+,.) \longrightarrow (\mathbb{C},+,.)$$
$$a \longmapsto (a,0)$$

Therefore, $\mathbb{R}$ can be considered as a subset of $\mathbb{C}$.

If we define $i = (0,1)$, then the ordered pair (a,b) can be written in the following form:

$(a,b) = a+bi$.

From this point we can omit the ordered pair for the complex numbers. That is,

$$\mathbb{C} = \{a+bi;\ a,b \in \mathbb{R}\}$$

Clearly, from the above construction, we easily see that $i^2 = -1$, so the equation $z^2+1=0$ has a root in $\mathbb{C}$. In fact, for each z in $\mathbb{C}$, $z^2+1 = (z+i)(z-i)$.

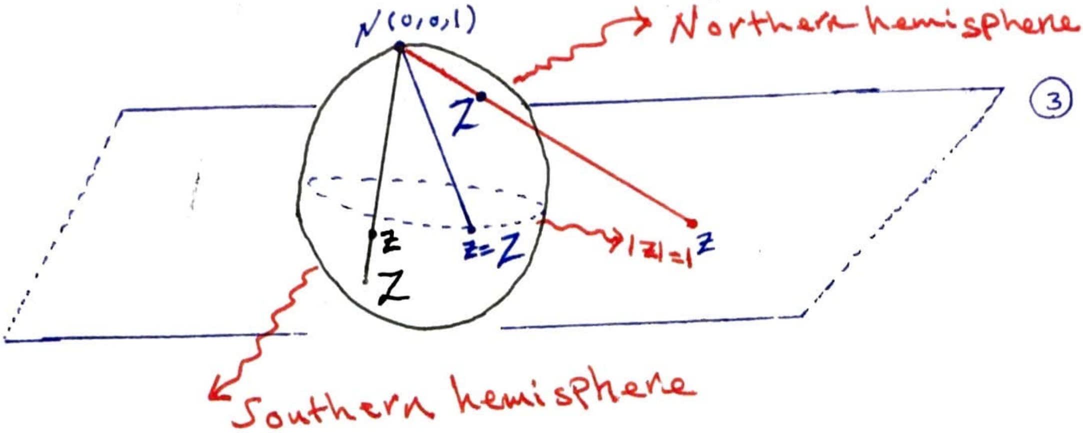
* The student should review pages (1–7) in the text book (Conway).

2. The extended plane and its spherical representation

The topology that have been used in the complex plane $\mathbb{C}$ is induced by the Euclidean metric $d(z,w) = |z-w|$. And this topology have been used to prove that the compact sets in $\mathbb{C}$ those are both closed and bounded. However, the set $S = \{1,2,3,\dots\}$ has no limit point in $\mathbb{C}$, therefore, the set S is not compact in $\mathbb{C}$. So we need to develop the notion of a limit point for unbounded sets by defining a new metric which works for unbounded sets in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

To accomplish this, we identify $\mathbb{C}_\infty$ by the unit sphere in $\mathbb{R}^3$ as follows:

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$



Let $N=(0,0,1)$, that is, N is the north pole on S. Also identify $\mathbb{C}$ with $\{(x_1, x_2, 0): x_1, x_2 \in \mathbb{R}\}$ so that $\mathbb{C}$ cuts S along the equator.

Now for each point $z \in \mathbb{C}$ consider the straight line in $\mathbb{R}^3$ through z and N. This line intersects S exactly in one point $Z \neq N$. Noticing that:

1. if $|z|>1$, then Z lies in the Northern hemisphere.
2. if $|z|<1$, then Z lies in the southern hemisphere
3. if $|z|=1$, then $z=Z$.

What happens to Z as $|z| \to \infty$? clearly Z approaches N. Hence we identify N as the point when $|z| \to \infty$ in $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ represented as the sphere S.

Let us explore the representation as follows:

Put $z = x+iy$ and $Z=(x_1, x_2, x_3)$ be the corresponding point on the sphere S. We will find the equations expressing x_1, x_2 and x_3 in terms of x and y.

Notice that; The straight line through N and z is given by the form:

$$\left\{ tN+(1-t)Z \,;\, -\infty < t < \infty \right\}, \text{ or by:}$$

$$\left\{ \left((1-t)x,\ (1-t)y,\ t\right) ;\ -\infty < t < \infty \right\} \quad \text{-----} \ (1)$$

Hence we can find the coordinates of Z if we can find the value of t at which the line intersects the sphere S. If t is this value, then:

$$(1-t)^2x^2+(1-t)^2y^2+t^2=1.$$

That is; $(1-t^2) = (1-t)^2(|z|^2)$.

since $t \neq 1$ (otherwise $z=\infty$)

Thus $\cancel{(1-t)}(1+t) = \cancel{(1-t)}(1-t)|z|^2$

$$\Rightarrow \quad t + t|z|^2 = |z|^2-1$$

$$\Rightarrow \quad t = \frac{|z|^2-1}{1+|z|^2}.$$

Thus: $x_1 = \dfrac{2x}{1+|z|^2}$, $x_2 = \dfrac{2y}{1+|z|^2}$, $x_3 = \dfrac{|z|^2-1}{1+|z|^2}$ ----- (2)

But this gives: ----- (3)

$$x_1 = \frac{z+\bar{z}}{1+|z|^2}, \quad x_2 = \frac{-i(z-\bar{z})}{1+|z|^2}, \quad x_3 = \frac{|z|^2-1}{1+|z|^2}$$

If the point Z is given ($Z \neq N$) and we wish to find z, then by setting $t = x_3$ and use (1) we arrive at:

$$\begin{aligned} x_1 &= (1-t)x \Rightarrow \\ x_2 &= (1-t)y \Rightarrow \\ x_3 &= t \end{aligned} \quad \boxed{\begin{aligned} x &= \frac{x_1}{1-x_3} \\ y &= \frac{x_2}{1-x_3} \end{aligned}} \Rightarrow \boxed{z = \frac{x_1+ix_2}{1-x_3}} \quad \text{----} \ (4)$$

Now, let use define the distance function in $\mathbb{C}_\infty$:

Let $z, z' \in \mathbb{C}_\infty$, we define the "chordal distance" $d(z,z')$ as $d(Z, Z')$; that is:

$$d(z, z') = d(Z, Z') = |Z - Z'| = \left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\right]^{\frac{1}{2}} \quad \text{---}(5)$$

Using the fact that $Z, Z' \in S$, we obtain the following

$$d^2(z, z') = 2 - 2(x_1 x_1' + x_2 x_2' + x_3 x_3') \quad \text{---} \quad (6)$$

By using equation (3) we get

$$d(z, z') = \begin{cases} \dfrac{2|z - z'|}{\left[(1 + |z|^2)(1 + |z'|^2)\right]^{1/2}} \ ; & z, z' \in \mathbb{C} \\[2ex] \dfrac{2}{\left[1 + |z|^2\right]^{1/2}} \ ; & z \in \mathbb{C},\ z' = \infty. \end{cases} \quad \text{....}(7)$$

(The derivation of (7) left as an exercise for the students)

Exercise show that $(\mathbb{C}_\infty, d)$ is a metric space.

Remark The correspondence between points of S and $\mathbb{C}_\infty$ is called the sterographic projection.