# University of Basrah 

Collage of Engineering
Mechanical Engineering Department

# Engineering Mechanics 

Dynamic

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## Syllabus: Dynamic

## Chapter One "Kinematics of Particles"

Introduction to Dynamics

## Rectilinear Motion of Particles

- Position, Velocity, and Acceleration
- Determination of the Motion of a Particle
- Uniform Rectilinear Motion
- Uniformly Accelerated Rectilinear Motion
- Motion of Several Particles


## Curvilinear Motion of Particles

-Position Vector, Velocity, and Acceleration

- Derivatives of Vector Functions
- Rectangular Components of Velocity and Acceleration


## Chapter Two "Kinetics of Particles"

- Newton's Second Law of Motion
- Systems of Units
- Equations of Motion
- Dynamic Equilibrium


## Chapter Three "Kinetics of Particles: Energy and Momentum Methods"

- Work of a Force
- Kinetic Energy of a Particle. Principle of Work and Energy
- Applications of the Principle of Work and Energy
- Power and Efficiency
- Potential Energy
- Conservative Forces
- Conservation of Energy
- Impact
- Direct Central Impact
- Oblique Central Impact



## Chapter One

## Kinematics of Particles

## 1- Dynamics includes:

1. Kinematics: which is the study of the geometry of motion? Kinematics is used to relate displacement, velocity, acceleration, and time, without reference to the cause of the motion.
2. Kinetics: which is the study of the relation existing between the forces acting on a body, the mass of the body, and the motion of the body? Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

## 2. Rectilinear Motion of Particles

## POSITION, VELOCITY, AND ACCELERATION

$$
\text { Average velocity }=\frac{\Delta x}{\Delta t} \quad v=\frac{d x}{d t}
$$


(a)

(b)

The velocity $v$ is represented by an algebraic number which can be Positive or negative
.Note: A positive value of v indicates that $x$ increases, i.e., that the particle moves in the positive direction (Fig. 11.3a); a negative value of $v$ indicates that $x$ decreases, i.e., that the particle moves in the negative direction (Fig. 11.3b).

(a)



Average acceleration $=\frac{\Delta v}{\Delta t}$

$$
a=\frac{d v}{d t}
$$



$$
a=v \frac{d v}{d x}
$$

The acceleration $a$ is represented by an algebraic number which can be positive or negative.
Note: A positive value of a indicates that the velocity (i.e., the algebraic number v) increases. This may mean that the particle is moving faster in the positive direction (Fig. 11.5a) A negative value of a indicates that the velocity decreases; either the particle is moving more slowly in the positive direction (Fig. 11.5c)

(a)

(c)

## 3- Uniform Rectilinear Motion

Uniform rectilinear motion is a type of straight-line motion which is frequently encountered in practical applications. In this motion, the acceleration a of the particle is zero for every value of $t$. The velocity $v$ is therefore constant.

$$
\frac{d x}{d t}=v=\text { constant }
$$

$$
\begin{aligned}
\int_{x_{0}}^{x} d v & =v \int_{0}^{t} d t \\
x-x_{0} & =v t \\
x & =x_{0}+v t
\end{aligned}
$$

## 4- Uniformly Acceleration Rectilinear Motion

Uniformly accelerated rectilinear motion is another common type of motion. In this motion, the acceleration a of the particle is constant

$$
\frac{d v}{d t}=a=\text { constant }
$$

The velocity $v$ of the particle is obtained by integrating this equation:

$$
\begin{align*}
\int_{v_{0}}^{v} d v & =a \int_{0}^{t} d t \\
v-v_{0} & =a t  \tag{11.6}\\
v & =v_{0}+a t
\end{align*}
$$

where $v_{0}$ is the initial velocity. Substituting for $v$ in (11.1), we write

$$
\frac{d x}{d t}=v_{0}+a t
$$

Denoting by $x_{0}$ the initial value of $x$ and integrating, we have

$$
\begin{align*}
\int_{x_{0}}^{x} d x & =\int_{0}^{t}\left(v_{0}+a t\right) d t \\
x-x_{0} & =v_{0} t+\frac{1}{2} a t^{2} \\
x & =x_{0}+v_{0} t+\frac{1}{2} a t^{2} \tag{11.7}
\end{align*}
$$

We can also use Eq. (11.4) and write

$$
\begin{aligned}
v \frac{d v}{d x}=a & =\text { constant } \\
v d v & =a d x
\end{aligned}
$$

Integrating both sides, we obtain

$$
\begin{align*}
\int_{v_{0}}^{v} v d v & =a \int_{x_{0}}^{x} d x \\
\frac{1}{2}\left(v^{2}-v_{0}^{2}\right) & =a\left(x-x_{0}\right) \\
v^{2} & =v_{0}^{2}+2 a\left(x-x_{0}\right) \tag{11.8}
\end{align*}
$$

## 3- Motion of Several Particles

Relative Motion of Two Particles: Consider two particles A and B moving along the same straight line. If the position coordinates $X_{A}$ and $X_{B}$ are measured from the same origin, the difference ( $X_{A}-X_{B}$ )defines the relative position coordinate of $B$ with respect to $A$ and is denoted by $X_{B / A}$. We write

$$
x_{B / A}=x_{B}-x_{A} \quad \text { or } \quad x_{B}=x_{A}+x_{B / A}
$$

The rate of change of $X_{B / A}$ is known as the relative velocity of $B$ with respect to $A$ and is denoted by $\mathrm{V}_{\mathrm{B} / \mathrm{A}}$.

$$
v_{B / A}=v_{B}-v_{A} \quad \text { or } \quad v_{B}=v_{A}+v_{B / A}
$$

The rate of change of $V_{B / A}$ is known as the relative acceleration of $B$ with respect to $A$ and is denoted by $a_{B / A}$

$$
a_{B / A}=a_{B}-a_{A} \quad \text { or } \quad a_{B}=a_{A}+a_{B / A}
$$

## Dependent Motions

Sometimes, the position of a particle will depend upon the position of another particle or of several other particles. The motions are then said to be dependent.


$$
x_{A}+2 x_{B}=\text { constant }
$$


$2 x_{A}+2 x_{B}+x_{C}=$ constant

Example-1: A ball is thrown vertically upward from the 12-m level in an elevator shaft with an initial velocity of $18 \mathrm{~m} / \mathrm{s}$. At the same instant an open-platform elevator passes the $5-\mathrm{m}$ level, moving upward with a constant velocity of $2 \mathrm{~m} / \mathrm{s}$. Determine (a) when and where the ball will hit the elevator, (b) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.

$$
\begin{array}{ll}
v_{B}=v_{0}+a t & v_{B}=18-9.81 t \\
y_{B}=y_{0}+v_{0} t+\frac{1}{2} a t^{2} & y_{B}=12+18 t-4.905 t^{2}
\end{array}
$$



$$
\begin{aligned}
& v_{E}=+2 \mathrm{~m} / \mathrm{s} \\
& y_{E}=y_{0}+v_{E} t \quad y_{E}=5+2 t
\end{aligned}
$$

$$
\begin{equation*}
y_{E}=y_{B} \tag{5}
\end{equation*}
$$

Substituting for $y_{E}$ and $y_{B}$ from (2) and (4) into (5), we have

$$
\begin{aligned}
5+2 t & =12+18 t-4.905 t^{2} \\
t & =-0.39 \mathrm{~s} \quad \text { and } \quad t=3.65 \mathrm{~s}
\end{aligned}
$$

Only the root $t=3.65 \mathrm{~s}$ corresponds to a time after the motion has begun.
Substituting this value into (4), we have

$$
\begin{aligned}
& y_{E}=5+2(3.65)=12.30 \mathrm{~m} \\
& \quad \text { Elevation from ground }=12.30 \mathrm{~m}
\end{aligned}
$$

The relative velocity of the ball with respect to the elevator is

$$
v_{B / E}=v_{B}-v_{E}=(18-9.81 t)-2=16-9.81 t
$$

When the ball hits the elevator at time $t=3.65 \mathrm{~s}$, we have

$$
v_{B / E}=16-9.81(3.65) \quad v_{B / E}=-19.81 \mathrm{~m} / \mathrm{s}
$$

Example-2: Collar A and block B are connected by a cable passing over three pulleys $C, D$, and $E$ as shown. Pulleys $C$ and $E$ are fixed, while $D$ is attached to a collar which is pulled downward with a constant velocity of $3 \mathrm{in} . / \mathrm{s}$. At $\mathrm{t}=0$, collar A starts moving downward from position K with a constant acceleration and no initial velocity. Knowing that the velocity of collar A is 12 in ./s as it passes through point $L$, determine the change in elevation, the velocity, and the acceleration of block B when collar A passes through L.

$$
\begin{gathered}
v_{A}^{2}=\left(v_{A}\right)_{0}^{2}+2 a_{A}\left[x_{A}-\left(x_{A}\right)_{0}\right] \quad(12)^{2}=0+2 a_{A}(8) \\
a_{A}=9 \mathrm{in} / \mathrm{s}^{2}
\end{gathered}
$$

The time at which collar A reaches point $L$ is obtained by writing

$$
v_{A}=\left(v_{A}\right)_{0}+a_{A} t \quad 12=0+9 t \quad t=1.333 \mathrm{~s}
$$

Motion of Pulley D. Recalling that the positive direction is downward, we write

$$
a_{D}=0 \quad v_{D}=3 \mathrm{in} / \mathrm{s} \quad x_{D}=\left(x_{D}\right)_{0}+v_{D} t=\left(x_{D}\right)_{0}+3 t
$$

When collar A reaches $L$, at $t=1.333 \mathrm{~s}$, we have

$$
x_{D}=\left(x_{D}\right)_{0}+3(1.333)=\left(x_{D}\right)_{0}+4
$$

Thus,

$$
x_{D}-\left(x_{D}\right)_{0}=4 \mathrm{in} .
$$

$$
\begin{gathered}
x_{A}+2 x_{D}+x_{B}=\left(x_{A}\right)_{0}+2\left(x_{D}\right)_{0}+\left(x_{B}\right)_{0} \\
{\left[x_{A}-\left(x_{A}\right)_{0}\right]+2\left[x_{D}-\left(x_{D}\right)_{0}\right]+\left[x_{B}-\left(x_{B}\right)_{0}\right]=0}
\end{gathered}
$$

But we know that $x_{A}-\left(x_{A}\right)_{0}=8 \mathrm{in}$. and $x_{D}-\left(x_{D}\right)_{0}=4 \mathrm{in}$.; substituting these values in (2), we find

$$
8+2(4)+\left[x_{B}-\left(x_{B}\right)_{0}\right]=0 \quad x_{B}-\left(x_{B}\right)_{0}=-16 \mathrm{in} .
$$

Thus:
Change in elevation of $B=16 \mathrm{in} . \uparrow$
Differentiating (1) twice, we obtain equations relating the velocities and the accelerations of $A, B$, and $D$. Substituting for the velocities and accelerations of $A$ and $D$ at $t=1.333 \mathrm{~s}$, we have

$$
\begin{array}{rrrr}
v_{A}+2 v_{D}+v_{B}=0: & 12+2(3)+v_{B}=0 & \\
v_{B}=-18 \mathrm{in} / \mathrm{s} & v_{B}=18 \mathrm{in} . / \mathrm{s} \uparrow \\
a_{A}+2 a_{D}+a_{B}=0: & 9+2(0)+a_{B}=0 & \\
a_{B}=-9 \mathrm{in} . \mathrm{s}^{2} & a_{B}=9 \mathrm{in} . / \mathrm{s}^{2} \uparrow
\end{array}
$$



Example -3: A stone is thrown vertically upward from a point on a bridge located 40 m above the water. Knowing that it strikes the water 4 s after release, determine (a) the speed with which the stone was thrown upward, (b) the speed with which the stone strikes the water.

## SOLUTION

Uniformly accelerated motion. Origin at water. $+\uparrow$

$$
\begin{aligned}
& y=y_{0}+v_{0} t+\frac{1}{2} a t^{2} \\
& v=v_{0}+a t
\end{aligned}
$$

where $y_{0}=40 \mathrm{~m}$ and $a=-9.81 \mathrm{~m} / \mathrm{s}^{2}$.
(a) Initial speed.

$$
\begin{aligned}
y & =0 \text { when } t=4 \mathrm{~s} . \\
0 & =40+v_{0}(4)-\frac{1}{2}(9.81)(4)^{2} \\
v_{0} & =9.62 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

(b) Speed when striking the water. ( $v$ at $t=4 \mathrm{~s}$ )

$$
v=9.62-(9.81)(4)=-29.62 \mathrm{~m} / \mathrm{s} \quad \mathbf{v}=29.6 \mathrm{~m} / \mathrm{s}
$$

## Example -4:

Example -5: A sprinter in a 100-m race accelerates uniformly for the first 35 m and then runs with constant velocity. If the sprinter's time for the first 35 m is 5.4 s , determine (a) his acceleration, (b) his final velocity, (c) his time for the race.

## SOLUTION

## Given:

$$
0 \leq x \leq 35 \mathrm{~m}, \quad a=\text { constant }
$$

$$
35 \mathrm{~m}<x \leq 100 \mathrm{~m}, \quad v=\text { constant }
$$

$$
\text { At } t=0, \quad v=0 \quad \text { when } \quad x=35 \mathrm{~m}, \quad t=5.4 \mathrm{~s}
$$

Find:
(a) $a$
(b) $\quad v$ when $x=100 \mathrm{~m}$
(c) $t$ when $x=100 \mathrm{~m}$

(a) We have

$$
\begin{aligned}
\text { At } t=5.4 \mathrm{~s}: & 35 \mathrm{~m} & =\frac{1}{2} a(5.4 \mathrm{~s})^{2} \\
\text { or } & a & =2.4005 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

$$
a=2.40 \mathrm{~m} / \mathrm{s}^{2}
$$

(b) First note that $v=v_{\text {max }}$ for $35 \mathrm{~m} \leq x \leq 100 \mathrm{~m}$.

Now

$$
v^{2}=0+2 a(x-0) \text { for } 0 \leq x \leq 35 \mathrm{~m}
$$

When $x=35 \mathrm{~m}: \quad v_{\max }^{2}=2\left(2.4005 \mathrm{~m} / \mathrm{s}^{2}\right)(35 \mathrm{~m})$
or
(c) We have

$$
x=x_{1}+v_{0}\left(t-t_{1}\right) \quad \text { for } \quad 35 \mathrm{~m}<x \leq 100 \mathrm{~m}
$$

When $x=100 \mathrm{~m}: \quad 100 \mathrm{~m}=35 \mathrm{~m}+(12.9628 \mathrm{~m} / \mathrm{s})\left(t_{2}-5.4\right) \mathrm{s}$
or

$$
v_{\max }=12.9628 \mathrm{~m} / \mathrm{s} \quad v_{\max }=12.96 \mathrm{~m} / \mathrm{s}
$$

$$
t_{2}=10.41 \mathrm{~s}
$$

## Problems

Q1: As relay runner $A$ enters the 20 -m-long exchange zone with a speed of $12.9 \mathrm{~m} / \mathrm{s}$, he begins to slow down. He hands the baton to runner $B 1.82 \mathrm{~s}$ later as they leave the exchange zone with the same velocity. Determine (a) the uniform acceleration of each of the runners, (b) when runner $B$ should begin to run.

Q2: Automobiles $A$ and $B$ are traveling in adjacent highway lanes and at t 50 have the positions and speeds shown. Knowing that automobile A has a constant acceleration of 1.8 $\mathrm{ft} / \mathrm{s} 2$ and that $B$ has a constant deceleration of 1.2
 $\mathrm{ft} / \mathrm{s} 2$, determine (a) when and where $A$ will overtake $B$, (b) the speed of each automobile at

Q3: Two rockets are launched at a fireworks display. Rocket A is launched with an initial velocity v0 5100 $\mathrm{m} / \mathrm{s}$ and rocket $B$ is launched $t 1 \mathrm{~s}$ later with the same initial velocity. The two rockets are timed to explode simultaneously at a height of 300 m as A is falling and $B$ is rising. Assuming a constant acceleration g 5 $9.81 \mathrm{~m} / \mathrm{s} 2$, determine (a) the time t 1 , (b) the velocity of

Q4: A sprinter in a 100-m race accelerates uniformly for the first 35 m and then runs with constant velocity. If the sprinter's time for the first 35 m is 5.4 s , determine (a) his acceleration, (b) his final velocity, (c) his time for the race.


Q5: Two automobiles $A$ and $B$ are approaching each other in adjacent highway lanes. At t 50 , A and B are 3200 ft apart, their speeds are vA $565 \mathrm{mi} / \mathrm{h}$ and vB $540 \mathrm{mi} / \mathrm{h}$, and they are at points $P$ and $Q$, respectively. Knowing that $A$ passes point Q 40 s after B was there and that B passes point P 42 s after A was there, determine (a) the uniform accelerations of $A$ and $B$, (b) when the vehicles pass each other. (c) the speed of $B$ at that time.


Q6: The elevator shown in the figure moves downward with a constant velocity of $4 \mathrm{~m} / \mathrm{s}$. Determine (a) the velocity of the cable C , (b) the velocity of the counterweight W , (c) the relative velocity of the cable $C$ with respect to the elevator, (d) the relative velocity of the counterweight W with respect to the elevator.


Q7: At the instant shown, slider block $B$ is moving with a constant acceleration, and its speed is $150 \mathrm{~mm} / \mathrm{s}$. Knowing that after slider block A has moved 240 mm to the right its velocity is $60 \mathrm{~mm} / \mathrm{s}$, determine (a) the accelerations of $A$ and $B$, (b) the acceleration of portion $D$ of the cable, (c) the velocity and the change in position of slider block B after 4 s .


Q8: The motor M reels in the cable at a constant rate of $100 \mathrm{~mm} / \mathrm{s}$. Determine (a) the velocity of load L, (b) the velocity of pulley $B$ with respect to load $L$.


## Curvilinear Motion of Partials

## Position Vector, Velocity, Acceleration

When a particle moves along a curve other than a straight we say that the particle is in curvilinear motion


The instantaneous velocity

$$
\mathbf{v}=\lim _{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} \quad \mathbf{v}=\frac{d \mathbf{r}}{d t}
$$

The vector $v$ obtained in the limit must therefore be tangen to the path of the particle (Fig. 11.14c).

The instantaneous acceleration

(b)


$$
\mathbf{a}=\frac{d \mathbf{v}}{d t}
$$

The curve described by the tip of $v$ and shown in Fig. 11.15c is called the hodograph of the motion


(b)

(c)

## Rectangular Components of Velocity and Acceleration

$$
\begin{aligned}
& \mathbf{r}=x \mathbf{i}+y \mathbf{j}+z \mathbf{k} \\
& \mathbf{v}=\frac{d \mathbf{r}}{d t}=\ddot{x} \mathbf{i}+\dot{y} \mathbf{j}+\dot{z} \mathbf{k} \\
& \mathbf{a}=\frac{d \mathbf{v}}{d t}=\ddot{x} \mathbf{i}+\ddot{y} \mathbf{j}+\ddot{z} \mathbf{k}
\end{aligned}
$$



## In the case of the motion of a projectile

$$
a_{x}=\ddot{x}=0 \quad a_{y}=\ddot{y}=-g \quad a_{z}=\ddot{z}=0
$$

In the Horizontal Motion

$$
\begin{aligned}
v_{x} & =\dot{x}=\left(v_{x}\right)_{0} \\
x & =x_{0}+\left(v_{x}\right)_{0} t
\end{aligned}
$$

In the Vertical Motion

(a) Motion of a projectile
$v_{y}=\dot{y}=\left(v_{y}\right)_{0}-g t$

$$
y=y_{0}+\left(v_{y}\right)_{0} t-\frac{1}{2} g t^{2}
$$

$V_{y}{ }^{2}=V_{y o}{ }^{2}-2 g\left(y-y_{0}\right)$

If the projectile is fired in the xy plane from the origin O , we have $\mathrm{xO}=\mathrm{y} 0=\mathrm{zO}=0$ and $(\mathrm{vz}) 0=0$, and the equations of motion reduce to

$$
\begin{array}{rlrl}
v_{x} & =\left(v_{x}\right)_{0} & v_{y} & =\left(v_{y}\right)_{0}-g t \\
x & =\left(v_{x}\right)_{0} t & y & =\left(v_{y}\right)_{0} t-\frac{1}{2} g t^{2}
\end{array} r z=0
$$

Example-1: A projectile is fired from the edge of a 150-m cliff with an initial velocity of $180 \mathrm{~m} / \mathrm{s}$ at an angle of $30^{\circ}$ with the horizontal. Neglecting air resistance, find (a) the horizontal distance from the gun to the point where the projectile strikes the ground, (b) the greatest elevation above the ground reached by the projectile.

Vertical Motion. Uniformly Accelerated Motion. Choosing the positive sense of the $y$ axis upward and placing the origin $O$ at the gun, we have

$$
\begin{aligned}
\left(v_{y}\right)_{0} & =(180 \mathrm{~m} / \mathrm{s}) \sin 30^{\circ}=+90 \mathrm{~m} / \mathrm{s} \\
a & =-9.81 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Substituting into the equations of uniformly accelerated motion, we have

$$
\begin{array}{rlrl}
v_{y} & =\left(v_{y}\right)_{0}+a t & v_{y} & =90-9.81 t \\
y & =\left(v_{y}\right)_{0} t+\frac{1}{2} a t^{2} & y & =90 t-4.90 t^{2} \\
v_{y}^{2} & =\left(v_{y}\right)_{0}^{2}+2 a y & v_{y}^{2} & =8100-19.62 y \tag{3}
\end{array}
$$

Horizontal Motion. Uniform Motion. Choosing the positive sense of the $x$ axis to the right, we have

$$
\left(v_{x}\right)_{0}=(180 \mathrm{~m} / \mathrm{s}) \cos 30^{\circ}=+155.9 \mathrm{~m} / \mathrm{s}
$$

Substituting into the equation of uniform motion, we obtain

$$
\begin{equation*}
x=\left(v_{x}\right)_{0} t \quad x=155.9 t \tag{4}
\end{equation*}
$$


a. Horizontal Distance. When the projectile strikes the ground, we have

$$
y=-150 m
$$

Carrying this value into Eq. (2) for the vertical motion, we write
$-150=90 t-4.90 t^{2} \quad t^{2}-18.37 t-30.6=0 \quad t=19.91 \mathrm{~s}$ Carrying $t=19.91 \mathrm{~s}$ into Eq. (4) for the horizontal motion, we obtain

$$
x=155.9(19.91) \quad x=3100 \mathrm{~m}
$$


b. Greatest Elevation. When the projectile reaches its greatest elevation, we have $v_{y}=0$; carrying this value into Eq. (3) for the vertical motion, we write

$$
0=8100-19.62 y \quad y=413 \mathrm{~m}
$$

Greatest elevation above ground $=150 \mathrm{~m}+413 \mathrm{~m}=563 \mathrm{~m}$

Example-2: A projectile is fired with an initial velocity of $800 \mathrm{ft} / \mathrm{s}$ at a target $B$ located 2000 ft above the gun $A$ and at a horizontal distance of $12,000 \mathrm{ft}$. Neglecting air resistance, determine the value of the firing angle $\alpha$.


Horizontal Motion. Placing the origin of the coordinate axes at the gun, we have

$$
\left(v_{x}\right)_{0}=800 \cos \alpha
$$

Substituting into the equation of uniform horizontal motion, we obtain

$$
x=\left(v_{x}\right)_{0} t \quad x=(800 \cos \alpha) t
$$

The time required for the projectile to move through a horizontal distance of $12,000 \mathrm{ft}$ is obtained by setting $x$ equal to $12,000 \mathrm{ft}$.

$$
\begin{aligned}
12,000 & =(800 \cos \alpha) t \\
t & =\frac{12,000}{800 \cos \alpha}=\frac{15}{\cos \alpha}
\end{aligned}
$$

## Vertical Motion

$$
\left(v_{y}\right)_{0}=800 \sin \alpha \quad a=-32.2 \mathrm{ft} / \mathrm{s}^{2}
$$

Substituting into the equation of uniformly accelerated vertical motion, we obtain

$$
y=\left(v_{y}\right)_{0} t+\frac{1}{2} a t^{2} \quad y=(800 \sin \alpha) t-16.1 t^{2}
$$

Projectile Hits Target. When $x=12,000 \mathrm{ft}$, we must have $y=2000 \mathrm{ft}$. Substituting for $y$ and setting $t$ equal to the value found above, we write

$$
2000=800 \sin \alpha \frac{15}{\cos \alpha}-16.1\left(\frac{15}{\cos \alpha}\right)^{2}
$$

Since $1 / \cos ^{2} \alpha=\sec ^{2} \alpha=1+\tan ^{2} \alpha$, we have

$$
\begin{gathered}
2000=800(15) \tan \alpha-16.1\left(15^{2}\right)\left(1+\tan ^{2} \alpha\right) \\
3622 \tan ^{2} \alpha-12,000 \tan \alpha+5622=0
\end{gathered}
$$



Solving this quadratic equation for $\tan \alpha$, we have

$$
\begin{array}{cc}
\tan \alpha=0.565 \quad \text { and } \quad \tan \alpha=2.75 \\
& \alpha=29.5^{\circ} \quad \text { and } \quad \alpha=70.0^{\circ}
\end{array}
$$

Example-3: An airplane used to drop water on brushfires is flying horizontally in a straight line at $180 \mathrm{mi} / \mathrm{h}$ at an altitude of 300 ft . Determine the distance d at which the pilot should release the water so that it will hit the fire at B.


## SOLUTION

First note

$$
v_{0}=180 \mathrm{~km} / \mathrm{h}=264 \mathrm{ft} / \mathrm{s}
$$

Place origin of coordinates at Point $A$.
Vertical motion. (Uniformly accelerated motion)

$$
y=0+(0) t-\frac{1}{2} g t^{2}
$$

At $B$ :

$$
-300 \mathrm{ft}=-\frac{1}{2}\left(32.2 \mathrm{ft} / \mathrm{s}^{2}\right) t^{2}
$$


or

$$
t_{B}=4.31666 \mathrm{~s}
$$

Horizontal motion. (Uniform)

At $B: \quad d=(264 \mathrm{ft} / \mathrm{s})(4.31666 \mathrm{~s})$
or

Example-4: In slow pitch softball the underhand pitch must reach a maximum height of between 1.8 m and 3.7 m above the ground. A pitch is made with an initial velocity of magnitude $13 \mathrm{~m} / \mathrm{s}$ at an angle of 33 with the horizontal. Determine (a) if the pitch meets the maximum height requirement, (b) the height of the ball as it reaches the batter


## SOLUTION

$$
v_{0}=13 \mathrm{~m} / \mathrm{s}, \alpha=33^{\circ}, x_{0}=0, y_{0}=0.6 \mathrm{~m}
$$

Vertical motion:

$$
\begin{aligned}
& v_{y}=v_{0} \sin \alpha-g t \\
& y=y_{0}+\left(v_{0} \sin \alpha\right) t-\frac{1}{2} g t^{2}
\end{aligned}
$$

At maximum height,

$$
v_{y}=0 \quad \text { or } \quad t=\frac{v_{0} \sin \alpha}{g}
$$

(a)

$$
\begin{gathered}
t=\frac{13 \sin 33^{\circ}}{9.81}=0.7217 \mathrm{~s} \\
y_{\max }=0.6+\left(13 \sin 33^{\circ}\right)(0.7217)-\frac{1}{2}(9.81)(0.7217)^{2} \quad y_{\max }=3.16 \mathrm{~m}
\end{gathered}
$$

$$
1.8 \mathrm{~m}<3.16 \mathrm{~m}<3.7 \mathrm{~m} \quad \text { yes } 4
$$

Horizontal motion:

$$
x=x_{0}+\left(v_{0} \cos \alpha\right) t \quad \text { or } \quad t=\frac{x-x_{0}}{v_{0} \cos \alpha}
$$

At $x=15.2 \mathrm{~m}$,

$$
t=\frac{15.2-0}{13 \cos 33^{\circ}}=1.3941 \mathrm{~s}
$$

(b) Corresponding value of $y$ :

$$
y=0.6+\left(13 \sin 33^{\circ}\right)(1.3941)-\frac{1}{2}(9.81)(1.3941)^{2}
$$

$$
y=0.937 \mathrm{~m}
$$

## Problems

Q1: A homeowner uses a snow blower to clear his driveway. Knowing that the snow is discharged at an average angle of $40^{\circ}$ with the horizontal, determine the initial velocity $\mathrm{v}_{0}$ of the snow.


Q2: A helicopter is flying with a constant horizontal velocity of $180 \mathrm{~km} / \mathrm{h}$ and is directly above point $A$ when a loose part begins to fall. The part lands 6.5 s later at point $B$ on an inclined surface. Determine (a) the distance d between
 points $A$ and $B$. (b) the initial height $h$.

Q3: A baseball-pitching machine "throws" baseballs with a horizontal velocity $\mathbf{v 0}$. Knowing that height $h$ varies between 788 mm and 1068 mm , determine (a) the range of values of $\omega,(b)$ the values of a corresponding to $h 5788 \mathrm{~mm}$ and $h 51068$ mm.


Q4: Water flows from a drain spout with an initial velocity of $2.5 \mathrm{ft} / \mathrm{s}$ at an angle of $15^{\circ}$ with the horizontal. Determine the range of values of the distance $d$ for which the water will enter the trough $B C$.


Q5: A golfer hits a golf ball with an initial velocity of $160 \mathrm{ft} / \mathrm{s}$ at an angle of $25^{\circ}$ with the horizontal. Knowing that the fairway slopes downward at an average angle of $5^{\circ}$, determine the distance $d$ between the golfer and point $B$ where the ball first lands.


Q6: A basketball player shoots when she is 16 ft from the backboard. Knowing that the ball has an initial velocity v 0 at an angle of $30^{\circ}$ with the horizontal, determine the value of $v 0$ when $d$ is equal to (a) 9 in., (b) 17 in.


Q7: A tennis player serves the ball at a height h 52.5 m with an initial velocity of $v 0$ at an angle of 58 with the horizontal. Determine the range of $v 0$ for which the ball will land in the service area that extends to 6.4 m beyond the net.


Q8: The initial velocity v0 of a hockey puck is $105 \mathrm{mi} / \mathrm{h}$. Determine (a) the largest value (less than $45^{\circ}$ ) of the angle a for which the puck will enter the net, (b) the corresponding time required for the puck to reach the net.


Q9: A worker uses high-pressure water to clean the inside of a long drainpipe. If the water is discharged with an initial velocity v 0 of $11.5 \mathrm{~m} / \mathrm{s}$, determine (a) the distance $d$ to the farthest point $B$ on the top of the pipe that the worker can wash from his position at $A$, (b) the corresponding angle a.


## 4-Tangential and Normal Component

that the velocity of a particle is a vector tangent to the path of the particle but that, in general, the acceleration is not tangent to the path. It is sometimes convenient to resolve the acceleration into components directed, respectively, along the tangent and the normal to the path of the particle.

$$
\begin{aligned}
& \mathbf{a}=\frac{d \mathbf{v}}{d t}=\frac{d v}{d t} \mathbf{e}_{t}+v \frac{d \mathbf{e}_{t}}{d t} \\
& \mathbf{a}=\frac{d v}{d t} \mathbf{e}_{t}+\frac{v^{2}}{\mathrm{r}} \mathbf{e}_{n}
\end{aligned}
$$


(a)

The relations obtained express that the tangential component of the acceleration is equal to the rate of change of the speed of the particle, while the normal component is equal to the square of the speed divided by the radius of curvature of the path at $P$. If the speed of the particle increases, at is positive and the vector component at points in the direction of motion. If the speed of the particle decreases, at is negative and at points against the direction of motion. The vector component an, on the other hand, is always directed toward the center of curvature $C$ of the path

Example-1: A motorist is traveling on a curved section of highway of radius 2500 ft at the speed of $60 \mathrm{mi} / \mathrm{h}$. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to $45 \mathrm{mi} / \mathrm{h}$, determine the acceleration of the automobile immediately after the brakes have been applied.


## SOLUTION

Tangential Component of Acceleration. First the speeds are expresseo in $\mathrm{ft} / \mathrm{s}$.

$$
\begin{aligned}
60 \mathrm{mi} / \mathrm{h}=\left(60 \frac{\mathrm{mi}}{\mathrm{~h}}\right)\left(\frac{5280 \mathrm{ft}}{1 \mathrm{mi}}\right)\left(\frac{1 \mathrm{~h}}{3600 \mathrm{~s}}\right) & =88 \mathrm{ft} / \mathrm{s} \\
45 \mathrm{mi} / \mathrm{h} & =66 \mathrm{ft} / \mathrm{s}
\end{aligned}
$$

Since the automobile slows down at a constant rate, we have

$$
a_{t}=\text { average } a_{t}=\frac{\Delta v}{\Delta t}=\frac{66 \mathrm{ft} / \mathrm{s}-88 \mathrm{ft} / \mathrm{s}}{8 \mathrm{~s}}=-2.75 \mathrm{ft} / \mathrm{s}^{2}
$$

Normal Component of Acceleration. Immediately after the brakes havi been applied, the speed is still $88 \mathrm{ft} / \mathrm{s}$, and we have

$$
a_{n}=\frac{v^{2}}{r}=\frac{(88 \mathrm{ft} / \mathrm{s})^{2}}{2500 \mathrm{ft}}=3.10 \mathrm{ft} / \mathrm{s}^{2}
$$

Magnitude and Direction of Acceleration. The magnitude and directior of the resultant $\mathbf{a}$ of the components $\mathbf{a}_{n}$ and $\mathbf{a}_{t}$ are

$$
\begin{array}{rlr}
\tan \mathrm{a} & =\frac{a_{n}}{a_{t}}=\frac{3.10 \mathrm{ft} / \mathrm{s}^{2}}{2.75 \mathrm{ft} / \mathrm{s}^{2}} & \mathrm{a}=48.4^{\circ} \\
a & =\frac{a_{n}}{\sin \mathrm{a}}=\frac{3.10 \mathrm{ft} \mathrm{~s}^{2}}{\sin 48.4^{\circ}} & \mathrm{a}=4.14 \mathrm{ft} / \mathrm{s}^{2}
\end{array}
$$

Example-2: Determine the maximum speed that the cars of the roller-coaster can reach along the circular portion $A B$ of the track if r 525 m and the normal component of their acceleration cannot exceed 3 g .

$$
\begin{aligned}
a_{n} & =\frac{v^{2}}{\rho} \\
\left(v_{\max }\right)_{A B}^{2} & =\left(3 \times 9.81 \mathrm{~m} / \mathrm{s}^{2}\right)(25 \mathrm{~m}) \\
\left(v_{\max }\right)_{A B} & =27.124 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



$$
\left(v_{\max }\right)_{A B}=97.6 \mathrm{~km} / \mathrm{h}
$$

Example-3: A robot arm moves so that $P$ travels in a circle about Point $B$, which is not moving. Knowing that $P$ starts from rest, and its speed increases at a constant rate of $10 \mathrm{~mm} / \mathrm{s} 2$, determine (a) the magnitude of the acceleration when $\mathrm{t}=4 \mathrm{~s}$, (b) the time for the magnitude of the acceleration to be $80 \mathrm{~mm} / \mathrm{s} 2$.

## SOLUTION

Tangential acceleration: $\quad a_{t}=10 \mathrm{~mm} / \mathrm{s}^{2}$
Speed:

$$
v=a_{t} t
$$

Normal acceleration:

$$
a_{n}=\frac{v^{2}}{\rho}=\frac{a_{t}^{2} t^{2}}{\rho}
$$


where

$$
\rho=0.8 \mathrm{~m}=800 \mathrm{~mm}
$$

(a) When $t=4 \mathrm{~s}$

$$
v=(10)(4)=40 \mathrm{~mm} / \mathrm{s}
$$

$$
a_{n}=\frac{(40)^{2}}{800}=2 \mathrm{~mm} / \mathrm{s}^{2}
$$

Acceleration:

$$
a=\sqrt{a_{t}^{2}+a_{n}^{2}}=\sqrt{(10)^{2}+(2)^{2}}
$$

$$
a=10.20 \mathrm{~mm} / \mathrm{s}^{2}
$$

(b) Time when $a=80 \mathrm{~mm} / \mathrm{s}^{2}$

$$
\begin{aligned}
a^{2} & =a_{n}^{2}+a_{t}^{2} \\
(80)^{2} & =\left[\frac{(10)^{2} t^{2}}{800}\right]^{2}+10^{2} \quad t^{4}=403200 \mathrm{~s}^{4}
\end{aligned}
$$

$$
t=25.2 \mathrm{~s}
$$

## Problems

Q1: Determine the smallest radius that should be used for a highway if the normal component of the acceleration of a car traveling at $72 \mathrm{~km} / \mathrm{h}$ is not to exceed $0.8 \mathrm{~m} / \mathrm{s} 2$.

Q2: To test its performance, an automobile is driven around a circular test track of diameter $d$. Determine (a) the value of $d$ if when the speed of the automobile is 45 $\mathrm{mi} / \mathrm{h}$, the normal component of the acceleration is $11 \mathrm{ft} / \mathrm{s} 2$, (b) the speed of the automobile if d 5600 ft and the normal component of the acceleration is measured to be 0.6 g .

Q3: Race car $A$ is traveling on a straight portion of the track while race car B is traveling on a circular portion of the track. At the instant shown, the speed of $A$ is increasing at the rate of $10 \mathrm{~m} / \mathrm{s} 2$, and the speed of $B$ is decreasing at the rate of $6 \mathrm{~m} / \mathrm{s} 2$. For
 the position shown, determine (a) the velocity of $B$ relative to $A$, (b) the acceleration of $B$ relative to $A$.

Q4: From a photograph of a homeowner using a snowblower, it is determined that the radius of curvature of the trajectory of the snow was 30 ft as the snow left the discharge chute at $A$. Determine
(a) the discharge velocity vA of the snow, (b) the radius of curvature of the trajectory at its maximum height.


## Kinetics of Particles

## Newton's Second Law

## 1- Newton second law Motion

If the resultant force acting on a particle is not zero, the particle will have acceleration proportional to the magnitude of the resultant and in the direction of this resultant force

(a)

(b)

$$
\Sigma \mathbf{F}=m \mathbf{a}
$$

where F represents the sum, or resultant, of all the forces acting on the particle.

## 2 -System of Units

## A-International System of Units (SI Units)

$$
1 \mathrm{~N}=(1 \mathrm{~kg})\left(1 \mathrm{~m} / \mathrm{s}^{2}\right)=1 \mathrm{~kg} \cdot \mathrm{~m} / \mathrm{s}^{2}
$$


$W=(1 \mathrm{~kg})\left(9.81 \mathrm{~m} / \mathrm{s}^{2}\right)=9.81 \mathrm{~N}$

$$
\begin{aligned}
& 1 \mathrm{~km}= 1000 \mathrm{~m} \quad 1 \mathrm{~mm}= \\
& 1 \mathrm{Mg}= 0.001 \mathrm{~m} \\
& 1000 \mathrm{~kg} \quad 1 \mathrm{~g}= 0.001 \mathrm{~kg} \\
& 1 \mathrm{kN}=1000 \mathrm{~N}
\end{aligned}
$$

## B- U.S. Customary Units

$$
\begin{aligned}
& F=m a \quad 1 \mathrm{lb}=(1 \mathrm{slug})\left(1 \mathrm{ft} / \mathrm{s}^{2}\right) \\
& 1 \mathrm{slug}=\frac{1 \mathrm{lb}}{1 \mathrm{ft} / \mathrm{s}^{2}}=1 \mathrm{lb} \cdot \mathrm{~s}^{2} / \mathrm{ft}
\end{aligned}
$$



Fig. 12.6

$$
\Sigma F_{x}=m \ddot{x} \quad \Sigma F_{y}=m \ddot{y} \quad \Sigma F_{z}=m \ddot{z}
$$

## B-Tangential and Normal Components

Resolving the forces and the acceleration of the particle into components along the tangent to the path (in the direction of motion) and the normal (toward the inside of


$$
\Sigma F_{t}=m a_{t} \quad \Sigma F_{n}=m a_{n}
$$

Substituting for $a_{t}$ and $a_{n}$ from Eqs. (11.40), we have

$$
\Sigma F_{t}=m \frac{d v}{d t} \quad \Sigma F_{n}=m \frac{v^{2}}{\rho}
$$

Example -1: A 200-lb block rests on a horizontal plane. Find the magnitude of the force $\mathbf{P}$ required to give the block an acceleration of $10 \mathrm{ft} / \mathrm{s} 2$ to the right. The coefficient of kinetic friction between the block and the plane is $\mu_{k}=0.25$.


The mass of the block is

$$
m=\frac{W}{g}=\frac{200 \mathrm{lb}}{32.2 \mathrm{ft} / \mathrm{s}^{2}}=6.21 \mathrm{lb} \cdot \mathrm{~s}^{2} / \mathrm{ft}
$$

We note that $F=\mu_{k} N=0.25 N$ and that $a=10 \mathrm{ft} / \mathrm{s}^{2}$. Expressing that the forces acting on the block are equivalent to the vector ma, we write

$$
\begin{array}{ll}
\xrightarrow{+} \Sigma F_{x}=m a: & P \cos 30^{\circ}-0.25 N=\left(6.21 \mathrm{lb} \cdot \mathrm{~s}^{2} / \mathrm{ft}\right)\left(10 \mathrm{ft} / \mathrm{s}^{2}\right) \\
& P \cos 30^{\circ}-0.25 N=62.1 \mathrm{lb} \\
+\uparrow \Sigma F_{y}=0: & N-P \sin 30^{\circ}-200 \mathrm{lb}=0 \tag{2}
\end{array}
$$

Solving (2) for $N$ and substituting the result into (1), we obtain

$$
N=P \sin 30^{\circ}+200 \mathrm{lb}
$$

$P \cos 30^{\circ}-0.25\left(P \sin 30^{\circ}+200 \mathrm{lb}\right)=62.1 \mathrm{lb} \quad P=151 \mathrm{lb}$

Example -2: The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.

$$
x_{B}=\frac{1}{2} x_{A}
$$

lect to $t$, we have

$$
a_{B}=\frac{1}{2} a_{A}
$$



Block A. Denoting by $T_{1}$ the tension in cord $A C D$, we write $\xrightarrow{+} \Sigma F_{x}=m_{A} a_{A}: \quad T_{1}=100 a_{A}$
Block B. Observing that the weight of block $B$ is


$$
W_{B}=m_{B} g=(300 \mathrm{~kg})\left(9.81 \mathrm{~m} / \mathrm{s}^{2}\right)=2940 \mathrm{~N}
$$

and denoting by $T_{2}$ the tension in cord $B C$, we write
$+\downarrow \Sigma F_{y}=m_{B} a_{B}: \quad 2940-T_{2}=300 a_{B}$ or, substituting for $a_{B}$ from (1),

$$
\begin{gathered}
2940-T_{2}=300\left(\frac{1}{2} a_{A}\right) \\
T_{2}=2940-150 a_{A}
\end{gathered}
$$

Pulley C. Since $m_{C}$ is assumed to be zero, we have
$+\downarrow \Sigma F_{y}=m_{C} a_{C}=0: \quad T_{2}-2 T_{1}=0$
Substituting for $T_{1}$ and $T_{2}$ from (2) and (3), respectively, into (4) we write

$$
\begin{aligned}
& 2940-150 a_{A}-2\left(100 a_{A}\right)=0 \\
& 2940-350 a_{A}=0 \quad a_{A}=8.40 \mathrm{~m} / \mathrm{s}^{2}
\end{aligned}
$$

Substituting the value obtained for $a_{A}$ into (1) and (2), we have

$$
\begin{array}{lr}
a_{B}=\frac{1}{2} a_{A}=\frac{1}{2}\left(8.40 \mathrm{~m} / \mathrm{s}^{2}\right) & a_{B}=4.20 \mathrm{~m} / \mathrm{s}^{2} \\
T_{1}=100 a_{A}=(100 \mathrm{~kg})\left(8.40 \mathrm{~m} / \mathrm{s}^{2}\right) & T_{1}=840 \mathrm{~N}
\end{array}
$$

Recalling (4), we write

$$
T_{2}=2 T_{1} \quad T_{2}=2(840 \mathrm{~N}) \quad T_{2}=1680 \mathrm{~N}
$$

We note that the value obtained for $T_{2}$ is not equal to the weight of block $B$.

Example -3: The bob of a 2-m pendulum describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and the acceleration of the bob in that position


The weight of the bob is $W=m g$; the tension in the cord is thus 2.5 mg . Recalling that $\mathbf{a}_{n}$ is directed toward $O$ and assuming $\mathbf{a}_{t}$ as shown, we apply Newton's second law and obtain

$$
\begin{array}{ccc}
+\angle \Sigma F_{t}=m a_{t}: & m g \sin 30^{\circ}=m a_{t} & \\
& a_{t}=g \sin 30^{\circ}=+4.90 \mathrm{~m} / \mathrm{s}^{2} & \mathrm{a}_{t}=4.90 \mathrm{~m} / \mathrm{s}^{2} \swarrow \\
+\nwarrow \Sigma F_{n}=m a_{n}: & 2.5 \mathrm{mg}-m g \cos 30^{\circ}=m a_{n} & \\
& a_{n}=1.634 \mathrm{~g}=+16.03 \mathrm{~m} / \mathrm{s}^{2} & \mathrm{a}_{n}=16.03 \mathrm{~m} / \mathrm{s}^{2 \pi}
\end{array}
$$

Since $a_{n}=v^{2} / \rho$, we have $v^{2}=\rho a_{n}=(2 \mathrm{~m})\left(16.03 \mathrm{~m} / \mathrm{s}^{2}\right)$

$$
v= \pm 5.66 \mathrm{~m} / \mathrm{s} \quad \mathrm{v}=5.66 \mathrm{~m} / \mathrm{s} \Omega(\text { up or down })
$$

## Problems

Q1: The acceleration of a package sliding at point $A$ is $3 \mathrm{~m} / \mathrm{s} 2$. Assuming that the coefficient of kinetic friction is the same for each section, determine the acceleration of the package at point $B$.

Q 2: A 20-kg package is at rest on an incline when a force $\mathbf{P}$ is applied to it. Determine the magnitude of $\mathbf{P}$ if 10 s is required for the package to travel 5 m up the incline. The kinetic coefficients of friction between the package and the incline are both equal to 0.3 .


Q 3: A 5000-lb truck is being used to lift a $1000-\mathrm{lb}$ boulder $B$ that is on a $200-\mathrm{lb}$ pallet A . Knowing the acceleration of the truck is $1 \mathrm{ft} / \mathrm{s} 2$, determine (a) the horizontal force between the tires and the
 pallet.

Q 4: The system shown is initially at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys, determine (a) the acceleration of each block, (b) the tension in each cable.

20 lb


Q 5: Each of the systems shown is initially at rest. Neglecting axle friction and the masses of the pulleys, determine for each system (a) the acceleration of block A, (b) the velocity of block A after it has moved through 10 ft , (c) the time required for block $A$ to reach a velocity of $20 \mathrm{ft} / \mathrm{s}$.


Q 6: The two blocks shown are originally at rest. Neglecting the masses of the pulleys and the effect of friction in the pulleys and between block A and the incline, determine (a) the acceleration of each block, (b) the tension in the cable.

Q 7: During a hammer thrower's practice swings, the 7.1-kg head A of the hammer revolves at a constant speed $v$ in a horizontal circle as shown. If $r=0.93 \mathrm{~m}$ and $u=60^{\circ}$, determine (a) the tension in wire BC, (b) the speed of the hammer's head

Q 8: To unload a bound stack of plywood from a truck, the driver first tilts the bed of the truck and then accelerates from rest. Knowing that the coefficients of friction between the bottom sheet of plywood and the bed are ms 50.40 and mk 50.30 , determine (a) the smallest acceleration of the truck
 which will cause the stack of plywood to slide, (b) the acceleration of the truck which causes corner $A$ of the stack to reach the end of the bed in

Q 9: A 450-g tetherball $A$ is moving along a horizontal circular path at a constant speed of $4 \mathrm{~m} / \mathrm{s}$. Determine (a) the angle $u$ that the cord forms with pole BC, (b) the tension in the cord.


Q 10: Two wires $A C$ and $B C$ are tied at $C$ to a sphere which revolves at a constant speed $v$ in the horizontal circle shown. Determine the range of values of $v$ for which both wires remain taut.

Q 11: During a high-speed chase, a $2400-\mathrm{lb}$ sports car traveling at a speed of $100 \mathrm{mi} / \mathrm{h}$ just loses contact with the road as it reaches the crest $A$ of a hill. (a) Determine the radius of curvature $r$ of the vertical profile of the road at $A$. (b) Using the value of $r$ found in part a, determine the force exerted on a $160-\mathrm{lb}$ driver by the seat of his $3100-\mathrm{lb}$ car as the car, traveling at a constant speed of $50 \mathrm{mi} / \mathrm{h}$, passes through A.


## Chapter Three

## Energy and Momentum Methods

## 1-WORK OF A FORCE

The work of the force $\mathbf{F}$ corresponding to the displacement dr is defined as the quantity

$$
d U=\mathbf{F} \cdot d \mathbf{r}
$$



$$
\begin{equation*}
d U=F d s \cos \alpha \tag{13.1'}
\end{equation*}
$$

Using formula (3.30), we can also express the work $d U$ in terms of the rectangular components of the force and of the displacement:

$$
\begin{equation*}
d U=F_{x} d x+F_{y} d y+F_{z} d z \tag{13.1"}
\end{equation*}
$$

In U.S. customary units are used, work should be expressed in( ft. lb or in . lb).
In SI units are used, work should be expressed in N. m. The unit of work N. $m$ is called a joule ( J )

$$
1 \mathrm{ft} \cdot \mathrm{lb}=(1 \mathrm{ft})(1 \mathrm{lb})=(0.3048 \mathrm{~m})(4.448 \mathrm{~N})=1.356 \mathrm{~J}
$$

The work of $\mathbf{F}$ during a finite displacement of the particle from A1 to A2 (Fig. ) is obtained by integrating Eq. (13.1) along the path described by the particle. This work, denoted by U1-2, is

$$
U_{1 \rightarrow 2}=\int_{A_{1}}^{A_{2}} \mathbf{F} \cdot d \mathbf{r}
$$



When the force F is defined by its rectangular components, the expression (13.1) can be used for the elementary work. We then write

$$
U_{1 \rightarrow 2}=\int_{A_{1}}^{A_{2}}\left(F_{x} d x+F_{y} d y+F_{z} d z\right)
$$

* Work of a Constant Force in Rectilinear Motion. When a particle moving in a straight line is acted upon by a force $F$ of constant magnitude and of constant direction (Fig.), formula (13.2) yields

$$
U_{1 \rightarrow 2}=(F \cos \alpha) \Delta x
$$


where a a angle the force forms with direction of motion
$x=$ displacement from A1 to A2

* Work of the Force of Gravity. The work of the weight W of a body, i.e., of the force of gravity exerted on that body, is obtained by substituting the components of W into (13.1) and (13.2). With the y axis chosen upward (Fig. 13.4), we have $F x=0, F y=-W$, and $F z=0$, and we write

$$
\begin{aligned}
d U & =-W d y \\
U_{1 \rightarrow 2} & =-\int_{y_{1}}^{y_{2}} W d y=W y_{1}-W y_{2} \\
U_{1 \rightarrow 2} & =-W\left(y_{2}-y_{1}\right)=-W \Delta y
\end{aligned}
$$


where $\mathbf{y}$ is the vertical displacement from A 1 to A 2 .
The work of the weight $W$ is thus equal to the product of $W$ and the vertical displacement of the center of gravity of the body. The work is positive when $y<0$, that is, when the body moves down.

* Work of the Force Exerted by a Spring. Consider a body A attached to a fixed point $B$ by a spring; it is assumed that the spring is undeformed when the body is at A0 (Fig.). Experimental evidence shows that the magnitude of the force $F$ exerted by the spring on body $A$ is proportional to the deflection $x$ of the spring measured from the position A0. We have


$$
F=k x
$$


where $k$ is the spring constant, expressed in $\mathrm{N} / \mathrm{m}$ or $\mathrm{kN} / \mathrm{m}$ if SI units are used and in lb/ft or lb/in. if U.S. customary units are used.

$$
\begin{aligned}
d U & =-F d x=-k x d x \\
U_{1 \rightarrow 2} & =-\int_{x_{1}}^{x_{2}} k x d x=\frac{1}{2} k x_{1}^{2}-\frac{1}{2} k x_{2}^{2}
\end{aligned}
$$

## 2- PRINCIPLE OF WORK AND ENERGY

$$
F_{t}=m a_{t} \quad \text { or } \quad F_{t}=m \frac{d v}{d t}
$$

where $v$ is the speed of the particle. Recalling from Sec. 11.9 that $v=d s / d t$, we obtain

$$
\begin{gathered}
F_{t}=m \frac{d v}{d s} \frac{d s}{d t}=m v \frac{d v}{d s} \\
F_{t} d s=m v d v
\end{gathered}
$$



Integrating from $A_{1}$, where $s=s_{1}$ and $v=v_{1}$, to $A_{2}$, where $s=s_{2}$ and $v=v_{2}$, we write

$$
\begin{equation*}
\int_{s_{1}}^{s_{2}} F_{t} d s=m \int_{v_{1}}^{v_{2}} v d v=\frac{1}{2} m v_{2}^{2}-\frac{1}{2} m v_{1}^{2} \tag{13.8}
\end{equation*}
$$

$A_{2}$; as indicated in Sec. 13.2 , the work $U_{1 \rightarrow 2}$ is a scalar quantity. The expression $\frac{1}{2} m v^{2}$ is also a scalar quantity; it is defined as the kinetic energy of the particle and is denoted by $T$. We write

$$
\begin{equation*}
T=\frac{1}{2} m v^{2} \tag{13.9}
\end{equation*}
$$

Substituting into (13.8), we have

$$
\begin{equation*}
U_{1 \rightarrow 2}=T_{2}-T_{1} \tag{13.10}
\end{equation*}
$$

which expresses that, when a particle moves from A1 to A2 under the action of a force $F$, the work of the force $F$ is equal to the change in kinetic energy of the particle. This is known as the principle of work and energy

$$
T_{1}+U_{1 \rightarrow 2}=T_{2}
$$

## * APPLICATIONS OF THE PRINCIPLE OF WORK AND ENERGY

$$
\begin{aligned}
& T_{1}+U_{1 \rightarrow 2}=T_{2} \quad 0+W l=\frac{1}{2} \frac{W}{g} v_{2}^{2} \\
& P-W=m a_{n}=\frac{W}{g} \frac{v_{2}^{2}}{l} \\
& P=W+\frac{W}{g} \frac{2 g l}{l}=3 W
\end{aligned}
$$


(a)

(b)


## 3- Power and Efficiency

Power is defined as the time rate at which work is done

$$
\text { Power }=\frac{d U}{d t}
$$

Substituting the scalar product $\mathbf{F} \cdot d \mathbf{r}$ for $d U$, we can also write

$$
\text { Power }=\frac{d U}{d t}=\frac{\mathbf{F} \cdot d \mathbf{r}}{d t}
$$

and, recalling that $d \mathbf{r} / d t$ represents the velocity $\mathbf{v}$ of the point of application of $\mathbf{F}$,

$$
\begin{equation*}
\text { Power }=\mathbf{F} \cdot \mathbf{v} \tag{13.13}
\end{equation*}
$$

The mechanical efficiency of a machine was defined as the ratio of the output work to the input work:

$$
\begin{equation*}
\mathrm{h}=\frac{\text { output work }}{\text { input work }} \tag{13.14}
\end{equation*}
$$

This definition is based on the assumption that work is done at a constant rate. The ratio of the output to the input work is therefore equal to the ratio of the rates at which output and input work are done, and we have

$$
\begin{equation*}
\mathrm{h}=\frac{\text { power output }}{\text { power input }} \tag{13.15}
\end{equation*}
$$

Example-1: An automobile weighing 4000 lb is driven down a $5^{\circ}$ incline at a speed of $60 \mathrm{mi} / \mathrm{h}$ when the brakes are applied, causing a constant total braking force (applied by the road on the tires) of 1500 lb . Determine the distance
 traveled by the automobile as it comes to a stop

## Kinetic Energy

Position 1: $\quad v_{1}=\left(60 \frac{\mathrm{mi}}{\mathrm{h}}\right)\left(\frac{5280 \mathrm{ft}}{1 \mathrm{mi}}\right)\left(\frac{1 \mathrm{~h}}{3600 \mathrm{~s}}\right)=88 \mathrm{ft} / \mathrm{s}$

$$
T_{1}=\frac{1}{2} m v_{1}^{2}=\frac{1}{2}(4000 / 32.2)(88)^{2}=481,000 \mathrm{ft} \cdot \mathrm{lb}
$$

Position 2:

$$
v_{2}=0 \quad T_{2}=0
$$

Work $\quad U_{1 \rightarrow 2}=-1500 x+\left(4000 \sin 5^{\circ}\right) x=-1151 x$

## Principle of Work and Energy

$$
T_{1}+U_{1 \rightarrow 2}=T_{2}
$$



$$
481,000-1151 x=0 \quad x=418 \mathrm{ft}
$$



Example-2: Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block $A$ after it has moved 2 m . Assume that the coefficient of kinetic friction between block $A$ and the plane is $\mathrm{m} k 50.25$ and that the pulley is weightless and frictionless.


## SOLUTION

Work and Energy for Block $A$. We denote the friction force by $\mathbf{F}_{A}$ and the force exerted by the cable by $\mathbf{F}_{C}$, and write

$$
\begin{array}{r}
m_{A}=200 \mathrm{~kg} \quad W_{A}=(200 \mathrm{~kg})\left(9.81 \mathrm{~m} / \mathrm{s}^{2}\right)=1962 \mathrm{~N} \\
F_{A}=\mathrm{m}_{k} N_{A}=\mathrm{m}_{k} W_{A}=0.25(1962 \mathrm{~N})=490 \mathrm{~N} \\
T_{1}+U_{1 \mathrm{y} 2}=T_{2}: \quad 0+F_{C}(2 \mathrm{~m})-F_{A}(2 \mathrm{~m})=\frac{1}{2} m_{A} v^{2} \\
F_{C}(2 \mathrm{~m})-(490 \mathrm{~N})(2 \mathrm{~m})=\frac{1}{2}(200 \mathrm{~kg}) v^{2}
\end{array}
$$

Work and Energy for Block B. We write

$$
\begin{aligned}
& m_{B}=300 \mathrm{~kg} \quad W_{B}=(300 \mathrm{~kg})\left(9.81 \mathrm{~m} / \mathrm{s}^{2}\right)=2940 \mathrm{~N} \\
& T_{1}+U_{1 \mathrm{y} 2}=T_{2}: \quad 0+W_{B}(2 \mathrm{~m})-F_{C}(2 \mathrm{~m})=\frac{1}{2} m_{B} v^{2} \\
&(2940 \mathrm{~N})(2 \mathrm{~m})-F_{C}(2 \mathrm{~m})=\frac{1}{2}(300 \mathrm{~kg}) v^{2}
\end{aligned}
$$

Adding the left-hand and right-hand members of (1) and (2), we observe that the work of the forces exerted by the cable on $A$ and $B$ cancels out:

$$
\begin{aligned}
(2940 \mathrm{~N})(2 \mathrm{~m})-(490 \mathrm{~N})(2 \mathrm{~m}) & =\frac{1}{2}(200 \mathrm{~kg}+300 \mathrm{~kg}) v^{2} \\
4900 \mathrm{~J} & =\frac{1}{2}(500 \mathrm{~kg}) v^{2} \quad v=4.43 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$


$\square$

Example-2: A spring is used to stop a $60-\mathrm{kg}$ package which is sliding on a horizontal surface. The spring has a constant $\mathrm{k}=20 \mathrm{kN} / \mathrm{m}$ and is held by cables so that it is initially compressed 120 mm . Knowing that the package has a velocity of $2.5 \mathrm{~m} / \mathrm{s}$ in the position shown and that
 the maximum additional deflection of the spring is 40 mm , determine (a) the coefficient of kinetic friction between the package and the surface, (b) the velocity of the package as it passes again through the position shown
a. Motion from Position 1 to Position 2

Kinetic Energy Position 1: $\quad v_{1}=2.5 \mathrm{~m} / \mathrm{s}$

$$
T_{1}=\frac{1}{2} m v_{1}^{2}=\frac{1}{2}(60 \mathrm{~kg})(2.5 \mathrm{~m} / \mathrm{s})^{2}=187.5 \mathrm{~N} \cdot \mathrm{~m}=187.5 \mathrm{~J}
$$

Position 2: (maximum spring deflection): $\quad v_{2}=0 \quad T_{2}=0$


Work
Friction Force F. We have

$$
F=\mu_{k} N=\mu_{k} W=\mu_{k} m g=\mu_{k}(60 \mathrm{~kg})\left(9.81 \mathrm{~m} / \mathrm{s}^{2}\right)=(588.6 \mathrm{~N}) \mu_{k}
$$

The work of $\mathbf{F}$ is negative and equal to

$$
\left(U_{1 \rightarrow 2}\right)_{f}=-F x=-(588.6 \mathrm{~N}) \mu_{k}(0.600 \mathrm{~m}+0.040 \mathrm{~m})=-(377 \mathrm{~J}) \mu_{k}
$$

Spring Force $\mathbf{P}$. The variable force $\mathbf{P}$ exerted by the spring does an amount of negative work equal to the area under the force-deflection curve of the spring force. We have

$$
\begin{aligned}
& P_{\min }=k x_{0}=(20 \mathrm{kN} / \mathrm{m})(120 \mathrm{~mm})=(20000 \mathrm{~N} / \mathrm{m})(0.120 \mathrm{~m})=2400 \mathrm{~N} \\
& P_{\max }=P_{\mathrm{mnn}}+k \Delta x=2400 \mathrm{~N}+(20 \mathrm{kN} / \mathrm{m})(40 \mathrm{~mm})=3200 \mathrm{~N} \\
& \left(\mathrm{U}_{1 \rightarrow 2}\right)_{e}=-\frac{1}{2}\left(P_{\operatorname{mnn}}+P_{\max }\right) \Delta x=-\frac{1}{2}(2400 \mathrm{~N}+3200 \mathrm{~N})(0.040 \mathrm{~m})=-112.0 \mathrm{~J}
\end{aligned}
$$

The total work is thus

$$
U_{1 \rightarrow 2}=\left(U_{1 \rightarrow 2}\right)_{f}+\left(U_{1 \rightarrow 2}\right)_{e}=-(377 \mathrm{~J}) \mu_{k}-112.0 \mathrm{~J}
$$

Principle of Work and Energy

$$
T_{1}+U_{1 \rightarrow 2}=T_{2}: \quad 187.5 \mathrm{~J}-(377 \mathrm{~J}) \mu_{k}-112.0 \mathrm{~J}=0 \quad \mu_{k}=0.20
$$

Example-3: A 2000-lb car starts from rest at point 1 and moves without friction down the track shown. (a) Determine the force exerted by the track on the car at point 2 , where the radius of curvature of the track is 20
 ft . (b) determine the minimum safe value of the radius of curvature at point 3 .
a. Force Exerted by the Track at Point 2. The principle of work and energy is used to determine the velocity of the car as it passes through point 2.

Kinetic Energy. $\quad T_{1}=0$

$$
T_{2}=\frac{1}{2} m v_{2}^{2}=\frac{1}{2} \frac{W}{g} v_{2}^{2}
$$

Work. The only force which does work is the weight $\mathbf{W}$. Since the vertical displacement from point 1 to point 2 is 40 ft downward, the work of the weight is

$$
U_{1 \rightarrow 2}=+W(40 \mathrm{ft})
$$

## Principle of Work and Energy

$$
\begin{gathered}
T_{1}+U_{1 \rightarrow 2}=T_{2} \quad 0+W(40 \mathrm{ft})=\frac{1}{2} \frac{W}{g} v_{2}^{2} \\
v_{2}^{2}=80 g=80(32.2) \quad v_{2}=50.8 \mathrm{ft} / \mathrm{s}
\end{gathered}
$$

Newton's Second Law at Point 2. The acceleration $\mathbf{a}_{n}$ of the car at point 2 has a magnitude $a_{n}=v_{2}^{2} / \rho$ and is directed upward. Since the external forces acting on the car are $\mathbf{W}$ and $\mathbf{N}$, we write

$$
\begin{aligned}
+\uparrow \Sigma F_{n}=m a_{n}: \quad-W+N & =m a_{n} \\
& =\frac{W}{g} \frac{v_{2}^{2}}{\rho} \\
& =\frac{W}{g} \frac{80 g}{20}
\end{aligned}
$$

$$
N=5 W \quad \mathrm{~N}=10,000 \mathrm{lb} \uparrow
$$

b. Minimum Value of $\rho$ at Point 3. Principle of Work and Energy. Applying the principle of work and energy between point 1 and point 3 , we obtain

$$
\begin{gathered}
T_{1}+U_{1 \rightarrow 3}=T_{3} \quad 0+W(25 \mathrm{ft})=\frac{1}{2} \frac{W}{g} v_{3}^{2} \\
v_{3}^{2}=50 g=50(32.2) \quad v_{3}=40.1 \mathrm{ft} / \mathrm{s}
\end{gathered}
$$

Newton's Second Law at Point 3. The minimum safe value of $\rho$ occurs when $\mathbf{N}=0$. In this case, the acceleration $\mathbf{a}_{n}$, of magnitude $a_{n}=v_{3}^{2} / \rho$, is directed downward, and we write

$$
\begin{aligned}
+\downarrow \Sigma F_{n}=m a_{n}: & =\frac{W}{g} \frac{v_{3}^{2}}{\rho} \\
& =\frac{W}{g} \frac{50 g}{\rho}
\end{aligned} \rho=50 \mathrm{ft}
$$

Example -5: The dumbwaiter D and its load have a combined weight of 600 lb , while the counterweight $C$ weighs 800 lb . Determine the power delivered by the electric motor M when the dumbwaiter (a) is moving up at a constant speed of $8 \mathrm{ft} / \mathrm{s}$, (b) has an instantaneous velocity of $8 \mathrm{ft} / \mathrm{s}$ and an acceleration of $2.5 \mathrm{ft} / \mathrm{s} 2$, both directed upward.
a. Uniform Motion. We have $\mathbf{a}_{C}=\mathbf{a}_{D}=0$; both bodies are in equilibrium.

Free Body $C: \quad+\uparrow \Sigma F_{y}=0: \quad 2 T-800 \mathrm{lb}=0 \quad T=400 \mathrm{lb}$
Free Body $D: \quad+\uparrow \Sigma F_{y}=0: \quad F+T-600 \mathrm{lb}=0$

$$
\begin{aligned}
& F=600 \mathrm{lb}-T=600 \mathrm{lb}-400 \mathrm{lb}=200 \mathrm{lb} \\
& F v_{D}=(200 \mathrm{lb})(8 \mathrm{ft} / \mathrm{s})=1600 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s} \\
& \text { Power }=(1600 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s}) \frac{\mathrm{lhp}}{550 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s}}=2.91 \mathrm{hp}
\end{aligned}
$$

b. Accelerated Motion. We have

$$
\mathrm{a}_{D}=2.5 \mathrm{ft} / \mathrm{s}^{2} \uparrow \quad \mathbf{a}_{C}=-\frac{1}{2} \mathbf{a}_{D}=1.25 \mathrm{ft} / \mathrm{s}^{2} \downarrow
$$

The equations of motion are
Free Body C: $\quad+\downarrow \Sigma F_{y}=m_{C} a_{C}: \quad 800-2 T=\frac{800}{32.2}(1.25) \quad T=384.5 \mathrm{lb}$


Free Body $D: \quad+\uparrow \Sigma F_{y}=m_{D} a_{D}: \quad F+T-600=\frac{600}{32.2}(2.5)$

$$
\begin{gathered}
F+384.5-600=46.6 \quad F=262.1 \mathrm{lb} \\
F v_{D}=(262.1 \mathrm{lb})(8 \mathrm{ft} / \mathrm{s})=2097 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s}
\end{gathered}
$$

$$
\text { Power }=(2097 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s}) \frac{1 \mathrm{hp}}{550 \mathrm{ft} \cdot \mathrm{lb} / \mathrm{s}}=3.81 \mathrm{hp}
$$




## Problems

Q1: A 90-lb package is at rest on an incline when a constant force P is applied to it. The coefficient of kinetic friction between the package and the incline is 0.35 . Knowing that the speed of the package is $2 \mathrm{ft} / \mathrm{s}$ after it has moved 3 ft up the incline, determine the magnitude of the force $P$.

Q2: A 7.5-lb collar is released from rest in the position shown, slides down the inclined rod, and compresses the spring. The direction of motion is reversed and the collar slides up the rod. Knowing that the maximum deflection of the spring is 5 in., determine (a) the coefficient of kinetic friction between the collar and the rod, (b) the maximum speed of the collar

Q3:A package is projected up a $15^{\circ}$ incline at $A$ with an initial velocity of $8 \mathrm{~m} / \mathrm{s}$. Knowing that the coefficient of kinetic friction between the package and the incline is 0.12 , determine (a) the maximum distance $d$ that the package will move up the incline, (b) the velocity of the
 package as it returns to its original position.

Q4: Packages are thrown down an incline at $A$ with a velocity of $1 \mathrm{~m} / \mathrm{s}$. The packages slide along the surface $A B C$ to a conveyor belt which moves with a velocity of $2 \mathrm{~m} / \mathrm{s}$. Knowing that $d 5$ 7.5 m and mk 50.25 between the packages and
 all surfaces, determine (a) the speed of the package at $C,(b)$ the distance a package will slide on the conveyor belt before it comes to rest relative to the belt.

Q5: A trailer truck enters a 2 percent uphill grade traveling at $72 \mathrm{~km} / \mathrm{h}$ and reaches a speed of $108 \mathrm{~km} / \mathrm{h}$ in 300 m . The cab has a mass of 1800 kg and the trailer 5400 kg . Determine (a) the average force at the wheels of the cab, (b) the average force in the coupling between the cab and the trailer


Q6: The system shown is at rest when a constant 30lb force is applied to collar B. (a) If the force acts through the entire motion, determine the speed of collar B as it strikes the support at C. (b) After what distance d should the $30-\mathrm{lb}$ force be removed if the collar is to reach support C with zero velocity?


Q7: Two blocks $A$ and $B$, of mass 4 kg and 5 kg , respectively, are connected by a cord which passes over pulleys as shown. A $3-\mathrm{kg}$ collar C is placed on block $A$ and the system is released from rest. After the blocks have moved 0.9 m , collar $C$ is removed and blocks $A$ and $B$ continue to move. Determine the speed of block $A$ just before it strikes the ground.


Q8: An 8-lb collar C slides on a horizontal rod between springs $A$ and $B$. If the collar is pushed to the right until spring $B$ is compressed 2 in . and released, determine the distance through which the collar will travel assuming (a) no friction between the collar and the rod,
 (b) a coefficient of friction $\mu \mathrm{k}=0.35$.

## 4- Potential Energy

the work of the force of gravity $\mathbf{W}$ during this displacement is

$$
U_{1 \rightarrow 2}=W y_{1}-W y_{2}
$$

This function is called the potential energy of the body with respect to the force of gravity W and is denoted by Vg . We writ

$$
U_{1 \rightarrow 2}=\left(V_{g}\right)_{1}-\left(V_{g}\right)_{2} \quad \text { with } V_{g}=W y
$$



We note that if $(\mathrm{Vg}) 2>(\mathrm{Vg}) 1$, that is, if the potential energy increases during the displacement (as in the case considered here), the work U1-2 is negative
*the work of the force F exerted by the spring on the body is

$$
U_{1 \rightarrow 2}=\frac{1}{2} k x_{1}^{2}-\frac{1}{2} k x_{2}^{2}
$$



This function is denoted by Ve and is called the potential energy of the body with respect to the elastic force F. We write

$$
U_{1 \rightarrow 2}=\left(V_{e}\right)_{1}-\left(V_{e}\right)_{2} \quad \text { with } V_{e}=\frac{1}{2} k x^{2}
$$

## 5- Conservation of Energy

the sum of the kinetic energy and of the potential energy of the particle remains constant. The sum $\mathrm{T}+\mathrm{V}$ is called the total mechanical energy of the particle and is denoted by E

$$
\begin{aligned}
& T_{1}+V_{1}=T_{2}+V_{2} \\
& T_{1}=0 \quad V_{1}=W l \quad T_{1}+V_{1}=W l
\end{aligned}
$$

Recalling that at $A_{2}$ the speed of the pendulum is $v_{2}=\sqrt{2 g l}$, we have

$$
\begin{aligned}
T_{2}=\frac{1}{2} m v_{2}^{2}= & \frac{1}{2} \frac{W}{g}(2 g l)=W l \quad V_{2}=0 \\
& T_{2}+V_{2}=W l
\end{aligned}
$$

Example-1: A 20-lb collar slides without friction along a vertical rod as shown. The spring attached to the collar has an undeformed length of 4 in . and a constant of 3 $\mathrm{lb} / \mathrm{in}$. If the collar is released from rest in position 1 , determine its velocity after it has moved 6 in. to position 2.


Position 1. Potential Energy. The elongation of the spring is

$$
x_{1}=8 \mathrm{in} .-4 \mathrm{in} .=4 \mathrm{in} .
$$

and we have

$$
V_{e}=\frac{1}{2} k x_{1}^{2}=\frac{1}{2}(3 \mathrm{lb} / \mathrm{in} .)(4 \mathrm{in} .)^{2}=24 \mathrm{in} \cdot \mathrm{lb}
$$

Choosing the datum as shown, we have $V_{g}=0$. Therefore,

$$
V_{1}=V_{e}+V_{\mathrm{g}}=24 \mathrm{in} \cdot \mathrm{lb}=2 \mathrm{ft} \cdot \mathrm{lb}
$$

Kinetic Energy. Since the velocity in position 1 is zero, $T_{1}=0$. Position 2. Potential Energy. The elongation of the spring is

$$
x_{2}=10 \mathrm{in} .-4 \mathrm{in} .=6 \mathrm{in} .
$$

and we have

$$
\begin{aligned}
& V_{e}=\frac{1}{2} k x_{2}^{2}=\frac{1}{2}(3 \mathrm{lb} / \mathrm{in} .)(6 \mathrm{in} .)^{2}=54 \mathrm{in} \cdot \mathrm{lb} \\
& V_{g}=W y=(20 \mathrm{lb})(-6 \mathrm{in} .)=-120 \mathrm{in} \cdot \mathrm{lb}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
V_{2}=V_{e}+V_{g}=54-120 & =-66 \mathrm{in} \cdot \mathrm{lb} \\
& =-5.5 \mathrm{ft} \cdot \mathrm{lb}
\end{aligned}
$$

Kinetic Energy

$$
T_{2}=\frac{1}{2} m v_{2}^{2}=\frac{1}{2} \frac{20}{32.2} v_{2}^{2}=0.311 v_{2}^{2}
$$

Conservation of Energy. Applying the principle of conservation of energy between positions 1 and 2, we write

$$
\begin{aligned}
& T_{1}+V_{1}=T_{2}+V_{2} \\
& 0+2 \mathrm{ft} \cdot \mathrm{lb}=0.311 v_{2}^{2}-5.5 \mathrm{ft} \cdot \mathrm{lb} \\
& v_{2}= \pm 4.91 \mathrm{ft} / \mathrm{s} \\
& \quad \mathrm{v}_{2}=4.91 \mathrm{ft} / \mathrm{s} \downarrow
\end{aligned}
$$

Example -2: The $0.5-\mathrm{lb}$ pellet is pushed against the spring at A and released from rest. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop ABCDE and remain at all times in contact with the loop.


$$
\begin{aligned}
& +\downarrow \Sigma F_{n}=m a_{n}: \quad W=m a_{n} \quad m g=m a_{n} \quad a_{n}=g \\
& a_{n}=\frac{v_{D}^{2}}{r}: \quad v_{D}^{2}=r a_{n}=r g=(2 \mathrm{ft})\left(32.2 \mathrm{ft} / \mathrm{s}^{2}\right)=64.4 \mathrm{ft}^{2} / \mathrm{s}^{2}
\end{aligned}
$$

Position 1. Potential Energy. Denoting by $x$ the deflection of the spring and noting that $k=3 \mathrm{lb} / \mathrm{in}$. $=36 \mathrm{lb} / \mathrm{ft}$, we write

$$
V_{e}=\frac{1}{2} k x^{2}=\frac{1}{2}(36 \mathrm{lb} / \mathrm{ft}) x^{2}=18 x^{2}
$$

Choosing the datum at $A$, we have $V_{g}=0$; therefore

$$
V_{1}=V_{e}+V_{g}=18 x^{2}
$$



Kinetic Energy. Since the pellet is released from rest, $v_{A}=0$ and we have $T_{1}=0$.

Position 2. Potential Energy. The spring is now undeformed; thus $V_{e}=0$. Since the pellet is 4 ft above the datum, we have

$$
\begin{aligned}
& V_{g}=W y=(0.5 \mathrm{lb})(4 \mathrm{ft})=2 \mathrm{ft} \cdot \mathrm{lb} \\
& V_{2}=V_{e}+V_{g}=2 \mathrm{ft} \cdot \mathrm{lb}
\end{aligned}
$$

Kinetic Energy. Using the value of $v_{D}^{2}$ obtained above, we write

$$
T_{2}=\frac{1}{2} m v_{D}^{2}=\frac{1}{2} \frac{0.5 \mathrm{lb}}{32.2 \mathrm{ft} / \mathrm{s}^{2}}\left(64.4 \mathrm{ft}^{2} / \mathrm{s}^{2}\right)=0.5 \mathrm{ft} \cdot \mathrm{lb}
$$

Conservation of Energy. Applying the principle of conservation of energy between positions 1 and 2 , we write

$$
\begin{aligned}
T_{1}+V_{1} & =T_{2}+V_{2} \\
0+18 x^{2} & =0.5 \mathrm{ft} \cdot \mathrm{lb}+2 \mathrm{ft} \cdot \mathrm{lb} \\
x & =0.3727 \mathrm{ft}
\end{aligned}
$$

Example 3: A 4-lb collar can slide without friction along a horizontal rod and is released from rest at $A$. The undeformed lengths of springs BA and CA are 10 in . and 9 in ., respectively, and the constant of each spring is $2800 \mathrm{lb} / \mathrm{in}$. Determine the velocity of
 the collar when it has moved 1 in . to the right.

$$
\begin{gathered}
k=2800 \mathrm{lb} / \mathrm{in} .=33,600 \mathrm{lb} / \mathrm{ft} \\
T_{1}=0 \quad V_{1}=\frac{1}{2} k\left(\Delta J_{1}\right)^{2}=\frac{1}{2}(33,600)\left(\frac{1}{12}\right)^{2} \\
=116.667 \mathrm{ft} \cdot \mathrm{lb} \\
\ell_{1}=\sqrt{6^{2}+9^{2}}=10.817 \mathrm{in}=0.9014 \mathrm{ft} \\
S_{1}=\text { Stretch }=10.817-10=0.817 \mathrm{in}=0.06808 \mathrm{ft} \\
\ell_{2}=\sqrt{6^{2}+7^{2}}=9.2195 \mathrm{in} . \\
S_{2}=\text { Stretch }=9.2195-9=0.2195 \mathrm{in} .=0.018295 \mathrm{ft} \\
T_{2}=\frac{1}{2} m v_{2}^{2}=\frac{1}{2}\left(\frac{4}{32.2}\right) v_{2}^{2}=0.0621 v_{2}^{2} \\
V_{2}=\frac{1}{2}(33,600)\left(S_{1}^{2}+S_{2}^{2}\right) \\
=16800\left[(0.06808)^{2}+(0.018295)^{2}\right] \\
=83.489 \mathrm{ft} \cdot 1 \mathrm{~b} \\
T_{1}{ }^{0} V_{1}=T_{2}+V_{2} \\
\mathrm{~T}_{16} .667=0.06211 v_{2}^{2}+83.489
\end{gathered}
$$

$$
v_{2}=23.1 \mathrm{ft} / \mathrm{s}
$$

## Problems

Q1: A 3-lb collar C may slide without friction along a horizontal rod. It is attached to three springs, each of constant k $52 \mathrm{lb} / \mathrm{in}$. and 6-in. undeformed length. Knowing that the collar is released from rest in the position shown, determine the maximum speed it will reach in the ensuing motion

Q2: A 500-g collar can slide without friction on the curved rod BC in a horizontal plane. Knowing that the undeformed length of the spring is 80 mm and that k 5 $400 \mathrm{kN} / \mathrm{m}$, determine (a) the velocity that the collar should be given at $A$ to reach $B$ with zero velocity, (b) the velocity of the collar when it eventually reaches $C$.

Q3: A 1-lb collar is attached to a spring and slides without friction along a circular rod in a vertical plane. The spring has an undeformed length of 5 in . and a constant k 510 $\mathrm{lb} / \mathrm{ft}$. Knowing that the collar is released from being held at A determine the speed of the collar and the normal
 force between the collar and the rod as the collar passes through B.

Q4: A thin circular rod is supported in a vertical plane by a bracket at A. Attached to the bracket and loosely wound around the rod is a spring of constant k $53 \mathrm{lb} / \mathrm{ft}$ and undeformed length equal to the arc of circle AB. An 8-oz collar C, not attached to the spring, can slide without friction along the rod. Knowing that the collar is released from rest at an angle $u$ with the vertical, determine (a) the smallest value of $u$ for which the collar will pass through $D$ and reach point $A$, (b) the velocitv of the collar as it reaches point $A$.

Q5: A spring is used to stop a $50-\mathrm{kg}$ package which is moving down a $20^{\circ}$ incline. The spring has a constant k $530 \mathrm{kN} / \mathrm{m}$ and is held by cables so that it is initially compressed 50 mm . Knowing that the velocity of the package is $2 \mathrm{~m} / \mathrm{s}$ when it is 8 m from the spring and neglecting friction, determine the maximum additional deformation of the spring in bringing the package to rest.

Q6: A section of track for a roller coaster consists of two circular arcs $A B$ and $C D$ joined by a straight portion $B C$. The radius of $A B$ is 27 m and the radius of $C D$ is 72 m . The car and its occupants, of total mass 250 kg , reach point A with practically no velocity and then drop freely along the track. Determine the normal force exerted by the track on the car as the car reaches point B. Ignore air resistance and rolling resistance.


## 6- Impact

A collision between two bodies which occur in a very small interval of time and during which the two bodies exert relatively large forces on each other is called an impact. The common normal to the surfaces in contact during the impact is called the line of impact. If the mass centers on the two colliding bodies are located on this line, the impact is a central impact. Otherwise, the impact is said to be eccentric

(a) Direct central impact

(b) Oblique central impact

## A- Direct Central Impact

## Step of solution


(a) Before impact

(b) At maximum deformation
e: the coefficient of restitution

(c) After impact

Two particular cases of impact are of special interest:

1. $\mathrm{e}=0$, Perfectly Plastic Impact.

$$
\begin{aligned}
& v_{B}^{\prime}=v_{A}^{\prime}=v^{-} \\
& m_{A} v_{A}+m_{B} v_{B} \\
&=\left(m_{A}+m_{B}\right) v^{\prime}
\end{aligned}
$$

$\mathrm{e}=1$, Perfectly Elastic Impact

$$
v_{B}^{\prime}-v_{A}^{\prime}=v_{A}-v_{B}
$$

## B- Oblique Central Impact



## Step of solution

1. The component along the $t$ axis of the momentum of each particle, considered separately, is conserved; hence the $t$ component of the velocity of each particle remains unchanged. We write

$$
\begin{equation*}
\left(v_{A}\right)_{t}=\left(v_{A}^{\prime}\right)_{t} \quad\left(v_{B}\right)_{t}=\left(v_{B}^{\prime}\right)_{t} \tag{13.47}
\end{equation*}
$$

2. The component along the $n$ axis of the total momentum of the two particles is conserved. We write

$$
\begin{equation*}
m_{A}\left(v_{A}\right)_{n}+m_{B}\left(v_{B}\right)_{n}=m_{A}\left(v_{A}^{\prime}\right)_{n}+m_{B}\left(v_{B}^{\prime}\right)_{n} \tag{13.48}
\end{equation*}
$$

3. The component along the $n$ axis of the relative velocity of the two particles after impact is obtained by multiplying the $n$ component of their relative velocity before impact by the coefficient of restitution. Indeed, a derivation similar to that given in Sec. 13.13 for direct central impact yields

$$
\begin{equation*}
\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n}=e\left[\left(v_{A}\right)_{n}-\left(v_{B}\right)_{n}\right] \tag{13.4}
\end{equation*}
$$

Example-1: A $20-\mathrm{Mg}$ railroad car moving at a speed of $0.5 \mathrm{~m} / \mathrm{s}$ to the right collides with a $35-\mathrm{Mg}$ car which is at rest. If after the collision the $35-\mathrm{Mg}$ car is observed to move to the right at a speed of $0.3 \mathrm{~m} / \mathrm{s}$, determine the coefficient of restitution between the two cars.


$$
m_{A} \mathbf{v}_{A}+m_{B} \mathbf{v}_{B}=m_{A} \mathbf{v}_{A}^{\prime}+m_{B} \mathbf{v}_{B}^{\prime}
$$

$$
\begin{gathered}
(20 \mathrm{Mg})(+0.5 \mathrm{~m} / \mathrm{s})+(35 \mathrm{Mg})(0)=(20 \mathrm{Mg}) v_{\mathrm{A}}^{\prime}+(35 \mathrm{Mg})(+0.3 \mathrm{~m} / \mathrm{s}) \\
v_{\mathrm{A}}^{\prime}=-0.025 \mathrm{~m} / \mathrm{s} \quad \mathbf{v}_{\mathrm{A}}^{\prime}=0.025 \mathrm{~m} / \mathrm{s} \leftarrow
\end{gathered}
$$

The coefficient of restitution is obtained by writing

$$
e=\frac{v_{B}^{\prime}-v_{A}^{\prime}}{v_{A}-v_{B}}=\frac{+0.3-(-0.025)}{+0.5-0}=\frac{0.325}{0.5} \quad e=0.65
$$

Example-2: The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming e=0.90, determine the magnitude and direction of the velocity of each ball after the impact.

$$
\begin{aligned}
\left(v_{A}\right)_{n} & =v_{A} \cos 30^{\circ}=+26.0 \mathrm{ft} / \mathrm{s} \\
\left(v_{A}\right)_{t} & =v_{A} \sin 30^{\circ}=+15.0 \mathrm{ft} / \mathrm{s} \\
\left(v_{B}\right)_{n} & =-v_{B} \cos 60^{\circ}=-20.0 \mathrm{ft} / \mathrm{s} \\
\left(v_{B}\right)_{t} & =v_{B} \sin 60^{\circ}=+34.6 \mathrm{ft} / \mathrm{s}
\end{aligned}
$$



$$
\left(\mathbf{v}_{A}^{\prime}\right)_{t}=15.0 \mathrm{ft} / \mathrm{s} \mathrm{x} \quad\left(\mathbf{v}_{B}^{\prime}\right)_{t}=34.6 \mathrm{ft} / \mathrm{s} \mathrm{x}
$$

Using the relation (13.49) between relative velocities, we write

$$
\begin{align*}
\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n}= & e\left[\left(v_{A}\right)_{n}-\left(v_{B}\right)_{n}\right] \\
\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n}= & (0.90)[26.0-(-20.0)] \\
& \quad\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n}=41.4 \tag{2}
\end{align*}
$$

Solving Eqs. (1) and (2) simultaneously, we obtain

$$
\begin{array}{ll}
\left(v_{A}^{\prime}\right)_{n}=-17.7 & \left(v_{B}^{\prime}\right)_{n}=+23.7 \\
\left(\mathbf{v}_{A}^{\prime}\right)_{n}=17.7 \mathrm{ft} / \mathrm{s} \mathrm{z} & \left(\mathbf{v}_{B}^{\prime}\right)_{n}=23.7 \mathrm{ft} / \mathrm{s} \quad \mathrm{y}
\end{array}
$$

Resultant Motion. Adding vectorially the velocity components of each ball, we obtain

$$
\mathbf{v}_{A}^{\prime}=23.2 \mathrm{ft} / \mathrm{s} \mathrm{~b} 40.3^{\circ} \quad \mathbf{v}_{B}^{\prime}=41.9 \mathrm{ft} / \mathrm{s} \text { a } 55.6^{\circ}
$$



Example-3: A 600-g ball $A$ is moving with a velocity of magnitude $6 \mathrm{~m} / \mathrm{s}$ when it is hit as shown by a 1kg ball B which has a velocity of magnitude $4 \mathrm{~m} / \mathrm{s}$. Knowing that the coefficient of restitution is 0.8 and assuming no friction, determine the velocity of each ball after impact.



Before
After

$$
\begin{aligned}
v_{A} & =6 \mathrm{~m} / \mathrm{s} \\
\left(v_{A}\right)_{n} & =(6)\left(\cos 40^{\circ}\right)=4.596 \mathrm{~m} / \mathrm{s} \\
\left(v_{A}\right)_{t} & =-6\left(\sin 40^{\circ}\right)=-3.857 \mathrm{~m} / \mathrm{s} \\
v_{B} & =\left(v_{B}\right)_{n}=-4 \mathrm{~m} / \mathrm{s} \\
\left(v_{B}\right)_{t} & =0
\end{aligned}
$$

$t$-direction:
Total momentum conserved:

$$
\begin{align*}
m_{A}\left(v_{A}\right)_{t}+m_{B}\left(v_{B}\right)_{t} & =m_{A}\left(v_{B}^{\prime}\right)_{t}+m_{B}\left(v_{B}^{\prime}\right)_{t} \\
(0.6 \mathrm{~kg})(-3.857 \mathrm{~m} / \mathrm{s})+0 & =(0.6 \mathrm{~kg})\left(v_{A}^{\prime}\right)_{t}+(1 \mathrm{~kg})\left(v_{B}^{\prime}\right) t \\
-2.314 \mathrm{~m} / \mathrm{s} & =0.6\left(v_{A}^{\prime}\right)_{t}+\left(v_{B}^{\prime}\right)_{t} \tag{1}
\end{align*}
$$

Ball $A$ alone:
Momentum conserved:

$$
\begin{align*}
m_{A}\left(v_{A}\right)_{t} & =m_{A}\left(v_{A}^{\prime}\right)_{t} \quad-3.857=\left(v_{A}^{\prime}\right)_{t} \\
\left(v_{A}^{\prime}\right)_{t} & =-3.857 \mathrm{~m} / \mathrm{s} \tag{2}
\end{align*}
$$

Replacing $\left(v_{A}^{\prime}\right)_{t}$ in (2) in Eq. (1)

$$
\begin{aligned}
-2.314 & =(0.6)(-3.857)+\left(v_{B}^{\prime}\right)_{t} \\
-2.314 & =-2.314+\left(v_{B}^{\prime}\right)_{t} \\
\left(v_{B}^{\prime}\right)_{t} & =0
\end{aligned}
$$

$n$-direction:
Relative velocities:

$$
\begin{align*}
{\left[\left(v_{A}\right)_{n}-\left(v_{B}\right)_{n}\right] e } & =\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n} \\
{[(4.596)-(-4)](0.8) } & =\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n} \\
6.877 & =\left(v_{B}^{\prime}\right)_{n}-\left(v_{A}^{\prime}\right)_{n} \tag{3}
\end{align*}
$$

Total momentum conserved:

$$
\begin{align*}
m_{A}\left(v_{A}\right)_{n}+m_{B}\left(v_{B}\right)_{n} & =m_{A}\left(v_{A}^{\prime}\right)_{n}+m_{B}\left(v_{B}^{\prime}\right)_{n} \\
(0.6 \mathrm{~kg})(4.596 \mathrm{~m} / \mathrm{s})+(1 \mathrm{~kg})(-4 \mathrm{~m} / \mathrm{s}) & =(1 \mathrm{~kg})\left(v_{B}^{\prime}\right)_{n}+(0.6 \mathrm{~kg})\left(v_{A}^{\prime}\right)_{n} \\
-1.2424 & =\left(v_{B}^{\prime}\right)_{n}+0.6\left(v_{A}^{\prime}\right)_{n} \tag{4}
\end{align*}
$$

Solving Eqs. (4) and (3) simultaneously,

$$
\begin{aligned}
& \left(v_{A}^{\prime}\right)_{n}=5.075 \mathrm{~m} / \mathrm{s} \\
& \left(v_{B}^{\prime}\right)_{n}=1.802 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$

Velocity of $A$ :

$$
\begin{aligned}
\tan \beta & =\frac{\left|\left(v_{A}\right)_{t}\right|}{\left|\left(v_{A}\right)_{n}\right|} \\
& =\frac{3.857}{5.075} \\
\beta & =37.2^{\circ} \quad \beta+40^{\circ}=77.2^{\circ} \\
v_{A}^{\prime} & =\sqrt{(3.857)^{2}+(5.075)^{2}} \\
& =6.37 \mathrm{~m} / \mathrm{s}
\end{aligned}
$$



$$
\begin{aligned}
& \mathbf{v}_{A}^{\prime}=6.37 \mathrm{~m} / \mathrm{s} \nearrow 77.2^{\circ} \\
& \mathbf{v}_{B}^{\prime}=1.802 \mathrm{~m} / \mathrm{s} \measuredangle 40^{\circ}
\end{aligned}
$$

## Problems

Q1: Packages in an automobile parts supply house are transported to the loading dock by pushing them along on a roller track with very little friction. At the instant shown, packages $B$ and $C$ are at rest and package $A$ has a velocity of $2 \mathrm{~m} / \mathrm{s}$. Knowing that the coefficient of restitution between the packages is 0.3, determine (a) the velocity of package $C$ after $A$ hits $B$ and $B$ hits $C$, (b) the velocity of $A$ after it hits $B$ for the second time

Q2: Two identical billiard balls can move freely on a horizontal table. Ball A has a velocity v0 as shown and hits ball $B$, which is at rest, at a Point $C$ defined by $\Theta=45^{\circ}$. Knowing that the coefficient of restitution between the two balls is $\mathrm{e}=0.8$ and assuming no friction, determine the velocity of each ball after impact.

Q3: The coefficient of restitution between the two collars
 is known to be 0.70 . Determine (a) their velocities after impact, (b) the energy loss during impact.

Q4: To apply shock loading to an artillery shell, a $20-\mathrm{kg}$ pendulum A is released from a known height and strikes impactor B at a known velocity v0. Impactor B then strikes the $1-\mathrm{kg}$ artillery shell C . Knowing the coefficient of restitution between all objects is e, determine the mass of B to maximize the impulse applied to the artillery shell C.


Q5: Two identical cars $A$ and $B$ are at rest on a loading dock with brakes released. Car C, of a slightly different style but of the same weight, has been pushed by dockworkers and hits car $B$ with a velocity of $1.5 \mathrm{~m} / \mathrm{s}$. Knowing that the coefficient of restitution is 0.8 between $B$ and $C$ and 0.5 between $A$ and $B$, determine the velocity of each car after all collisions have taken place.


Q6: The coefficient of restitution is 0.9 between the two 2.37-in.- diameter billiard balls A and B. Ball $A$ is moving in the direction shown with a velocity of $3 \mathrm{ft} / \mathrm{s}$ when it strikes ball $B$, which is at rest. Knowing that after impact $B$ is moving in the $x$ direction, determine (a) the angle $u$, (b) the velocity of $B$ after impact


Q7: A 600-g ball A is moving with a velocity of magnitude $6 \mathrm{~m} / \mathrm{s}$ when it is hit as shown by a 1-kg ball B which has a velocity of magnitude $4 \mathrm{~m} / \mathrm{s}$. Knowing that the coefficient of restitution is 0.8 and assuming no friction, determine the velocity of each ball after impact.


Q8: Two identical hockey pucks are moving on a hockey rink at the same speed of $3 \mathrm{~m} / \mathrm{s}$ and in perpendicular directions when they strike each other as shown. Assuming a coefficient of restitution $\mathrm{e}=0.9$, determine the magnitude and direction of the velocity of each puck after impact.


