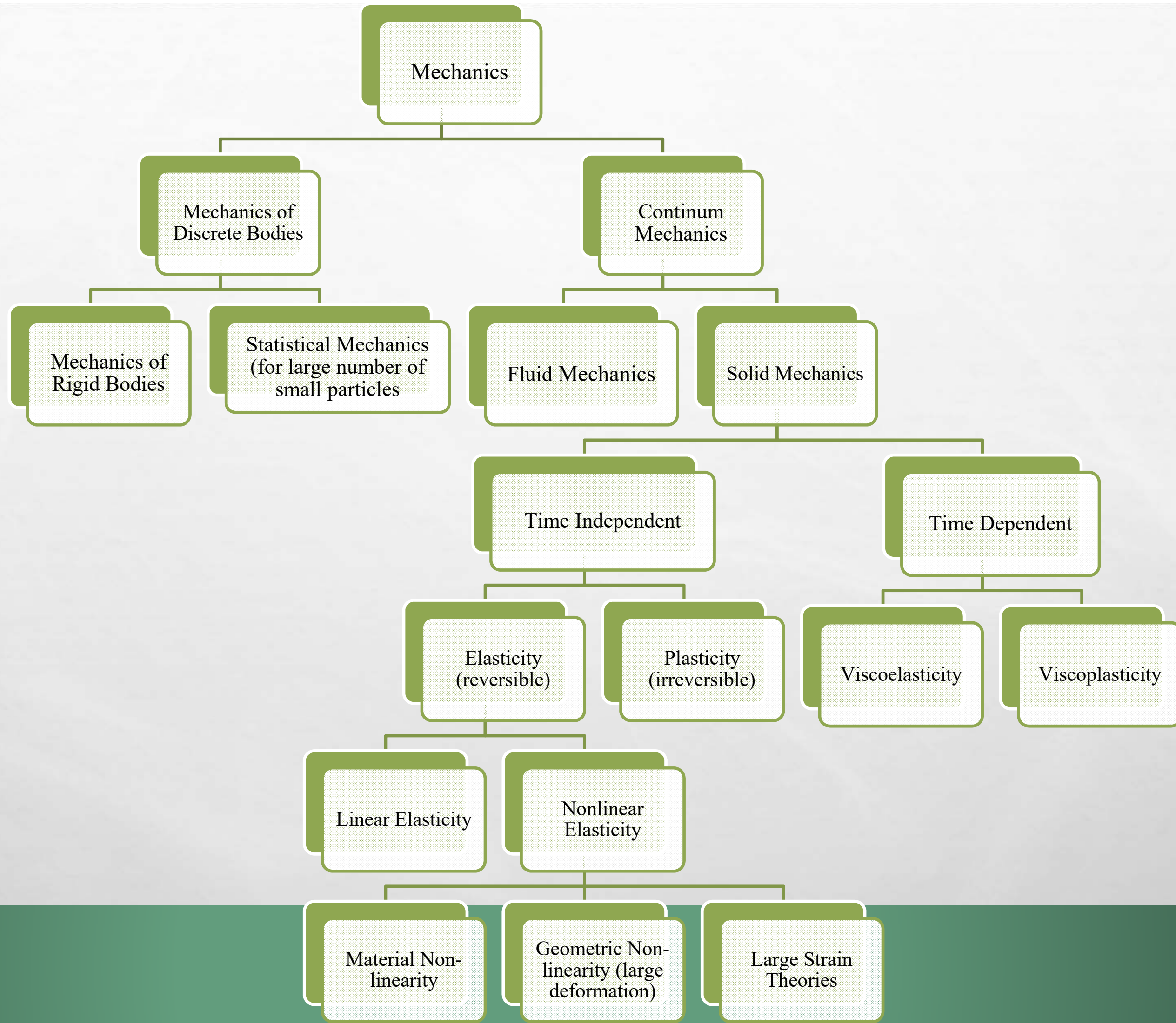
SOLID MECHANICS


Mechanics
Mechanics of Discrete Bodies
Continuum Mechanics
Mechanics of Rigid Bodies
Statistical Mechanics (for large number of small particles
Fluid Mechanics
Solid Mechanics
Time Independent
Time Dependent
Elasticity (reversible)
Plasticity (irreversible)
Viscoelasticity
Viscoplasticity
Linear Elasticity
Nonlinear Elasticity
Material Non-linearity
Geometric Non-linearity (large deformation)
Large Strain Theories

Mechanics of Solids (Theory of Elasticity)

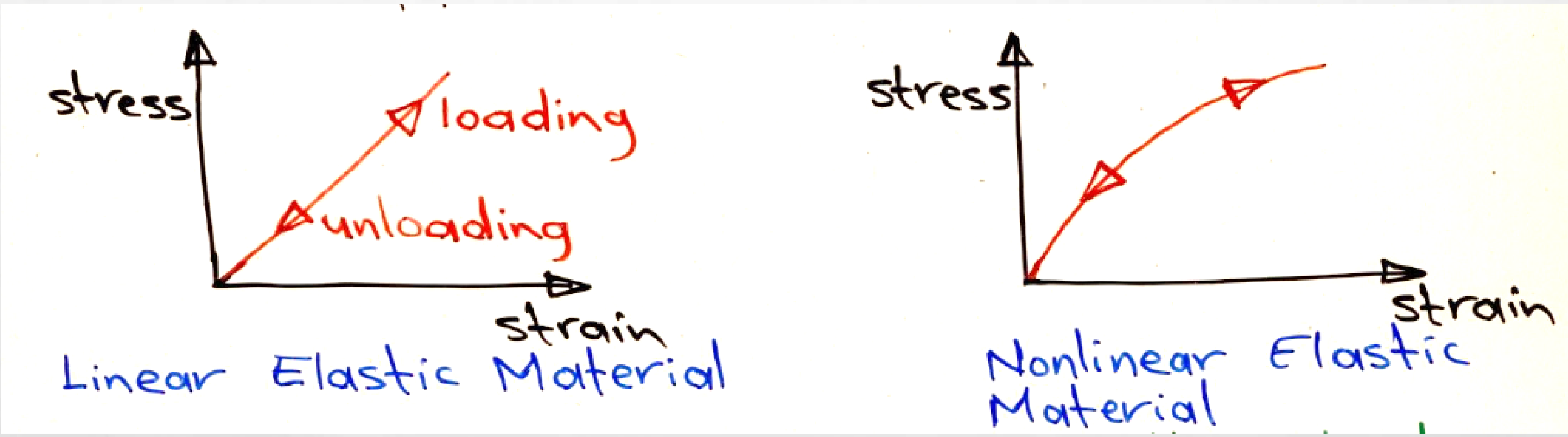
References:

1. **Theory of Elasticity by S.P. Timoshenko and J.N. Goodier**
2. **Advanced Strength and Applied Elasticity by A.C. Ugural and S.K. Fenster**
3. **Advanced Solid Mechanics by P.R. Lancaster and D. Mitchell**

Definitions:

Elasticity:

The material returns to its original (unloaded) shape upon the removal of applied forces.



The graphs follow the same line whether loading or unloading

Homogeneous Material: properties do not vary with location.

Isotropic Material: properties are identical in all directions at a point.

External Forces:

there are two kinds

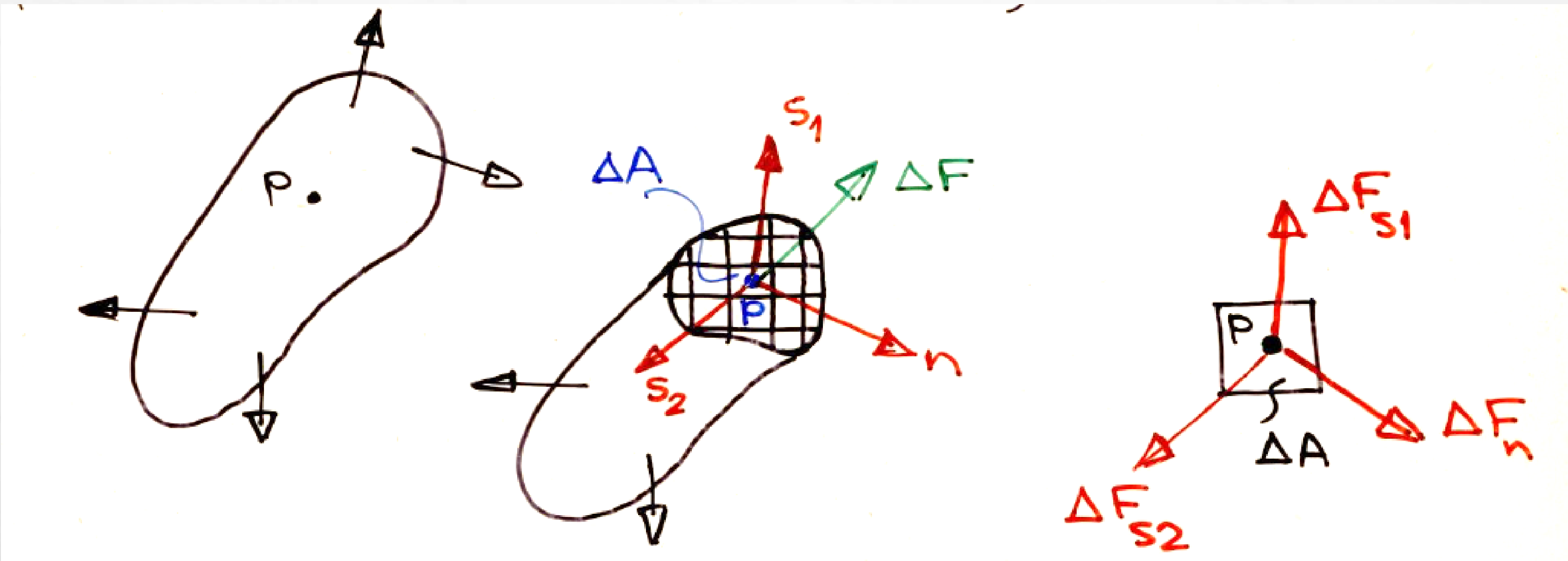
a) **Surface Forces:** Forces distributed over the surface of the body. (Atmospheric pressure, Hydraulic pressure)

b) **Body Forces:** Forces distributed over the volume of the body. (Gravitational forces, Centrifugal forces, Inertia forces)

Note : A body responds to the application of external forces by deforming and by developing internal forces.

Stresses:

Stress is a measure of the internal forces per unit area within a body.



ΔF is the force acting on an element of area ΔA.

n, s_1 and s_2 constitute a set of orthogonal axes, origin placed at the point P, with n normal and s_1, s_2 tangent to ΔA.

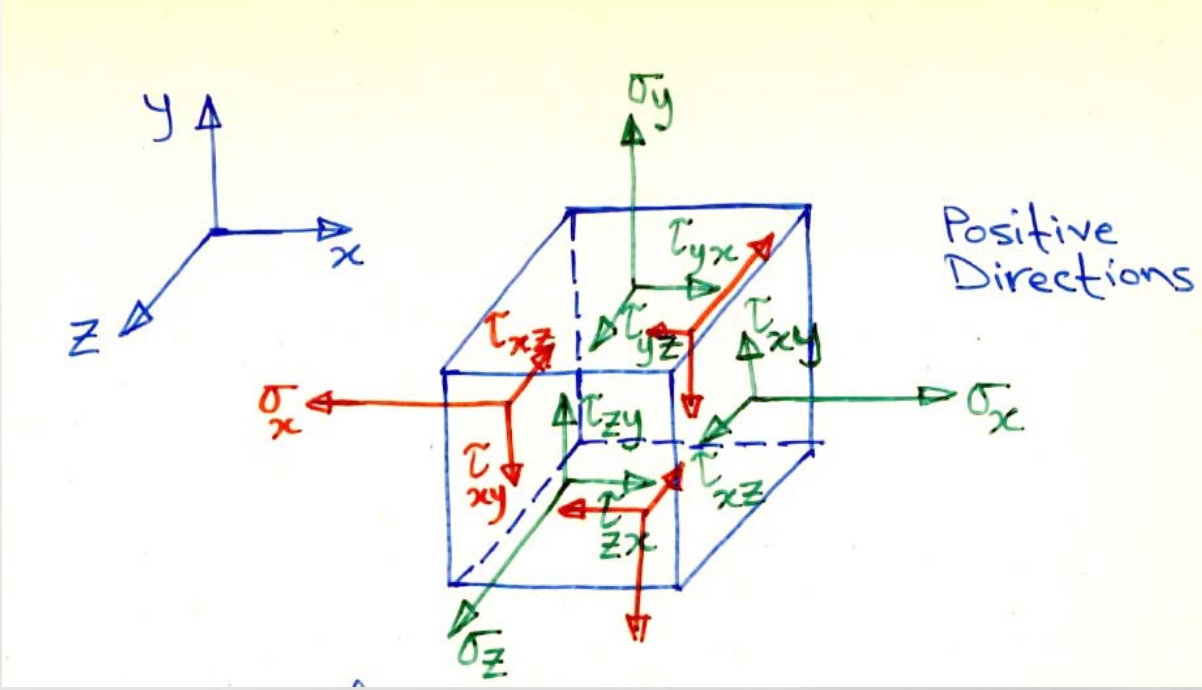
Decomposition of ΔF in to components parallel to n, s_1 and s_2, then the normal stress σ_n and the shear stresses are given by:

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$

$$\tau_{s1} = \lim_{\Delta A \to 0} \frac{\Delta F_{s1}}{\Delta A}$$

$$\tau_{s2} = \lim_{\Delta A \to 0} \frac{\Delta F_{s2}}{\Delta A}$$

A Set of stresses on an infinite number of planes passing through a point forms the state of the stress at the point. In order to define the state of stress at a point an elementary volume in the form of a right parallelepiped in the vicinity of the point in question is isolated.

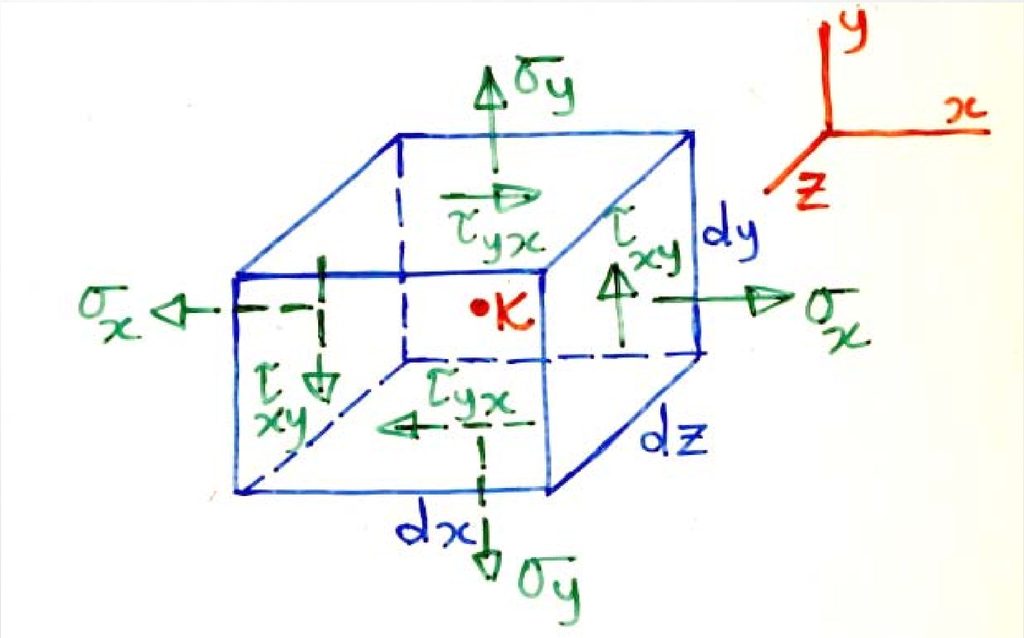


The equilibrium of moments in xy, yz and zx planes yield

$\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{zx} = \tau_{xz}$

For example:

Considering xy-plane



$\sum M_k = 0$

$\therefore (\tau_{xy} . d_y . d_z). d_x - (\tau_{yx} . d_x . d_z). d_y = 0$

$\therefore \tau_{xy} = \tau_{yx}$

Note:

On two planes at right angles to each other the components of shearing stresses perpendicular to the common edge are equal and directed either both toward the edge or both away from the edge.

Relationships Between Stress and Strain
(Generalized Hooke's Law)

Hooke's Law:

For an elastic material the strain produced is proportional to the applied stress.

For a linear elastic material the principle of superposition applies.

The effect of normal stress is to produce normal strains. The normal strains are unaffected by the shear stresses but shear strains are produced by shear stresses.

The compete stress-strain relationships are:

$$\epsilon_x = \frac{1}{\epsilon}[\sigma_x - \boldsymbol{\upsilon}(\sigma_y + \sigma_z)] \qquad \epsilon_y = \frac{1}{\epsilon}[\sigma_y - \boldsymbol{\upsilon}(\sigma_x + \sigma_z)] \qquad \epsilon_z = \frac{1}{\epsilon}[\sigma_z - \boldsymbol{\upsilon}(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

Relationships Between Stress and Strain
(Generalized Hooke's Law)

Where ϵ **= Young's modulus of elasticity**

υ **= Poisson's ratio**

G = Modulus of rigidity

The elastic constants ϵ**,** υ**, G are related by :**

$$G = \frac{\epsilon}{2\,(1+\upsilon)}$$

Strain and Strain Displacement Relations (Cauchy's Equations)

Strain is a measure of the change in shape of a body.

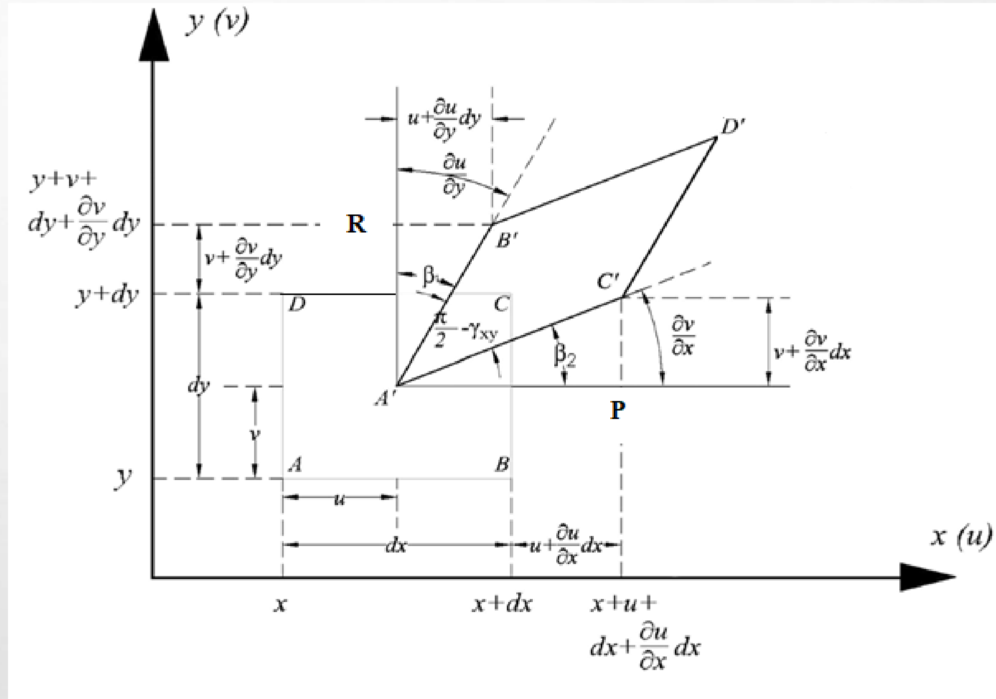
For simplicity, consider a two dimensional case :

ABCD is the element before deformation.

$A'B'C'D'$ is the element after deformation.

u= the displacement in the X-direction

v= the displacement in the Y-direction



Physically, small displacement means A'P $\approx A'C'$ and $\measuredangle$ P $A'C'$ is also small (i.e. , tan $\theta \approx \theta$)

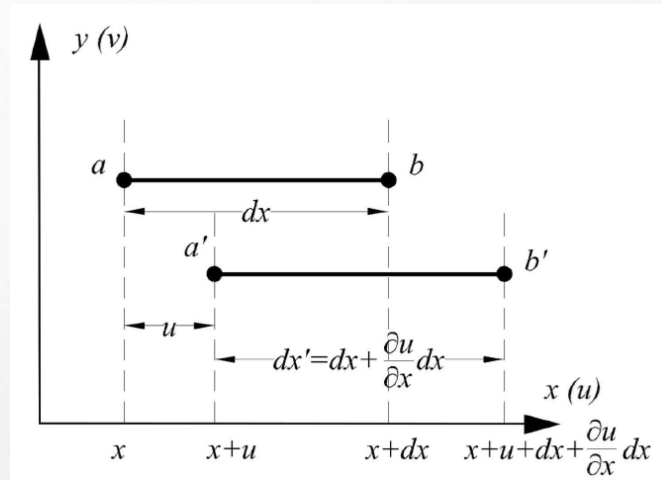
Point	x-coordinate	y-coordinate	z-coordinate
A	x	y	z
B	$x + dx$	$y + dy$	$z + dz$
A'	x' or $x + u$	y' or $y + v$	z' or $z + w$
B'	$x' + dx'$ or $x + u + dx + du$	$y' + dy'$ or $y + v + dy + dv$	$z' + dz'$ or $z + w + dz + dw$

The original length of element, which is also the distance between A and B, say dS can be expressed through Pythagoras theorem:

$$dS^2 = AB^2$$

$$dS^2 = [(x + dx) - x]^2 + [(y + dy) - y]^2 + [(z + dz) - z]^2$$

The normal (linear) strain is defined as the ratio of change in length to the initial length. The normal strain in X-direction ϵ_x is



$$\epsilon_x = \frac{A'C'}{AC} - 1 \approx \frac{A'P}{AC} - 1 = \frac{\mathbf{d}_x - u + u + \frac{\partial u}{\partial x}\cdot \mathbf{d}_x}{\mathbf{d}_x} - \mathbf{1}$$

$$= 1 + \frac{\partial u}{\partial x} - 1 = \frac{\partial u}{\partial x}$$

Similarly $\quad \boldsymbol{\epsilon_y} = \frac{\partial v}{\partial y}$

Shear strain is defined as the change in angle between two lines originally at right angles. The shear strain is $(\measuredangle RA'B' + \measuredangle PA'C')$

Angle $PA'C'$= **arctan** $\dfrac{\frac{\partial v}{\partial x}\cdot \mathbf{d}_x}{A'p}$ = **arctan** $\dfrac{\frac{\partial v}{\partial x}\mathbf{d}_x}{\mathbf{d}_x + \frac{\partial u}{\partial x} d_x}$

$$= \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} = \frac{\partial v}{\partial x} \qquad \textbf{since } \frac{\partial u}{\partial x} \ll \mathbf{1}$$

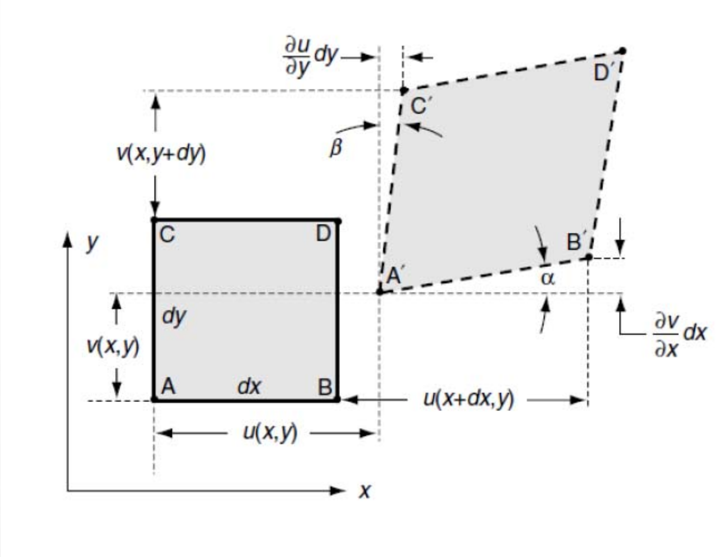
Similarly

Angle $RA'B' = \frac{\partial u}{\partial y}$

So

Shear strain $\gamma_{xy} = \measuredangle PA'B' + \measuredangle RA'B'$

$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ **(Engineering shear strain)**



Mathematical shear strain is equal to half engineering shear strain.

In the case of three dimensions, w **is the displacement in the Z- direction, then**

$\epsilon_x = \frac{\partial u}{\partial x}$, $\epsilon_y = \frac{\partial v}{\partial y}$, $\epsilon_z = \frac{\partial w}{\partial z}$,

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$, $\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$ **(these relationships are called Cauchy's equations)**

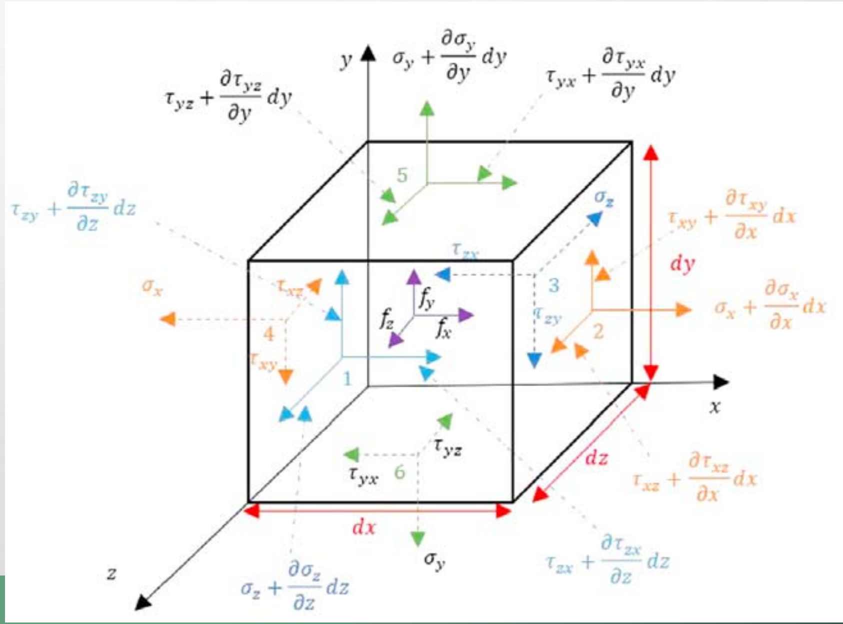
Equation of Equilibrium (Navier's Equation)

Consider a small parallelepiped with sides of lengths d_x d_y and d_z .

Consider the equilibrium of forces in the X-direction

$$X.\, d_x \,.\, d_y \,.\, d_z - \sigma_x \,.\, d_y \,.\, d_z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} d_x\right) d_y \,.\, d_z - \tau_{yx} \,.\, d_z \,.\, d_x + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \,.\, d_y\right) d_x \,.\, d_z - \tau_{zx} \,.\, d_y \,.\, d_x + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} .\, d_z\right) d_y \,.\, d_x = 0$$

Where X is the X component of the body forces per unit volume.



Canceling (d_x . d_y . d_z)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Similarly

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Equation of Equilibrium (Navier's Equation)

Here Y and Z are the y and z components of the body forces per unit volume.

Equations 1 to 3 are Navier's Equation of equilibrium for an elastic solid.

Compatibility Equations

(Saint-Venant's Equations)

If u (x,y,z), v (x,y,z), and w (x,y,z) are given, then the whole six strain components can be derived by differentiation. Compatibility is satisfied automatically. If the six strains ϵ_x (x,y,z), ϵ_y (x,y,z), ϵ_z (x,y,z), γ_{xy}(x,y,z), γ_{yz}(x,y,z), and γ_{zx}(x,y,z) are given, the three displacement components u (x,y,z), v (x,y,z), and w (x,y,z) can be determined uniquely if and only if relations exist among the strains.

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \therefore \ \frac{\partial}{\partial y}(\epsilon_x) = \frac{\partial^2 u}{\partial y \partial x} \qquad \therefore \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\epsilon_x\right) = \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x}$$

Similarly $\dfrac{\partial^2 \epsilon_y}{\partial x^2} = \dfrac{\partial^3 v}{\partial x^2 \partial y}$

But since $\dfrac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \dfrac{\partial^3 u}{\partial x\, \partial y^2} + \dfrac{\partial^3 v}{\partial y\, \partial x^2}$

It follows that $\dfrac{\partial^2 \epsilon_x}{\partial y^2} + \dfrac{\partial^2 \epsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x\, \partial y}$ **………………… (1)**

Similarly $$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

$$\frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z\, \partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}$$

Also from Cauchy's equations

$$\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} \qquad , \frac{\partial}{\partial x} \gamma_{yz} = \frac{\partial^2 v}{\partial x\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y}$$

$$\frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y} \qquad , \frac{\partial}{\partial z} \gamma_{xy} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial z}$$

$$\therefore \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial z} - \frac{\partial^2 \gamma_{yz}}{\partial x^2} + \frac{\partial^2 \gamma_{zx}}{\partial x\, \partial y} = 2\, \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 2\, \frac{\partial^2 \epsilon_x}{\partial y\, \partial z}$$

Or $$\frac{\partial}{\partial x} \left[\frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} \right] = 2\, \frac{\partial^2 \epsilon_x}{\partial y\, \partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(4)}$$

Similarly $$\frac{\partial}{\partial y} \left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right] = 2\, \frac{\partial^2 \epsilon_y}{\partial z\, \partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(5)}$$

$$\frac{\partial}{\partial z} \left[\frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} \right] = 2\, \frac{\partial^2 \epsilon_z}{\partial x\, \partial y} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(6)}$$

In terms of stresses, if there are no body forces or if the body forces are constants, the compatibility equation 1 to 6 become

$$\Delta\, \sigma_x + \frac{1}{1+v}\frac{\partial^2\theta}{\partial x^2} = 0$$

$$\Delta\, \sigma_y + \frac{1}{1+v}\frac{\partial^2\theta}{\partial y^2} = 0$$

$$\Delta\, \sigma_z + \frac{1}{1+v}\frac{\partial^2\theta}{\partial z^2} = 0$$

$$\Delta\, \tau_{yz} + \frac{1}{1+v}\frac{\partial^2\theta}{\partial y\, \partial z} = 0$$

$$\Delta\, \tau_{zx} + \frac{1}{1+v}\frac{\partial^2\theta}{\partial z\, \partial x} = 0$$

$$\Delta\, \tau_{xy} + \frac{1}{1+v}\frac{\partial^2\theta}{\partial x\, \partial y} = 0$$

Where $\theta = \sigma_x + \sigma_y + \sigma_z$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ **(Laplacian operator)**

Two – Dimensional Elasticity

There are two cases : Plane stress and plane strain problems.

Plane stress

One has a state of plane stress when the stresses satisfy the following conditions:

$\sigma_z = \tau_{xz} = \tau_{yz} = 0$ …………..……………. (1)

In this case the equations of equilibrium become:

$$\left.\begin{aligned}&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0\\ &\text{and}\\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0\end{aligned}\right\} \quad \text{………………..} \ (2)$$

The compatibility equations reduce to

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} \quad \text{..........................(3)}$$

The strain – stress relations are

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}(\sigma_x - v\,\sigma_y) \\ \epsilon_y &= \frac{1}{E}(\sigma_y - v\,\sigma_x) \\ \epsilon_z &= \frac{-v}{E}(\sigma_x + \sigma_y) \end{aligned}\right\} \quad \text{........................(4)}$$

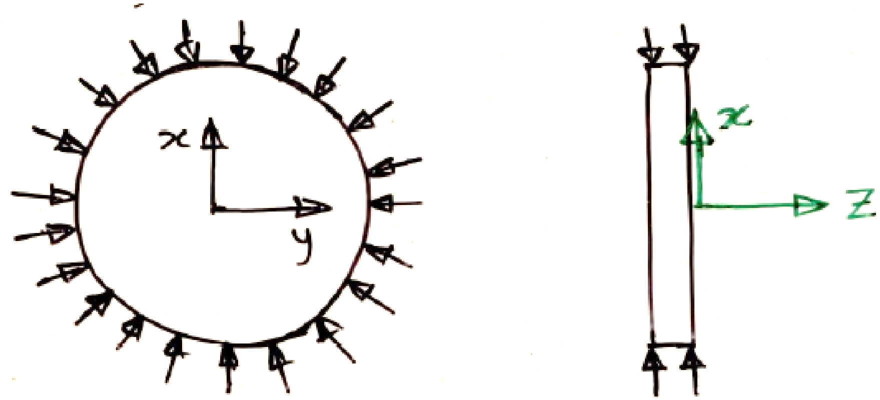
$$\left.\begin{aligned} \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \gamma_{yz} = 0 \end{aligned}\right\} \quad \text{..........................(5)}$$

Substituting the first two of Eqs.(4) and the first of Eqs.(5) into Eq.(3)

$$\therefore \frac{\partial^2}{\partial x^2}(\sigma_y - v\,\sigma_x) + \frac{\partial^2}{\partial y^2}(\sigma_x - v\,\sigma_y) = 2(1+v)\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} \quad \ldots(6)$$

Differentiating the first of Eqs. (2) w.r.t. x, the second w.r.t. y, and adding them together,

$$\therefore \frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} = -\frac{1}{2}\left[\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$



Substituting Eq. (7) into Eq. (6)

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -(1+v)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \ldots\ldots\ldots\ldots\ldots(8)$$

The case of plane stress represents, with only a very small error, the state of stress in a thin plate which is subjected to forces applied at the boundary, parallel to the plane of the plate, and uniformly distributed over the thickness.

Plane strain

One has the state of plane strain when the following conditions are satisfied

$$\left.\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \\ \epsilon_z &= \frac{\partial w}{\partial z} = 0 \end{aligned}\right\} \quad \text{.....................(1)}$$

Equilibrium equations become:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \end{aligned}\right\} \quad \text{.................... (2)}$$

(the same as those for the plane stress case)

The stress – strain relations are

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}\,[\sigma_x - v\,(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}\,[\sigma_y - v\,(\sigma_x + \sigma_z)] \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned}\right\} \text{.............. (3)}$$

Since $\epsilon_z = \frac{1}{E}\,[\sigma_z - v\,(\sigma_x + \sigma_y)]$

$\therefore\ \sigma_z = v\,(\sigma_x + \sigma_y)$

Thus Eqs. (3) become

$$\left.\begin{aligned} \epsilon_x &= \frac{1+v}{E}\,[\sigma_x\,(1-v) - v\,\sigma_y] \\ \epsilon_y &= \frac{1+v}{E}\,[\sigma_y\,(1-v) - v\,\sigma_x] \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \end{aligned}\right\} \text{.............. (4)}$$

The compatibility Eq. is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \quad \text{.....................(5)}$$

Using Eqs. (4) and (5)

$$\therefore \frac{\partial^2}{\partial y^2} [(1-v)\, \sigma_x - v\, \sigma_y] + \frac{\partial^2}{\partial x^2} [(1-v)\, \sigma_y - v\, \sigma_x] = 2\, \frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} \quad \text{.................. (6)}$$

Differentiating the first and second of Eqs.(2) w.r.t. x and y respectively, and adding:

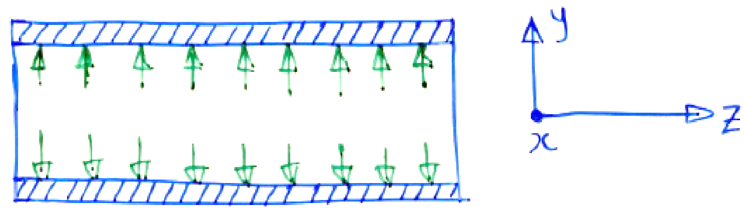
$$2\, \frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} = - \left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) - \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \text{.................. (7)}$$

Substituting Eq.(7) into Eq.(6)

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (\sigma_x + \sigma_y) = - \frac{1}{1-v} \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \text{............(8)}$$

In the absence of body forces or in the case of constant body forces, the compatibility equations for plane strain and plane stress are the same, i.e.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

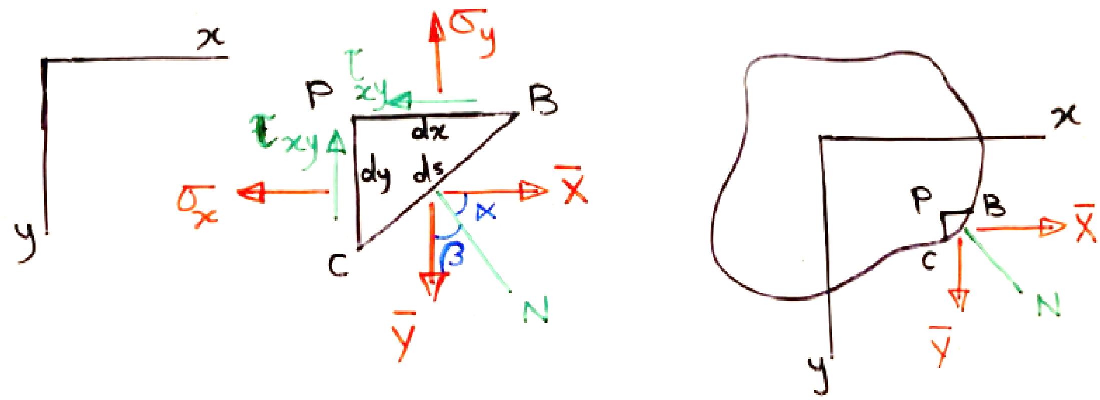


The case of plane strain is very closely approximated in the case of a long cylinder subjected to internal pressure.

The external forces are functions of the x and y coordinates. As a consequence all cross sections experience identical deformation.

At every cross section w=0

Boundary Conditions



Taking the small triangular prism PBC so that the side BC coincides with the boundary of the body, and denoting by $\bar{x}$ and $\bar{y}$ the components of the surface forces by unit area, then

$$\bar{x}\,(1^*\,d_s) = \sigma_x\,(1^*\,d_y) + \tau_{xy}\,(1^*\,d_x)$$

$$\therefore \bar{x} = \sigma_x \frac{d_y}{d_s} + \tau_{xy} \frac{d_x}{d_s} = \sigma_x\,.\cos\alpha + \tau_{xy}\,.\cos\beta$$

$$\therefore \bar{x} = \sigma_x\,\ell + \tau_{xy}\,m$$

Where $\ell = \cos\alpha$, $m = \cos\beta$

Similarly

$$\bar{y} = m\,\sigma_y + \ell\,\tau_{xy}$$

Here ℓ and m are the direction cosines of the normal N to the boundary.

Saint – Venant Principle

If there is a disturbance in one locality of a stable field, then this disturbance will not spread far from its locality when the resultant of disturbance is zero. In elasticity, if the applied force or moment does not produce stresses according to the laws of stress distributions, then this disturbance is usually localized.

Stress Function

The solution of two dimensional problems in elasticity requires integration of the differential equations of equilibrium together with the compatibility equation and the boundary conditions. For no body forces

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0$$

$$\bar{x} = \ell\, \sigma_x + m\, \tau_{xy}\,, \qquad \bar{y} = \ell\, \tau_{xy} + m\, \sigma_y$$

These equations are the same for both plane stress and plane strain problems.

The equations of equilibrium are identically satisfied by a new function ϕ (x,y), called the stress function, introduced by G.B. Airy, related to stresses as :

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad , \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad , \tau_{xy} = \frac{\partial^2 \phi}{\partial x \, \partial y}$$

Substituting the above expressions into the compatibility equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Or $\nabla^4 \phi = 0$

$\nabla^2 \nabla^2 \phi = 0$ **(Biharmonic Equation)**

Where

$$\nabla^4 = \nabla^2 \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2$$

For a more general case of a body forces and when the body force components X and Y are given by

$$X = -\frac{\partial V}{\partial x} \quad , Y = -\frac{\partial V}{\partial y}$$

in which V is the potential function. The equilibrium equations become

$$\frac{\partial}{\partial x}(\sigma_x - V) + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad , \quad \frac{\partial}{\partial y}(\sigma_y - V) + \frac{\partial \tau_{xy}}{\partial x} = 0$$

These equations can be satisfied by taking

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + V \quad , \sigma_y = \frac{\partial^2 \phi}{\partial x^2} + V \quad , \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \, \partial y}$$

Substituting into the compatibility equation for plane stress distribution

$$\therefore \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1-v)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

Or $\nabla^4 \phi = -(1-v)\, \nabla^2 V$

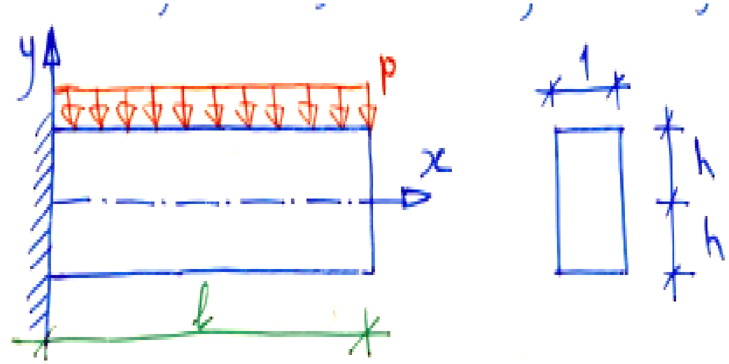
An analogous equation can be obtained for the case of plane strain.

Note: When the body force is simply the weight, the potential V is (p.g.y).

Example:

Show that the given function gives the stresses correctly on all boundaries except at the end $x=\ell$

$$\phi = \frac{p}{40h^3}(-10\,h^3 x^2 + 30\,h^2.\ell.x.y - 15\,h^2 x^2.y + 5\,\ell^2.y^3 + 2\,h^2.y^3 - 10\,\ell.x.y^3 + 5\,x^2 y^3 - y^5)$$



solution:

Check first that the given ϕ satisfies $\nabla^4\phi=0$

$$\frac{\partial\phi}{\partial x} = \frac{p}{40h^3}(-20\,h^3.x + 30\,h^2.\ell.y - 30\,h^2.X.y - 10\,\ell.y^3 + 10\,x.y^3)$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{p}{40h^3}(-20\,h^3 - 30\,h^2.y + 10\,y^3)$$

$$\frac{\partial^3\phi}{\partial x^3} = 0 \qquad , \frac{\partial^4\phi}{\partial x^4} = 0$$

$$\frac{\partial \phi}{\partial y} = \frac{p}{40h^3}(30\, h^2.\, \ell.\, \mathrm{x} - 15\, h^2\, x^2 + 15\, \ell^2\, .\, y^2 + 6\, h^2\, .y^2 - 30\, \ell.\, \mathrm{x}\, .\, y^2 + 15\, x^2\, y^2 - 5\, y^4)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{p}{40h^3}(30\, \ell^2.\, \mathrm{y} + 12\, h^2.y - 60\, \ell.\, \mathrm{x}\, .\, \mathrm{y} + 30\, x^2.\, \mathrm{y} - 20\, y^3)$$

$$\frac{\partial^3 \phi}{\partial y^3} = \frac{p}{40h^3}(30\, \ell^2 + 12\, h^2 - 60\, \ell.\, \mathrm{x} + 30\, x^2 - 60\, y^2)$$

$$\frac{\partial^4 \phi}{\partial y^4} = \frac{p}{40h^3}(-120\mathrm{y}) = -\frac{3p.y}{h^3}$$

$$\frac{\partial}{\partial y}\frac{\partial^2 \phi}{\partial x^2} = \frac{p}{40h^3}(-30\, h^2 + 30\, y^2)$$

$$\frac{\partial^2}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} = \frac{p}{40h^3}(60\, \mathrm{y}) = \frac{3}{2}\frac{p.y}{h^3}$$

$$\nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}$$

$$= 0 + 2\mathrm{x}\, \frac{3}{2}\frac{p.y}{h^3} - \frac{3p.y}{h^3} = 0$$

∴ **biharmonic equation is satisfied.**

Thus ϕ is a stress function.

To find the stresses σ_x , σ_y and τ_{xy} (here the body forces, or the weight of the cantilever is neglected):

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = \frac{p}{40h^3} (30\,\ell^2.\,y + 12\,h^2.y - 60\,\ell.\,x\,.\,y + 30\,x^2.\,y - 20\,y^3)$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = \frac{p}{40h^3} (-20\,h^3 - 30\,h^2.\,y + 10\,y^3)$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x\,\partial y} = -\frac{p}{40h^3} (\,30\,h^2.\,\ell - 30\,h^2.x - 30\,\ell.\,y^2 + 30\,x.\,y^2\,)$$

The boundary conditions which must be satisfied are:

1) $\left(\sigma_y\right)_{y=+h} = -p$ **(for all x)**

2) $\left(\sigma_y\right)_{y=-h} = 0$ **(for all x)**

3) $\left(\tau_{xy}\right)_{y=+h} = 0$ **(for all x)**

4) $\left(\tau_{xy}\right)_{y=-h} = 0$ **(for all x)**

5) $(\sigma_x)_{x=\ell} = 0$ **(for all y)**

6) $\left(\tau_{xy}\right)_{x=\ell} = 0$ **(for all y)**

For checking

1) $\left(\sigma_y\right)_{y=h} = \frac{p}{40h^3}$ (-20 h^3- 30 h^3+10 h^3) = - p ok

2) $\left(\sigma_y\right)_{y=-h} = \frac{p}{40h^3}$ (-20 h^3 + 30 h^3 −10 h^3)= 0 ok

3) $\left(\tau_{xy}\right)_{y=h}$ = - $\frac{p}{40h^3}$ (30 h^2. ℓ – 30 h^2.x – 30 ℓ. h^2 +30 x. h^2) = 0 ok

4) $\left(\tau_{xy}\right)_{y=-h}$ = - $\frac{p}{40h^3}$ (30 h^2. ℓ – 30 h^2.x – 30 ℓ. h^2 +30 x. h^2) = 0 ok

5) $(\sigma_x)_{x=\ell} = \frac{p}{40h^3}$ (30 ℓ^2. y +12 h^2.y - 60 ℓ^2. y+ 30 ℓ^2. y-20 y^3) = p $\{\frac{3}{10}\frac{y}{h} - \frac{1}{2}(\frac{y}{h})^3\} \neq 0$ **Not ok**

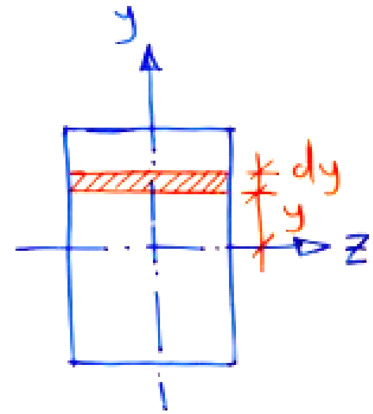
6) $\left(\tau_{xy}\right)_{x=\ell}$ = - $\frac{p}{40h^3}$ (30 h^2. ℓ – 30 h^2. ℓ – 30 ℓ. y^2 +30 ℓ. y^2) =0 ok

Thus all boundary conditions are satisfied except for the distribution of σ_x at the free end x= ℓ. A check must be mode that this disturbance in σ_x at the free edge is localized.

The normal force at the free end is

$$(N_x)_{x=\ell} = \int_{-h}^{h} (\sigma_x)_{x=\ell} \, . \, (1.\, d_y)$$

$$\int_{-h}^{h} p\left\{\frac{3}{10}\,\frac{y}{h} - \frac{1}{2}\left(\frac{y}{h}\right)^3\right\} . \, d_y = 0 \qquad \text{ok}$$



The moment force at the free end is

$$(m)_{x=\ell} = \int_{-h}^{h} y.\,(\sigma_x)_{x=\ell} \, . \, (1.\, d_y)$$

$$\therefore\ (m)_{x=\ell} = \int_{-h}^{h} y.\,(\sigma_x)_{x=\ell} \, . \, (1.\, d_y)$$

$$= p\int_{-h}^{h} \left(\frac{3}{10}\,\frac{y^2}{h} - \frac{1}{2}\left(\frac{y^4}{h^3}\right)\right) . \, d_y = 0 \qquad \text{ok}$$

Thus the disturbance is localized. To show the distribution of stresses at the section at distance $x = \frac{\ell}{2}$ from the fixed edge

$$(\sigma_x)_{x=\frac{\ell}{2}} = \frac{p}{40h^3}\,(12\,h^2.\,y + 7.5\,\ell^2.y - 20\,y^3)$$

$$(\sigma_y)_{x=\frac{\ell}{2}} = \frac{p}{40h^3}\,(-\,20\,h^3 - 30\,h^2.y + 10\,y^3)$$

$$(\tau_{xy})_{x=\frac{\ell}{2}} = -\,\frac{p}{40h^3}\,(15\,h^2.\ell - 15\,\ell.\,y^2)$$

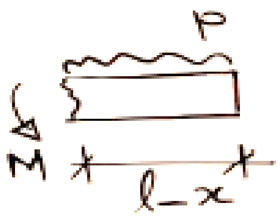
To draw the distribution of $(\sigma_x)_{x=\frac{\ell}{2}}$ assuming h=1 and ℓ=3 units

y	σ_x
0	0
$\frac{h}{8}$	0.244 p
$\frac{h}{4}$	0.490 p
$\frac{h}{2}$	0.934 p
$\frac{3h}{4}$	1.283 p
h	1.492 p

From the simple bending theory (Euler – Bernoulli theory)

$$\sigma_x = \frac{M}{I} .y$$

The B.M at a section at x from the fixed end is

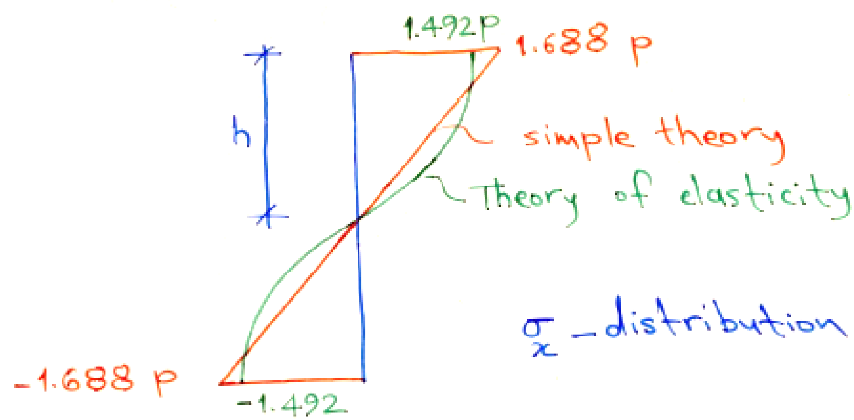


$$M = \frac{P}{2} (\ell - x)^2$$

$$\therefore \sigma_x = \frac{\frac{P}{2} (\ell - x)^2}{\frac{1}{12} (2h)^3} \text{ y} \quad = \frac{3P}{4h^3} (\ell - x)^2 \text{ .y}$$

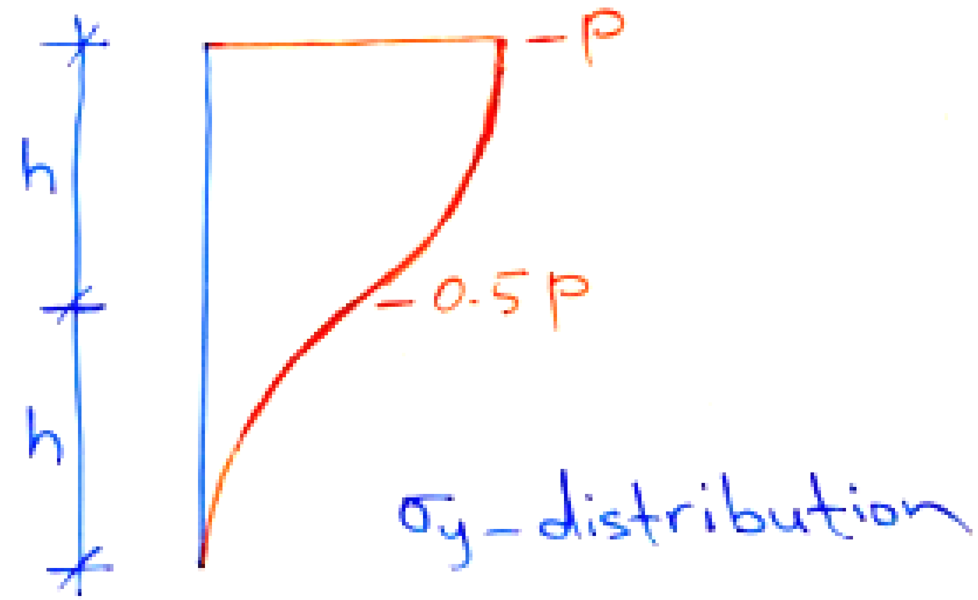
At $x = \frac{\ell}{2}$, $(\sigma_x)_{x=\frac{\ell}{2}} = \frac{3}{16} \frac{P\ell^2}{h^3} .y$

At $y = h$, $\sigma_x = 1.688$ **p** **(h=1 unit , ℓ = 3 units)**



To draw the σ_y at x= $\frac{\ell}{2}$

y	σ_y
- h	0
- $\frac{3h}{4}$	-0.043 p
- $\frac{h}{2}$	- 0.156 p
- $\frac{h}{4}$	- 0.316 p
0	- 0.5 p
$\frac{h}{4}$	- 0.684 p
$\frac{h}{2}$	- 0.844 p
$\frac{3h}{4}$	- 0.957 p
h	- p

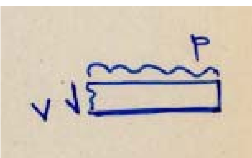


To draw the τ_{xy} at x= $\frac{\ell}{2}$

$$\left(\tau_{xy}\right)_{x=\frac{\ell}{2}} = -\frac{p}{40h^3}\left(15\, h^2.\ell - 15\, \ell.\, y^2\right) = 0.357\,\frac{p\ell}{h^3}\;.\,y^2 - 0.375\,\frac{p\ell}{h}$$

From the simple theory $\tau_{xy} = \frac{V.Q}{b.I}$

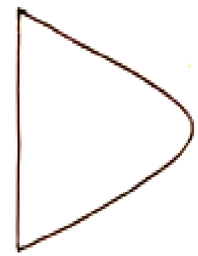
The shearing force at a distance x from the fixed end is



V = - p (ℓ –x)

$$\therefore\ \tau_{xy} = \frac{-\,p\,(\ell - x)}{1 * \frac{\ell}{12}(2h)^3}\,[\,(h - y) * 1\,] * \left(\frac{h + y}{2}\right) = -\,\frac{3}{4}\,\frac{p}{h^3}\,(\ell - x)\,(h^2 - y^2)$$

at $x=\frac{\ell}{2}$, $\tau_{xy}= -\frac{3}{8}\frac{p\ell}{h^3}(h^2 - y^2)$



$\tau_{max} = -\frac{3}{8}\frac{P.\ell}{h}$

Exact and approximate τ_{xy} distribution

$$= 0.375\,\frac{p\ell}{h^3}\,y^2 - 0.375\,\frac{p\ell}{h}$$

This is the same expression obtained from the exact solution

To find the displacements

$$\epsilon_x = \frac{1}{E}(\sigma_x - v\,\sigma_y)$$

$$= \frac{1}{E}\frac{p}{40h^3}\{30\,\ell^2.y + 12\,h^2.y - 60\,\ell.x.y + 30\,x^2.y - 20\,y^3 - v(-20\,h^3 - 30\,h^2.y + 10\,y^3)\}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - v\,\sigma_x)$$

$$= \frac{1}{E}\frac{p}{40h^3}\{-20\,h^3 - 30\,h^2.y + 10\,y^3 - v(30\,\ell^2.y + 12\,h^2.y - 60\,\ell.x.y + 30\,x^2.y - 20\,y^3)\}$$

$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2\,(1+v)}{E}$ $\{- \frac{p}{40h^3}$ (30 $h^2.\,\ell$ – 30 h^2.x – 30 $\ell.\,y^2$ +30 x. y^2)}

Let m= $\frac{1}{E}\frac{p}{40h^3}$

$\therefore \epsilon_x = \frac{\partial u}{\partial x}$ = m { 30 ℓ^2 .y + (12+30 v) h^2.y -60 ℓ. x. y + 30 x^2. y – (20 +10 v) y^3 + 20 $v.\,h^3$ }

Integrating

$\frac{1}{m}u$= 30 ℓ^2.x .y + (12+30 v) h^2.y .x -30 ℓ. x^2. y + 10 x^3. y – (20 +10 v) y^3.x + 20 $v.\,h^3$ .x +f(y)

$\epsilon_y = \frac{\partial v}{\partial y}$ = m { - 20 h^3 - (30+12 v) h^2.y + (10 +20 v) y^3 - 30 v . ℓ^2 .y + 60 v .ℓ. x. y – 30 v . x^2. y }

Integrating

$\frac{1}{m}$ v= - 20 h^3 .y - (15+6 v) h^2.y^2 + ($\frac{5}{2}$ +5 v) y^4 - 15 v . ℓ^2 . y^2 + 30 v .ℓ. x. y^2 – 15 v . x^2. y^2 +g(x)

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= m\,\{30\,\ell^2 . x + (12+30\,v)\,h^2 . x - 30\,\ell . x^2 + 10\,x^3 - (60+30\,v)\,y^2 . x + \frac{df}{dy}\} + m\,\{30\,v . \ell . y^2 - 30\,v . x . y^2 + \frac{dg}{dx}\}$$

$$\therefore \frac{1}{m}\gamma_{xy} = 30\,\ell^2 . x + 12\,h^2 . x + 30\,v . h^2 . x - 30\,\ell . x^2 + 10\,x^3 - 60\,y^2 . x - 30\,v . y^2 . x + \frac{df}{dy} + 30\,v . \ell . y^2 - 30\,v . x . y^2 + \frac{dg}{dx} \quad \ldots\ldots(1)$$

But

$$\frac{1}{m}\gamma_{xy} = (2+2\,v)\,\{-30\,\ell . h^2 + 30\,x . h^2 + 3\,\ell . y^2 - 30\,x . y^2\} \quad \ldots\ldots\ldots\ldots(2)$$

Equating (1) and (2)

$$\therefore \{(30\,\ell^2 - 48\,h^2 - 30\,v . h^2)\,x - 30\,\ell . x^2 + 10\,x^3 + \frac{dg}{dx}\} + \{\frac{df}{dy} - (60\,\ell + 30\,v . \ell)\,y^2\} = -60\,h^2 . \ell - 60\,v\,h^2 . \ell$$

$$\therefore (30\,\ell^2 - 48\,h^2 - 30\,v . h^2)\,x - 30\,\ell . x^2 + 10\,x^3 + \frac{dg}{dx} = a$$

And

$$\frac{df}{dy} - (60\,\ell + 30\,v . \ell)\,y^2 = c$$

Where $\quad a+c = -60\,\ell.\,h^2 - 60\,v.\,\ell.\,h^2$

Integrating

$\therefore\ g(x) = ax - (15\,\ell^2 - 24\,h^2 - 15\,v.\,h^2)\,x^2 + 10\,\ell.\,x^3 - \frac{5}{2}\,x^4 + b$

And

$F(y) = c.y + (20\,\ell + 10\,v.\,\ell^2)\,y^3 + d$

$\therefore\ u = m\{\,30\,\ell^2.x\,.y + (12+30\,v)\,h^2.y\,.x - 30\,\ell.\,x^2.\,y + 10\,x^3.\,y - (20+10\,v)\,y^3.x + 20\,v.\,h^3\,.x + c.y + (20\,\ell + 10\,v.\,\ell^2)\,y^3 + d\}$

$\therefore\ v = m\{-20\,h^3\,.y - (15+6\,v)\,h^2.y^2 + (\frac{5}{2} + 5\,v)\,y^4 - 15\,v\,.\,\ell^2\,.\,y^2 + 30\,v\,.\ell.\,x.\,y^2 - 15\,v\,.\,x^2.\,y^2 + ax - (15\,\ell^2 - 24\,h^2 - 15\,v.\,h^2)\,x^2 + 10\,\ell.\,x^3 - \frac{5}{2}\,x^4 + b\}$

Boundary conditions:

u=0 at x=0 and y=0

v=0 at x=0 and y=0

∴ d=0 & b=0

$\frac{\partial u}{\partial y} = 0$ at x=0 and y=0

$\frac{\partial v}{\partial x}$ =0 at x=0 and y=0

From $\frac{\partial u}{\partial y} = 0$ ∴ c=0 ∴ a= - 60 ℓ. h^2 - 60 v. ℓ. h^2 .... (3)

From $\frac{\partial v}{\partial x} = 0$ ∴ a=0 ∴ c= - 60 ℓ. h^2 - 60 v. ℓ. h^2 .... (4)

Using the first solution (3)

$$V = \frac{p}{40\,E\,h^3}\{-20\,h^3 .y - (15\,h^2 + 6\,v\,h^2 + 15\,v\,.\,\ell^2)\,y^2 + (\frac{5}{2} + 5\,v)\,y^4 + 30\,v\,.\ell.\,x.\,y^2 - 15\,v\,.\,x^2.\,y^2 + ax - (15\,\ell^2 - 24\,h^2 - 15\,v.\,h^2)\,x^2 + 10\,\ell.\,x^3 - \frac{5}{2}\,x^4 + (-60\,h^2.\,\ell - 60\,v\,.h^2.\,\ell)\,x\}$$

$$\therefore \quad (v)_{\substack{y=0\\ x=1}} = -\frac{7.5\,p\,\ell^4}{40\,E\,h^3} - \frac{p}{40\,E\,h^3}(36\,h^2.\,\ell^2 + 45\,v\,.h^2.\,\ell^2)$$

For simple theory:

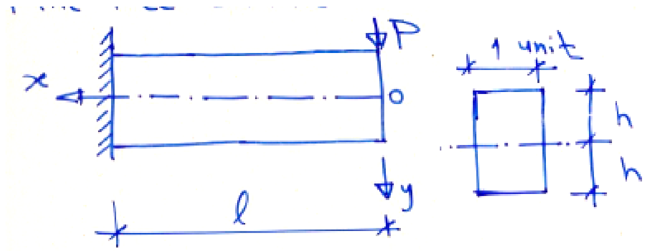
Deflection at free end $= \dfrac{w.\ell^2}{8\,E\,I} = \dfrac{7.5\,p\,\ell^4}{40\,E\,h^3}$

Example:

For the given cantilever show that

$\boldsymbol{\phi} = \mathbf{A . x. y + B . x.} \boldsymbol{y^3}$

is the proper stress function. Determine the constants A and B so that the Shear stress is zero on the top and bottom faces while the resultant vertical Force on the free surface is P



solution:

by direct substitution

$$\nabla^4\boldsymbol{\phi} = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)(\mathbf{A . x. y + B . x.}\, \boldsymbol{y^3}) = 0$$

Thus the given ϕ is a proper stress function. Next is to prove that this function is for the given problem.

The stresses are:

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6\,B\,.\,x.\,y$$

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = -A - 3\,B\,.\,y^2$$

The constants A and B must be determined.

The boundary conditions are:

1) $\left(\tau_{xy}\right)_{y=\pm h} = 0$

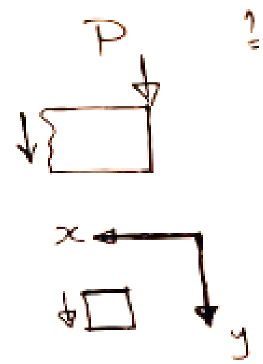
$\therefore -A - 3B\,h^2 = 0$

$\therefore A + 3B\,h^2 = 0$ ………….. (1)

2) **On any x section** $\int_{-h}^{h} \tau_{xy} * 1 * d_y = -P$

$\therefore \int_{-h}^{h} (-A - 3B \cdot y^2) * 1 * d_y = -P$

$\therefore$ 2Ah + 2B y^3 =P ……………………………….. (2)



Solving Eqs. (1) and (2)

$\therefore A = \frac{3p}{4h}$, $B = -\frac{p}{4h^3}$

$\therefore \sigma_x = -\frac{3}{2}\frac{p}{h^3}$ x.y

$\sigma_y = 0$

$\tau_{xy} = \frac{3p}{4h}(1 - \frac{y^2}{h^2})$

The other boundary conditions at any x-section:

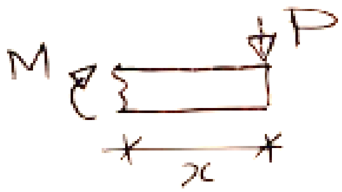
1. $\int_{-h}^{h} \sigma_x * 1 * d_y = 0$

$\therefore \int_{-h}^{h} -\frac{3}{2}\frac{p}{h^3}\, x.y\, .\, d_y = -\frac{3}{2}\frac{p}{h^3}\, x \left[\frac{y^2}{2}\right]_{-h}^{+h} = 0$ **ok**

2. $\int_{-h}^{h} y\,(\sigma_x * 1 * d_y) = M$

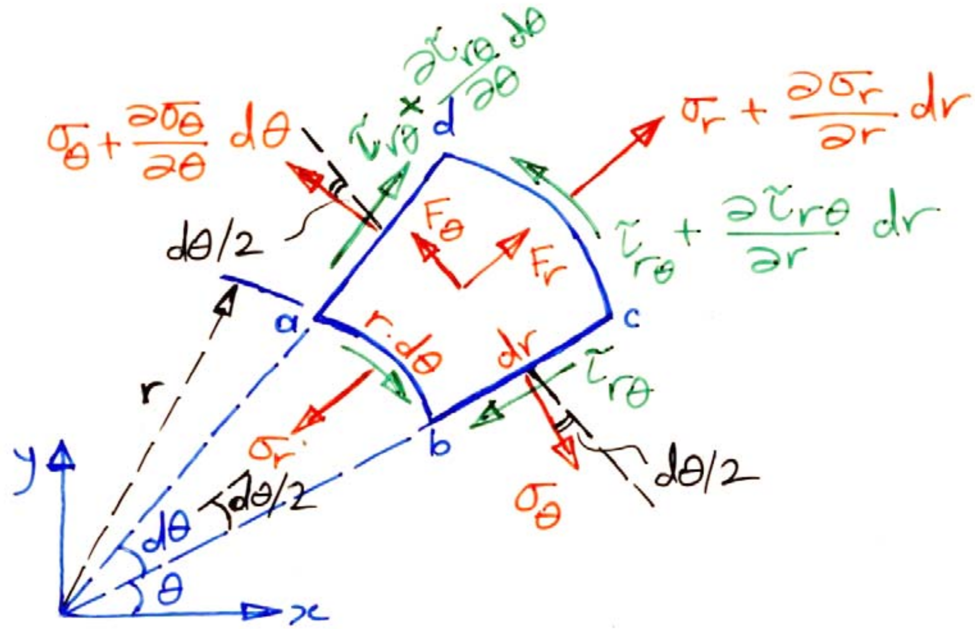
Where M= -P .x

$\therefore \int_{-h}^{h} \left(-\frac{3}{2}\frac{p}{h^3}\, x.y\right) y\, .\, d_y = -P\, .x$ **ok**



Equations of Equilibrium

Consider the state of stress on an infinitesimal element a b c d of unit thickness



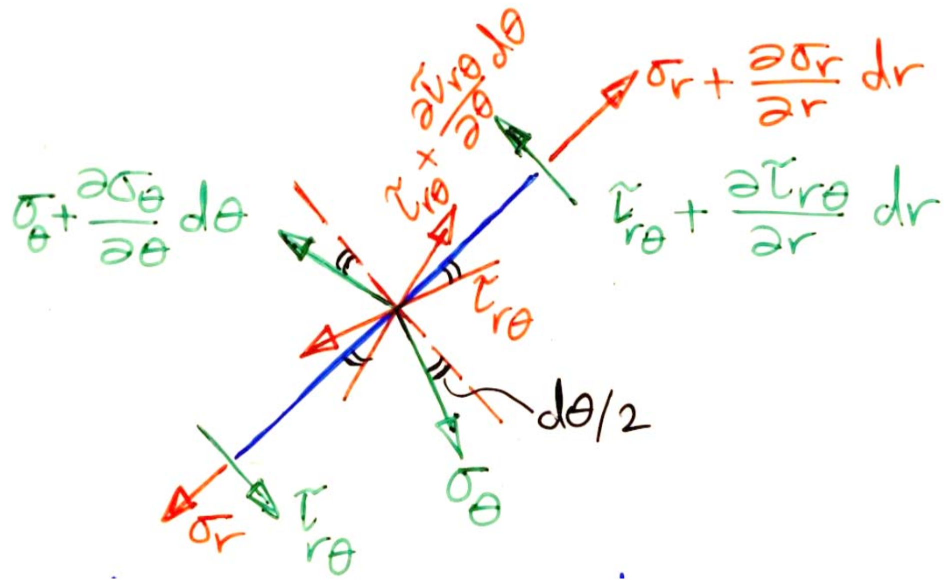
The stresses on the faces (ab and bc) of the element are:

σ_r - the normal stress component in the radial direction

σ_θ - the normal stress component in the circumferential direction

$\tau_{r\theta} = \tau_{\theta r}$ - the shearing stress components

The r and θ directed body forces are denoted by F_r and F_θ



Transfer of forces on sides the d_r to

the center of the element

Equilibrium in radial direction:

$(\sigma_r + \frac{\partial \sigma_r}{\partial r} d_r)$ (r + d_r) d_θ * 1- σ_r .r. d_θ * 1- σ_θ. d_r * 1 . sin $\frac{d_\theta}{2}$ - $(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d_\theta)$. d_r * 1 . sin $\frac{d_\theta}{2}$ + $(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d_\theta)$ d_r * 1 cos $\frac{d_\theta}{2}$ - $\tau_{r\theta}$. d_r * 1 . cos $\frac{d_\theta}{2}$ + F_r .r. d_θ . d_r .1 =0

As d_θ is small,

sin $\frac{d_\theta}{2} = \frac{d_\theta}{2}$ and cos $\frac{d_\theta}{2}$ =1

Also, dropping terms containing higher order infinitesimals $({d_r}^2, {d_\theta}^2)$

$\therefore \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$ .... (1)

Equilibrium in θ – direction:

$(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d_\theta)$ d_r * 1 . cos $\frac{d_\theta}{2}$ - σ_θ . d_r * 1 . cos $\frac{d_\theta}{2}$ + $(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} d_r)$ (r + d_r) d_θ * 1 - $\tau_{r\theta}$. r. d_θ.1 + $(\tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} d_\theta)$. d_r *1

sin $\frac{d_\theta}{2}$ + $\tau_{r\theta}$. d_r *1. sin $\frac{d_\theta}{2}$ + F_θ . d_r .r. d_θ .1

$\therefore \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + 2\frac{\tau_{r\theta}}{r} + F_\theta = 0$ ....(2)

Strain – Displacement Relations

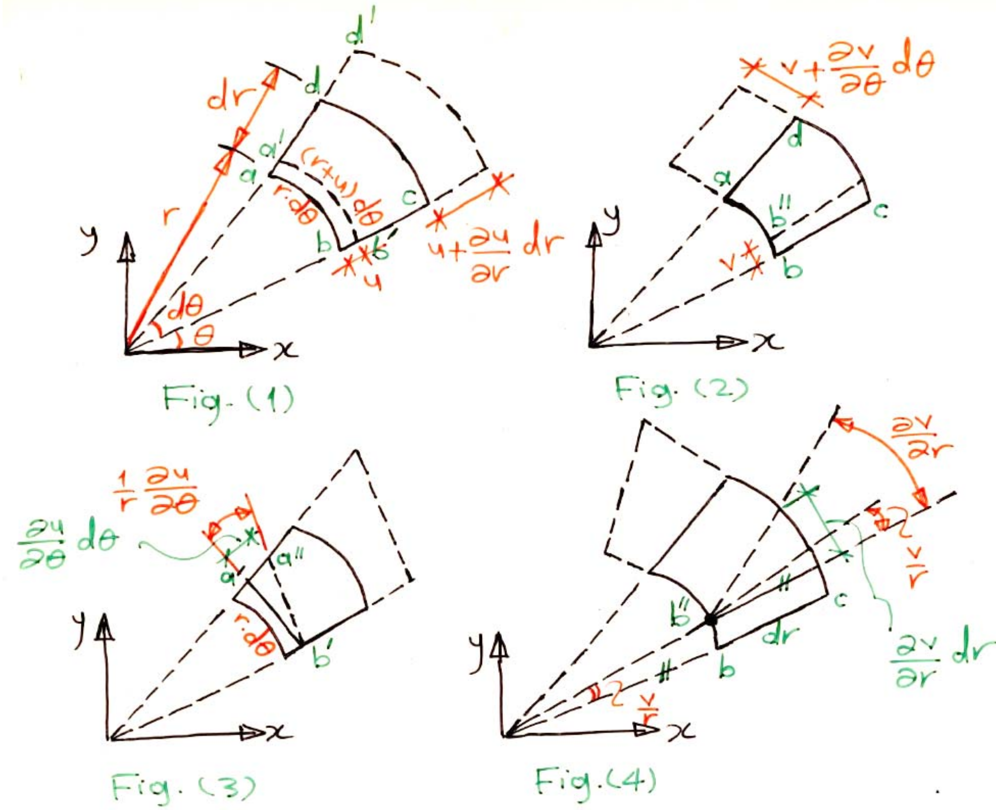
The general deformation experienced by an element abcd may be regarded as composed of :

A change in length of the sides, as in Figs. (1) and (2).

Rotation of the sides, as in Figs. (3) and (4).

The r and θ displacements are denoted

by u and v, respectively.



Referring to Fig.(1), a u displacement od side ab results in both radial and tangential strain. The radial strain ϵ_r is

$$\epsilon_r = \frac{a'd' - ad}{ad} = \frac{\left(dr - u + u\frac{\partial u}{\partial r}.dr\right) - dr}{dr} = \frac{\frac{\partial u}{\partial r}dr}{dr} = \frac{\partial u}{\partial r} \quad \text{............(1)}$$

The tangential strain is

$$(\epsilon_\theta)_u = \frac{(r+u)d_\theta - r.d_\theta}{r.d_\theta} = \frac{u}{r} \quad \text{...................................(2)}$$

A v displacement produces a tangential strain (Fig.2)

$$(\epsilon_\theta)_v = \frac{\left(r.d_\theta - v + v + \frac{\partial v}{\partial \theta}d_\theta\right) - r.d_\theta}{r.d_\theta} = \frac{\left(\frac{\partial v}{\partial \theta}d_\theta\right)}{r.d_\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{..............(3)}$$

The resultant tangential strain is

$$\epsilon_\theta = (\epsilon_\theta)_u + (\epsilon_\theta)_v$$

$$\therefore\ \epsilon_\theta = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

In Fig.(3), and due to a u displacement

$$(\gamma_{r\theta})_u = \frac{\left(\frac{\partial u}{\partial \theta} d_\theta\right)}{r.d_\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} \ \ldots\ldots\ldots\ldots\ldots (5)$$

The rotation od side bc associated with a v displacement alone (Fig.4) is

$$(\gamma_{r\theta})_v = \frac{\frac{\partial v}{\partial r} dr}{dr} - \frac{v}{r} = \frac{\partial v}{\partial r} - \frac{v}{r} \ \ldots\ldots\ldots\ldots (6)$$

The total shearing strain is

$$\gamma_{r\theta} = (\gamma_{r\theta})_u + (\gamma_{r\theta})_v$$

$$= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} \ \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Stress – Strain Relationship

In the case of plane stress:

$$\left.\begin{aligned} \epsilon_r &= \frac{1}{E}(\sigma_r - v\,\sigma_\theta) \\ \epsilon_\theta &= \frac{1}{E}(\sigma_\theta - v\,\sigma_r) \\ \gamma_{r\theta} &= \frac{1}{G}\tau_{r\theta} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

For plane strain case:

$$\epsilon_r = \frac{1+v}{E}[(1-v)\,\sigma_r - v\,\sigma_\theta]$$

$$\epsilon_\theta = \frac{1+v}{E}[(1-v)\,\sigma_\theta - v\,\sigma_r]$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

Compatibility Equation

By eliminating u and v from the expressions of strains, the compatibility equation is

$$\frac{\partial^2 \epsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \epsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \epsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \epsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta}$$

Or

$$\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \epsilon_\theta}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\epsilon_r$$

Solution by Stress Function

In the absence of body forces, the equations of equilibrium are satisfied by stress function ϕ (r,θ) for which

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r\,\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

The compatibility equation can be expressed in the alternative form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi=0$$

Or

$$\nabla^4\phi=0$$

Where in polar coordinates, the Laplacian operator is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

If ϕ and the components of stress are independent of the variable θ (axi – symmetric problems), then

$$\sigma_r=\frac{1}{r}\frac{\partial\phi}{\partial r}$$

$$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2}$$

$$\tau_{r\theta}=0$$

The compatibility equation can be expressed in the alternative form

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0$$

Or

$$\frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

To solve Eq.(1), introduce a new variable t such that $r=e^t$

$$\therefore \frac{dr}{dt}=e^t \qquad \text{and } \frac{dt}{dr}=e^{-t}$$

Also

$$\frac{d\phi(r)}{dr}=\frac{d\phi(r)}{dt}\cdot\frac{dt}{dr}=\frac{d\phi(t)}{dt}\,e^{-t}\,,$$

$$\frac{d^2\phi(r)}{dr^2}=\frac{d}{dr}\left[\frac{d\phi(r)}{dr}\right]=\frac{d}{dt}\left[\frac{d\phi(r)}{dr}\right]\frac{dt}{dr}$$

$$= \frac{d}{dt}\,[\frac{d\phi(t)}{dt}\,\mathrm{e}^{-t}\,]\,.\,\mathrm{e}^{-t}$$

$$= [\frac{d^2\phi(t)}{dt^2} - \frac{d\phi(t)}{dt}]\,.\,\mathrm{e}^{-2t}$$

$$\frac{d^3\phi(r)}{dr^3} = \frac{d}{dt}\,[\frac{d^2\phi(r)}{dr^2}\,]\,\frac{dt}{dr}$$

$$= \mathrm{e}^{-t}\,\frac{d}{dt}\,[\{\frac{d^2\phi(t)}{dt^2} - \frac{d\phi(t)}{dt}\}\,\mathrm{e}^{-2t}\,]$$

$$= [\frac{d^3\phi(t)}{dt^3} - 3\,\frac{d^2\phi(t)}{dt^2} + 2\,\frac{d\phi(t)}{dt}]\,\mathrm{e}^{-3t}$$

$$\frac{d^4\phi(r)}{dr^4} = \frac{d}{dt}\,[\frac{d^3\phi(r)}{dr^3}]\,\frac{dt}{dr}$$

$$= [\frac{d^4\phi(t)}{dt^4} - 6\,\frac{d^3\phi(t)}{dt^3} + 11\,\frac{d^2\phi(t)}{dt^2} - 6\,\frac{d\phi(t)}{dt}]\,\mathrm{e}^{-4t}$$

Substituting the above expressions in Eq.(1)

$\therefore \frac{d^4\phi(t)}{dt^4}$ - 4 $\frac{d^3\phi(t)}{dt^3}$ + 4 $\frac{d^2\phi(t)}{dt^2}$ =0 …………..(2)

To integrate Eq.(2) let $\phi(t)$= $e^{\lambda t}$ ………..(3)

Substituting Eq.(3) into Eq.(2)

$\therefore \lambda^4$- 4 λ^3 + 4 λ^2 =0

or $\lambda^2 (\lambda - 2)^2$ =0

$\therefore \lambda_1$=0 , λ_2=0 , λ_3=2 **and** λ_4=2

The general solution of Eq.(2) is

$\phi(t)$= a.t + b.t. e^{2t} + c. e^{2t} +d ………….(4)

Where a,b,c and d are constants .

Since r= e^t , i.e. , t=ln r , then the general solution of Eq.(1) is

$\phi(r)$= A. ln r + B. r^2.ln r + C. r^2.D ……………… (5)

Where A,B,C and D are constants.

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B\,(1 + 2.\ln r) + 2C \\ \sigma_\theta &= \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B\,(3 + 2.\ln r) + 2C \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Displacements for Symmetrical Stress Distribution

The Hooke's law for plane stress is:

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_r - v\,\sigma_\theta)$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}(\sigma_\theta - v\,\sigma_r)$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{\tau_{r\theta}}{G} = 0$$

Substituting Eqs. (6) in the first of eqs. (7)

$$\frac{\partial u}{\partial r} = \frac{1}{E}\left[\frac{(1+v)\,A}{r^2} + 2\,(1-v)\,B.\ln r + (1\text{-}3\,v)\,B + 2(1\text{-}\,v)\,C\right]$$

After integration

$$u = \frac{1}{E}\left[-\frac{(1+v)\,A}{r} + 2(1\text{-}\,v)\,B.\,r.\,\ln r - (1+v)\,B.r + 2(1\text{-}\,v)\,C.r + f(\theta)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots..\ (8)$$

In which f(θ) is a function of θ only. The second of Eqs.(7) can be written as

$$\frac{\partial v}{\partial \theta} = \frac{r}{E}(\sigma_\theta - v\,\sigma_r) - u$$

$$= \frac{4\,B.r}{E} - f(\theta)$$

After integration

$$v = \frac{4\,B.r.\theta}{E} - \int_0^\theta f(\theta)\,.\,d_\theta + f_1(r) \quad \text{………………….. (9)}$$

Substituting Eqs. (8) and (9) into the third of Eqs. (7)

$$\frac{1}{r}\frac{df(\theta)}{d\theta} + \frac{df_1(r)}{dr} + \frac{1}{r}\int_0^\theta f(\theta)\,.\,d_\theta - \frac{f_1(r)}{r} = 0$$

$$\therefore \frac{df(\theta)}{d\theta} + \int_0^\theta f(\theta)\,.\,d_\theta = f_1(r) - r\frac{df_1(r)}{dr} = N = \text{constant}$$

Which yields

$$\left.\begin{array}{l} \dfrac{df(\theta)}{d\theta} + \displaystyle\int_0^\theta f(\theta)\,.\,d_\theta = N \quad \text{and} \\ f_1(r) - r\dfrac{df_1(r)}{dr} = N \end{array}\right\} \quad \text{…………… (10)}$$

Differentiating the first of Eqs. (10) w.r.t. θ

$$\therefore \frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

$$\therefore f(\theta) = J.\ \text{Cos}\theta + K.\ \text{Sin}\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (11)$$

Where J and K are the constants of integration.

Substituting Eq. (11) into the first of Eqs.(10)

$$\therefore J.\ \text{Sin}\theta + K.\ \text{Cos}\theta + J.\ \text{Sin}\theta - K.\ \text{Cos}\theta + K = N$$

Or $K=N$

$$\therefore f(\theta) = J.\ \text{Cos}\theta + N.\ \text{Sin}\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (12)$$

Differentiating the second of Eqs. (10) w.r.t.r

$$\therefore \frac{df_1(r)}{dr} - r\frac{d^2 f_1(r)}{dr^2} - \frac{df_1(r)}{dr} = 0$$

Or $$\frac{d^2 f_1(r)}{dr^2} = 0$$

$\therefore f_1(r)$ = H.r + G ……………. (13)

Where H and G are the constants of integration.

Substituting Eq.(13) into the second of Eqs. (10)

$\therefore$ G=N

$f_1(r)$= H.r + N ……………….(14)

Substituting Eqs. (12) and (14) into Eqs. (8) and (9)

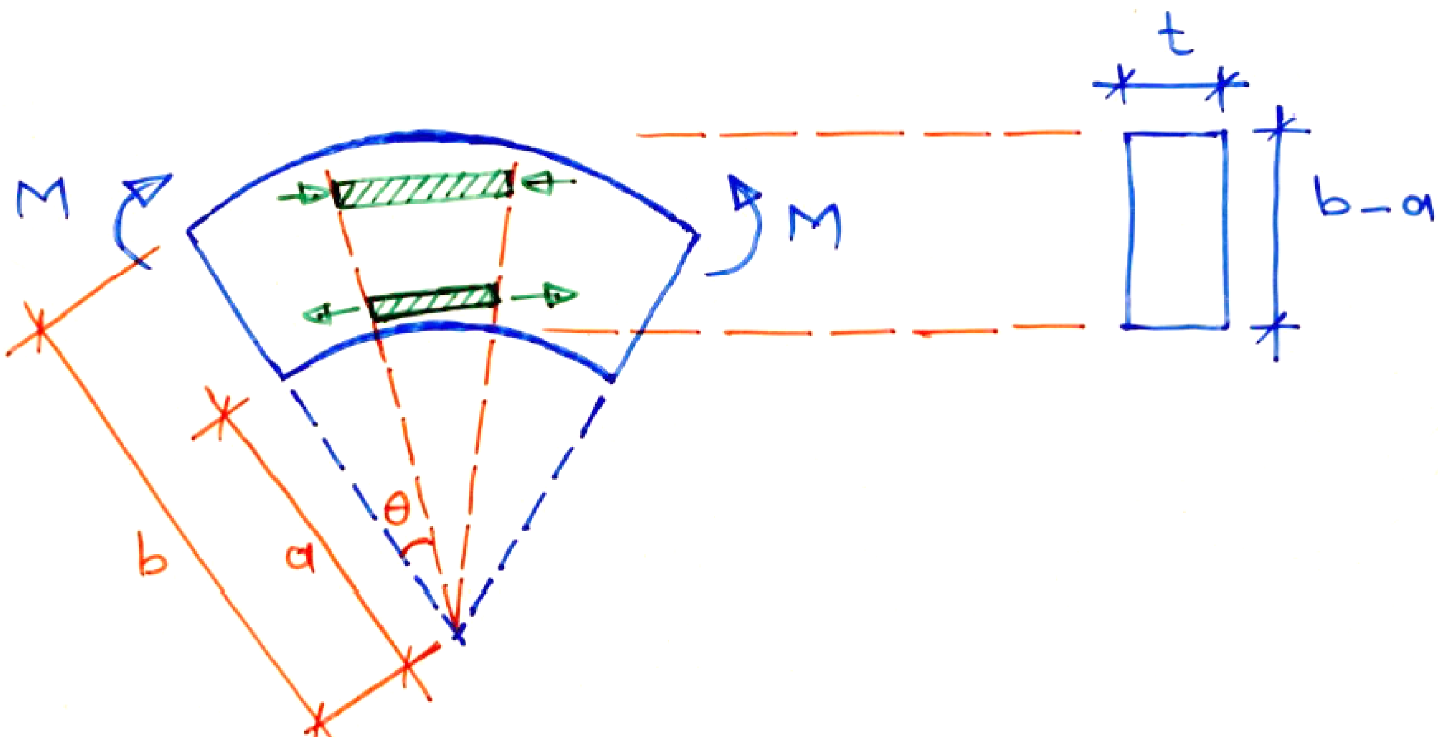
$$\left.\begin{aligned} &\therefore u=\frac{1}{E}\left[-\frac{(1+v)\,A}{r}+2(1-v)\,B.r.\ln r-(1+v)\,B.r+2(1-v)\,C.r\right]+J.\cos\theta+N.\sin\theta \\ &\text{And} \\ &v=\frac{4\,B.r.\theta}{E}-J.\sin\theta+N.\cos\theta+H.r \end{aligned}\right\} \quad \text{………………} \; (15)$$

Where A,B,C,N,H, and J are constants to be determined.

Applications

(1) Pure Bending of Curved Bars

Consider a curved bar with constant narrow rectangular cross section and a circular axis bent in the plane of curvature by two equal and opposite couples M applied at the ends.



The bending moment constant along the length of the bar and the stress distribution is the same in all radial cross sections.

The solution of the problem can be obtained by using the stress function ϕ(r) given by Eq. (5). The stresses are given by Eqs. (6)

The boundary conditions are:

1) **No normal forces act along the curved boundaries**

at r=a and r=b

$\therefore\ (\sigma_r)_{r=a} = 0 \qquad , (\sigma_r)_{r=b} = 0$ …………………….. (16)

2) **The normal stresses acting at the straight edge of the bar must yield a zero resultant force**

$t \int_a^b \sigma_\theta \cdot d_r = 0$ ……………………………………..…(17)

2) **The normal stresses at the ends must produce a couple M**

$t \int_a^b r \cdot \sigma_\theta \cdot d_r = -\ M$ ……………………………..(18)

The shearing stresses have been assumed zero throughout the beam, and $\tau_{r\theta}$ =0 is thus satisfied at the boundaries, where no tangential forces exist. Using the first boundary condition leads to

$$\left.\begin{array}{l} \frac{A}{a^2} + B\,(1+ 2.\ln a)+ 2C=0 \\ \\ \frac{A}{b^2} + B\,(1+ 2.\ln b)+ 2C=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (19)$$

From the second B.C.

$$\int_a^b \sigma_\theta.\, d_r = \int_a^b \frac{d^2\phi}{dr^2}\, d_r = \int_a^b d\left(\frac{d\phi}{dr}\right) = \left[\frac{d\phi}{dr}\right]_a^b$$

$$= [r.\,\sigma_r.]_a^b \quad = b\,\overbrace{(\sigma_r)_{r=b}}^{=0} - a\,\overbrace{(\sigma_r)_{r=a}}^{=0} = 0 - 0 = 0$$

Using the third B.C. gives

$$\int_a^b \sigma_{\theta}.\text{r. d}_r = -\frac{M}{t}$$

$$\therefore \int_a^b \frac{d^2\phi}{dr^2}\, r.\,\text{d}_r = -\frac{M}{t}$$

$$\therefore \int_a^b r.\,d\left(\frac{d\phi}{dr}\right) = -\frac{M}{t}$$

$$\therefore \left[\text{r} . \frac{d\phi}{dr}\right]_a^b - \int_a^b \frac{d\phi}{dr}\,\text{d}_r = -\frac{M}{t}$$

$$\therefore [r^2 .\sigma_{\text{r}}]_a^b - [\phi]_a^b = -\frac{M}{t}$$

$$\text{Since } [r^2 .\sigma_{\text{r}}]_a^b = b^2\, \underbrace{(\cancel{\sigma_r})_{r=b}}_{=0} - a^2\, \underbrace{(\cancel{\sigma_r})_{r=a}}_{=0} = 0 - 0 = 0$$

$$\therefore [\phi]_a^b = \frac{M}{t}$$

Substituting for ϕ from Eq.(5)

$$\therefore A\, ln\frac{b}{a} + B\,(b^2.\,ln\,b\; - \;a^2.\,ln\,a\,) + C\,(b^2 - a^2) = M \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Solving Eqs.(19) and (20) for A,B, and C

$$A = -\frac{4\,M}{tk}\,.\,a^2\,.\,b^2.\,ln\frac{b}{a}$$

$$B = -\frac{2\,M}{tk}\,(b^2 - a^2)$$

$$C = \frac{M}{tk}[b^2 - a^2 + 2(b^2.\,ln\,b\; - \;a^2.\,ln\,a\,)]$$

Where $k = (b^2 - a^2)^2 - 4.\,a^2.\,b^2\;(ln\frac{b}{a})^2$

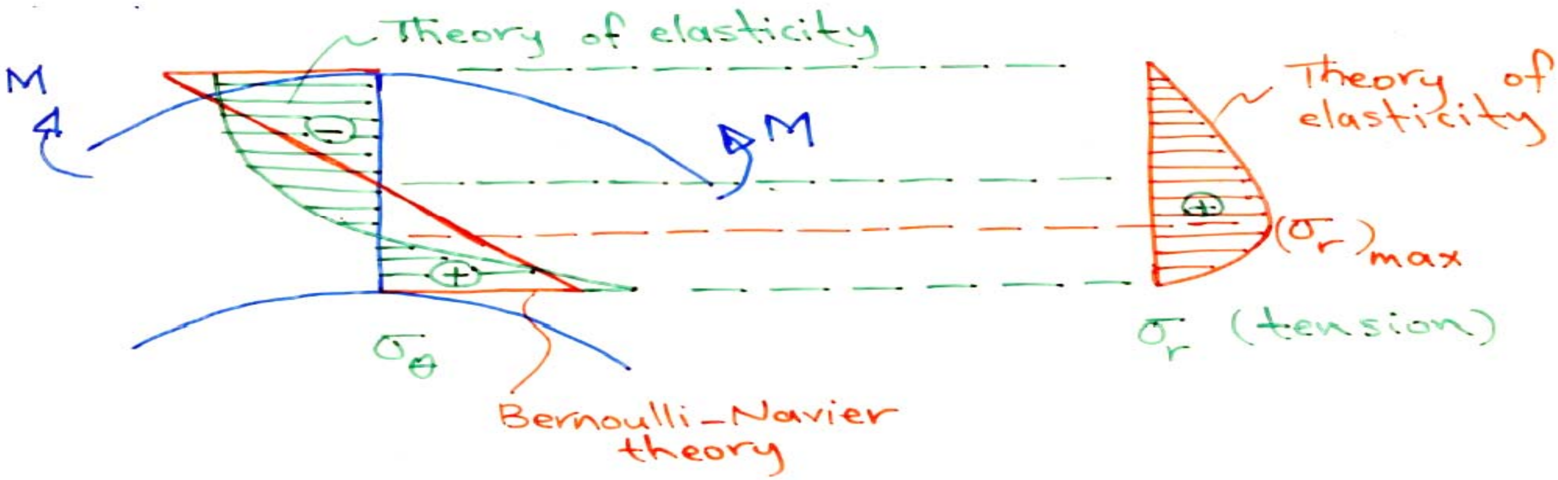
The stresses are

$$\sigma_r = -\frac{4M}{tk}\left[\frac{a^2 . b^2}{r^2}\, ln\frac{b}{a} + b^2\, ln\frac{r}{b} + a^2\, ln\frac{a}{r}\right]$$

$$\sigma_\theta = -\frac{4M}{tk}\left[-\frac{a^2 . b^2}{r^2}\, ln\frac{b}{a} + b^2\, ln\frac{r}{b} + a^2\, ln\frac{a}{r} + b^2 - a^2\right]$$

$$\tau_{r\theta} = 0$$

To find the neutral axis, put $\sigma_\theta = 0$



To compute the values of the three constant N,H and J,it will be assumed that the centroid of the cross section and also an element of the radius at this point, is rididly fixed.

$\therefore u = v = \frac{dv}{dr} = 0$

for $r = r_{\circ} = \frac{b+a}{2}$ and $\theta = 0$

Applying these to Eqs. (15) gives

N =H =0

$$J = \frac{1}{E}\left[\frac{(1+v)}{r_{\circ}} A - 2(1-v) B . r_{\circ} . \ln r_{\circ} (1+v) B . r_{\circ} - 2(1-v) C . r_{\circ}\right] \quad \text{.......................... (21)}$$

Component v of the elastic displacement is then

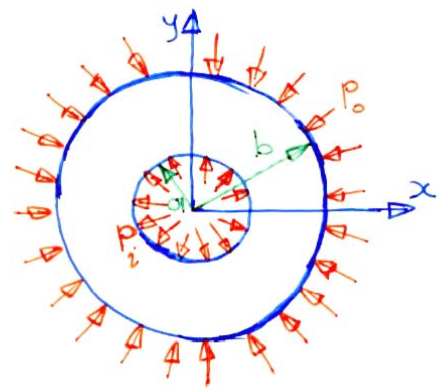
$$v = \frac{4 B . r . \theta}{E} - J . \sin\theta$$

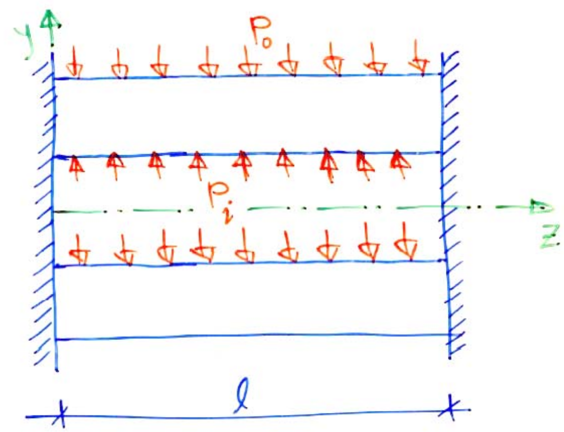
2) Thick Cylinder Subjected to Uniformly Distributed Pressures

Consider a thick cylinder of length ℓ, internal radius a, and external radius b. the cylinder is subjected to internal and external uniformly distributed pressures P_i and $P_\circ$. The bases of the cylinder at z=0 and z= ℓ are assumed to be completely restrained ; therefore, w=0 and it follows that the problem is a case of plane strain.

Because of symmetry, stresses σ_r and σ_θ will be independent of θ and the stress $\tau_{r\theta}$= 0. the stresses will be expressed by Eqs.(6) , i.e.,

$$\left.\begin{aligned} \sigma_r &= \frac{A}{r^2} + B\,(1 + 2\ln r) + 2C \\ \sigma_\theta &= -\frac{A}{r^2} + B\,(3 + 2\ln r) + 2C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$





Also for symmetry, component u of the displacement

will be independent of θ and component v will be zero

In this case

$$\epsilon_r = \frac{du}{dr} \quad , \epsilon_\theta = \frac{u}{r} \quad , \gamma_{r\theta} = 0 \quad \text{.................... (23)}$$

Hook's law is given by

$$\left.\begin{aligned} \epsilon_r &= \frac{1}{E}\left[\sigma_r - v(\sigma_\theta + \sigma_z)\right] \\ \epsilon_\theta &= \frac{1}{E}\left[\sigma_\theta - v(\sigma_r + \sigma_z)\right] \\ \epsilon_z &= \frac{1}{E}\left[\sigma_z - v(\sigma_r + \sigma_\theta)\right] \end{aligned}\right\} \text{..........................(24)}$$

Since $\epsilon_z = 0$, then from the third of Eqs.(24)

$$\therefore \sigma_z = v(\sigma_r + \sigma_\theta) \text{ ....(25)}$$

Substituting Eq.(25) into the first two of Eqs. (24)

$$\therefore \epsilon_r = \frac{1}{E}\left[(1-v^2)\sigma_r - v(1+v)\sigma_\theta\right]$$

$$\epsilon_\theta = \frac{1}{E}\left[(1-v^2)\sigma_\theta - v(1+v)\sigma_r\right] \quad \text{.......................... (26)}$$

Using Eqs. (22) , the first of Eqs. (23), and the first of Eqs. (26)

$$\therefore \frac{du}{dr} = \frac{1}{E}\left[(1-v^2)\left\{\frac{A}{r^2} + \text{B}(1+2\ln r) + 2\text{C}\right\} - v(1+v)\left\{-\frac{A}{r^2} + \text{B}(3+2\ln r) + 2\text{C}\right\}\right] \text{............ (27)}$$

Combining the second of Eqs.(23) with the second of Eqs.(24) and using Eqs.(22)

$$u = \frac{r}{E}\left[(1-v^2)\left\{-\frac{A}{r^2} + \text{B}(3+2\ln r) + 2\text{C}\right\} - v(1+v)\left\{\frac{A}{r^2} + \text{B}(1+2\ln r) + 2\text{C}\right\}\right]$$

Differentiating w.r.t.r

$$\frac{du}{dr} = \frac{1}{E}\left[(1-v^2)\left\{-\frac{A}{r^2} + \text{B}(3+2\ln r) + 2\text{C}\right\} - v(1+v)\left\{\frac{A}{r^2} + \text{B}(1+2\ln r) + 2\text{C}\right\}\right] + \frac{r}{E}\left[(1-v^2)\left\{\frac{2A}{r^3} + \frac{2B}{r} - v(1+v)\left\{-\frac{2A}{r^3} + \frac{2B}{r}\right\}\right]\right] \text{...................... (28)}$$

Equating Eqs.(27) and (28)

$4\,(1+v)\,B = 0$

$\therefore B = 0$

Equating (22) becomes

$$\left.\begin{aligned} \sigma_r &= \frac{A}{r^2} + 2C \\ \sigma_\theta &= -\frac{A}{r^2} + 2C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(29)}$$

The boundary conditions are

$\sigma_r = -P_i$ **at r=a**

$\sigma_r = -P_\circ$ **at r=b**

Using these B.Cs.

$$\therefore A = \frac{a^2 . b^2 (P_\circ - P_i)}{b^2 - a^2}$$

$$2C = \frac{a^2 . P_i - b^2 . P_\circ}{b^2 - a^2}$$

Hence Eq.(29) becomes

$$\left.\begin{aligned} \sigma_r &= \frac{a^2 . b^2 (P_\circ - P_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{a^2 . P_i - b^2 . P_\circ}{b^2 - a^2} \\ \sigma_\theta &= - \frac{a^2 . b^2 (P_\circ - P_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{a^2 . P_i - b^2 . P_\circ}{b^2 - a^2} \end{aligned}\right\} \qquad \text{.................................. (30)}$$

Substituting Eqs.(30) into Eq.(25)

$$\therefore \sigma_z = \frac{2v\,(a^2 . P_i - b^2 . P_\circ)}{b^2 - a^2} = \text{constant}$$

If $P_i \neq 0$ and $P_\circ = 0$, Eqs.(30) become

$$\left.\begin{aligned}\sigma_r &= \frac{a^2 . P_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \\ \\ \sigma_\theta &= \frac{a^2 . P_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)\end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(31)}$$

From the second of Eqs.(31) and when the thickness t= b –a becomes very small in comparison with the radius

$r_\circ = \frac{a+b}{2}$ **, then**

$$\sigma_\theta = \frac{a^2 . P_i}{b^2 - a^2}\left(1 + \frac{b^2}{a^2}\right) = \frac{r_\circ^{\,2} . p}{(b+a)(b-a)}\left(1 + \frac{b^2}{r_\circ^{\,2}}\right) = \frac{r_\circ^{\,2} . p\,(1+1)}{(2r_\circ) . t} = p . \frac{r_\circ}{t}$$

Line Load Acting on the Free Surface of a Plate
(Boussinesque – Flamant Solution)

For this case thr stress function is assumed to be

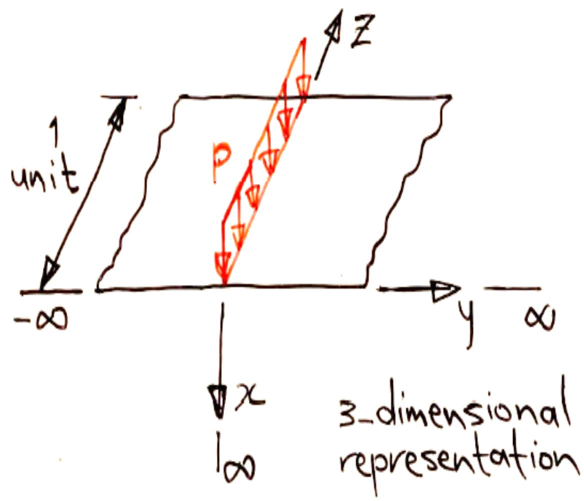
$$\phi\ (r,\theta) = C\,.\,r\,.\,\theta\,.\,Sin\,\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

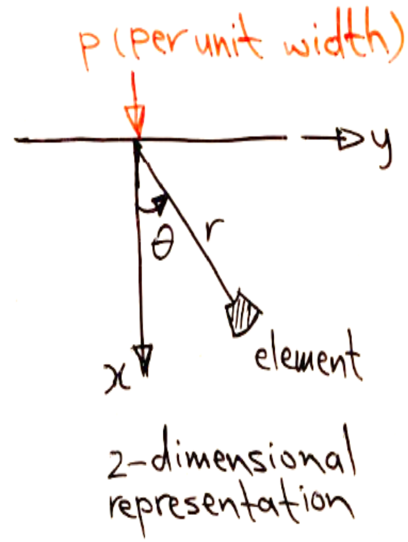
where C is a constant

Equation (1) can be shown to satisfy the compatibility equation

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\ )\ (\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\ ) = 0$$

The components of stress in this case are given by:





$$\left.\begin{array}{l} \sigma_r = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2} = \dfrac{2\,C}{r}\cos\theta \\[2ex] \sigma_\theta = \dfrac{\partial^2 \phi}{\partial r^2} = 0 \\[2ex] \tau_{r\theta} = \dfrac{1}{r^2}\dfrac{\partial \phi}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 \phi}{\partial r\,\partial \theta} = 0 \end{array}\right\} \ldots\ldots\ldots(2)$$

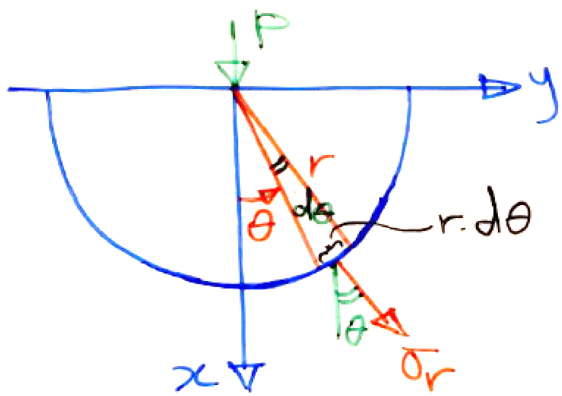
The boundary conditions which must be satisfied are:

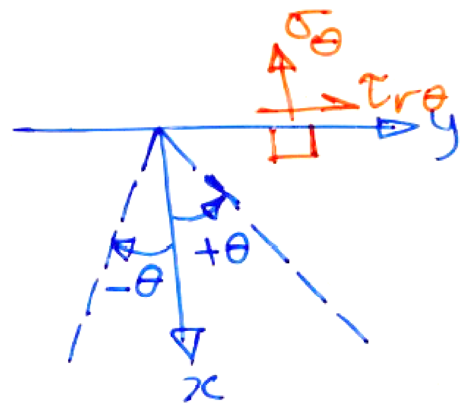
1) for $\theta = \pm\frac{\pi}{2}$ $\quad \sigma_\theta = \tau_{r\theta} = 0$

2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,(\sigma_r \, . \, d_A) = -p$

$$\left.\begin{aligned} &\sigma_\theta = \tau_{r\theta} = 0 \\ &\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\,(\sigma_r \, . \, d_A) = -p \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The first boundary condition is identically satisfied by the last two of Eqs. (2). The second of Eqs.(3) determines the constant C





The second of Eqs. (3) can be obtained by considering the vertical equilibrium of semi –circular portion of constant

r $(\sum f_x = 0)$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \,(\sigma_r \,.\, d_A) = -p$$

$$\therefore -p = 2\int_0^{\frac{\pi}{2}} \cos\theta \left(\frac{2c}{r}\cos\theta . r . d_\theta\right)$$

$$\therefore -p = 4\,C \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\, d_\theta$$

$$\therefore C = -\frac{p}{\pi}$$

$$\therefore \sigma_r = -\frac{2\,p}{\pi}\,\frac{\cos\theta}{r}$$

Stresses in Plate With Circular Holes

Consider an infinite plate subjected to a uniform tensile stress of intensity $\sigma_\circ$ in the x-direction. If there is no hole in the plate the state of stress will be given by

$$(\sigma_x)_1 = \sigma_\circ$$

$$(\sigma_y)_1 = (\tau_{xy})_1 = 0$$

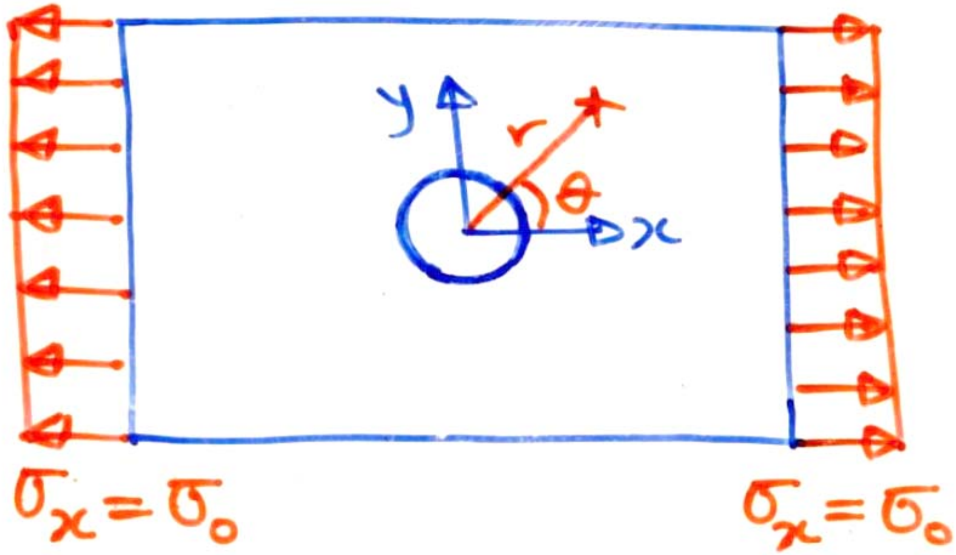
Where the suffix 1 denotes the case of no hole.

This state of stress can be derived from the stress function

$$\phi_1 = \frac{1}{2}\sigma_\circ \, . \, y^2$$

From which

$$\sigma_x = \frac{\partial^2 \phi_1}{\partial y^2} \quad , \sigma_y = \frac{\partial^2 \phi_1}{\partial x^2} \quad , \tau_{xy} = -\frac{\partial^2 \phi_1}{\partial x \, \partial y}$$



In polar coordinates

$$\phi_1 = \frac{1}{2}\sigma_\circ . r^2 . sin^2\theta = \frac{\sigma_\circ}{4} r^2 \text{ (1- cos}2\theta)$$

The corresponding stress components are

$$(\sigma_r)_1 = \frac{1}{r}\frac{\partial\phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_1}{\partial\theta^2} = \frac{\sigma_\circ}{2}\text{ (1+ cos}2\theta)$$

$$(\sigma_\theta)_1 = \frac{\partial^2\phi_1}{\partial r^2} = \frac{\sigma_\circ}{2}\text{ (1- cos}2\theta)$$

$$(\tau_{r\theta})_1 = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\right) = -\frac{\sigma_\circ}{2}\sin 2\theta$$

Suppose that a hole of radius a is drilled through the plate. The new system of stresses σ_r, σ_θ and $\tau_{r\theta}$ must satisfy the following B.CS.

1) $\sigma_r = 0$ at r=a (for all θ)

2) $\tau_{r\theta} = 0$ at r=a (for all θ)

3) When r= ∞ , $\sigma_r = (\sigma_r)_1$, $\sigma_\theta = (\sigma_\theta)_1$, $\tau_{r\theta} = (\tau_{r\theta})_1$

The stresses must be derived from a stress function ϕ which must satisfy the compatibility equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}\right)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Assume this unknown stress function ϕ to be composed of a function $f_1(r)$ plus a function $f_2(r)$ multiplied by $\cos 2\theta$,

thus

$$\phi(r,\theta) = f_1(r) + f_2(r) \,.\, \mathrm{Cos}2\theta$$

Substituting ϕ into Eq.(1)

$$\therefore \left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2 f_1}{dr^2}+\frac{1}{r}\frac{df_1}{dr}\right]+\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right]\left[\frac{d^2 f_2}{dr^2}+\frac{1}{r}\frac{df_2}{dr}-\frac{4f_2}{r^2}\right]\cos 2\theta = 0$$

$$\left.\begin{aligned} &\therefore \left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2 f_1}{dr^2}+\frac{1}{r}\frac{df_1}{dr}\right]=0 \\ &\text{And} \\ &\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{4}{r^2}\right]\left[\frac{d^2 f_2}{dr^2}+\frac{1}{r}\frac{df_2}{dr}-\frac{4f_2}{r^2}\right]=0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The solutions of Eqs.(2) are

$$f_1(r) = C_1 . r^2 . \ln r + C_2 . r^2 + C_3 . \ln r + C_4$$

$$f_2(r) = C_5 . r^2 + C_6 . r^4 + C_7 . \frac{1}{r^2} + C_8$$

These solutions may be obtained by two methods:

1) by rewriting Eqs.(2) in the following forms

$$\frac{1}{r}\frac{d}{dr}\{ r . \frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(r . \frac{df_1}{dr})]\} = 0$$

$$r . \frac{d}{dr}(\frac{1}{r^3} . \frac{d}{dr}\{r^3 . \frac{d}{dr}[\frac{1}{r^3}\frac{d}{dr}(r^2 . f_2)]\}) = 0$$

And by integrating the above.

2) by expanding Eqs.(2), setting t= ln r , and thereby transforming the resulting expressions into two ordinary differential equations with constant coefficients.

The resulting stress function ϕ (r , θ) is

$$\phi = [C_1 . r^2 . \ln r + C_2 . r^2 + C_3 . \ln r + C_4] + [C_5 . r^2 + C_6 . r^4 + C_7 . \frac{1}{r^2} + C_8] \cos 2\theta$$

The stresses will be

$$\sigma_r = C_1 (1 + 2 . \ln r) + 2C_2 + \frac{C_3}{r^2} - (2 C_5 + \frac{C_7}{r^4} + \frac{4C_8}{r^2}) \cos 2\theta$$

$$\sigma_\theta = C_1 (3 + 2 . \ln r) + 2C_2 - \frac{C_3}{r^2} +(2 C_5 + 12 C_6 . r^2 + \frac{6 C_7}{r^4}) \cos 2\theta$$

$$\tau_{r\theta} = (2 C_5 + 6 C_6 . r^2 - \frac{6 C_7}{r^4} - \frac{2 C_8}{r^2}) \sin 2\theta$$

Where C_1 to C_8 are constants to be determined

1) **The absence of C_4 in the expressions of stresses indicates that it has no influence upon the solution. Thus $C_4 = 0$ may be used.**

2) **Since, for $r \rightarrow \infty$, σ_r, σ_θ , and $\tau_{r\theta}$ must be finite { or $\sigma_r \rightarrow (\sigma_r)_1$, $\sigma_\theta \rightarrow (\sigma_\theta)_1$, and $\tau_{r\theta} \rightarrow (\tau_{r\theta})_1$ }, it follows that $C_1 = C_6 = 0$**

3) Also at $r \to \infty$, $\sigma_r \to (\sigma_r)_1$, thus $2\,C_2 - 2\,C_5 . \cos 2\theta = \frac{\sigma_\circ}{2} + \frac{\sigma_\circ}{2} \cos 2\theta$

$\therefore 2\,C_2 = \frac{\sigma_\circ}{2}$ and $-2\,C_5 = \frac{\sigma_\circ}{2}$

$\sigma_\theta = (\sigma_\theta)_1$, thus

$2\,C_2 + 2\,C_5 . \cos 2\theta = \frac{\sigma_\circ}{2} - \frac{\sigma_\circ}{2} \cos 2\theta$

$\therefore 2\,C_2 = \frac{\sigma_\circ}{2}$ and $2\,C_5 = -\frac{\sigma_\circ}{2}$ as before

$\tau_{r\theta} \to (\tau_{r\theta})_1$, thus

$2\,C_5 . \mathrm{Sin} 2\theta = -\frac{\sigma_\circ}{2} . \mathrm{Sin} 2\theta$

$2\,C_5 = -\frac{\sigma_\circ}{2}$ as before

$C_2 = \frac{\sigma_\circ}{4}$ and $C_5 = -\frac{\sigma_\circ}{4}$

4) **To find C_3 , C_7 and C_8 use**

$$\left.\begin{array}{l}\sigma_r = 0 \\ \\ \tau_{r\theta} = 0\end{array}\right\}\quad \text{at } r=a$$

$$\therefore 0 = 2\,C_2 + \frac{C_3}{a^2} - (2\,C_5 + 6\,\frac{C_7}{a^4} + \frac{4\,C_8}{a^2})\cos 2\theta$$

$$\therefore 2\,C_2 + \frac{C_3}{a^2} = 0 \qquad \text{and} \qquad 2\,C_5 + 6\,\frac{C_7}{a^4} + \frac{4\,C_8}{a^2} = 0$$

and for $\tau_{r\theta} = 0$

$$0 = (2\,C_5 + 6\,\frac{C_7}{a^4} + \frac{4\,C_8}{a^2})\sin 2\theta$$

$$\therefore 2\,C_5 - 6\,\frac{C_7}{a^4} - \frac{2\,C_8}{a^2} = 0$$

Hence $C_3 = -\frac{a^2\,\sigma_\circ}{2}$, $C_7 = -\frac{a^4\,\sigma_\circ}{2}$, $C_8 = \frac{a^2\,\sigma_\circ}{2}$

Substituting , then

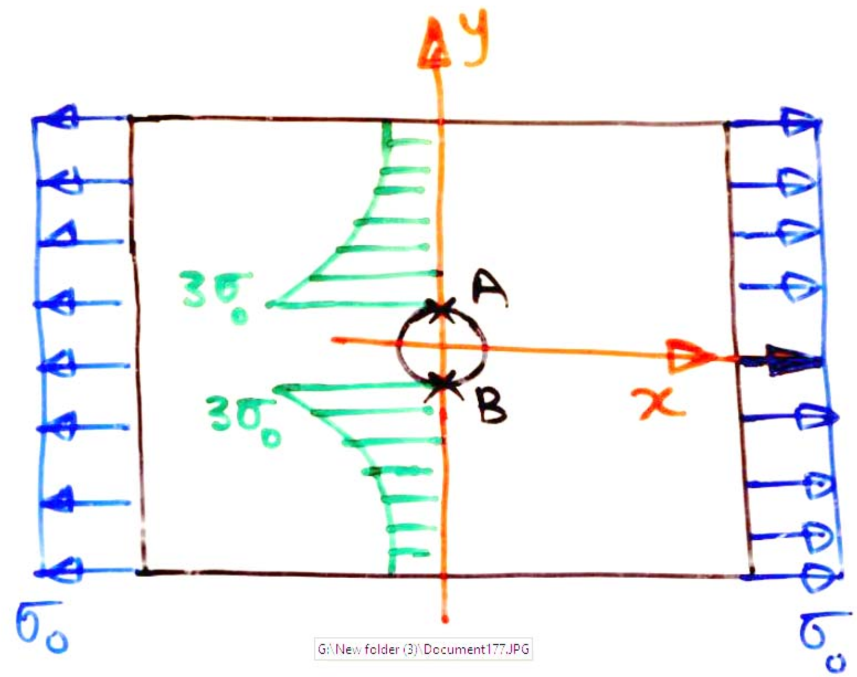
$$\sigma_\theta = \frac{\sigma_\circ}{2} \; [(1+ \frac{a^2}{r^2}) - (1+ \frac{3\,a^4}{r^4}) \cos 2\theta]$$

$$\therefore (\sigma_\theta)_{r=a} = \frac{\sigma_\circ}{2} \; [(2) - (4) \cos 2\theta] = \sigma_\circ - 2\,\sigma_\circ \,.\, \mathrm{Cos}2\theta$$

From which $(\sigma_\theta)_{max} = 3\,\sigma_\circ$ **for** $\theta = \begin{cases} \frac{\pi}{2} \\ \\ \frac{3\pi}{2} \end{cases}$

Thus due to the presence of the hole one has at the two points A and B a stress concentration, i.e., the stress is increased three times its average value. This phenomenon is localized near the hole. In fact

$$(\sigma_\theta)_{\theta=\frac{\pi}{2},\frac{3\pi}{2}} = \frac{\sigma_\circ}{2}\left[\,2+\frac{a^2}{r^2}+\frac{3\,a^4}{r^4}\,\right]\approx \sigma_\circ \quad \text{for} \quad r > 10\,a$$



The stress concentration factor, defined as the ratio of

maximum stress $(\sigma_\theta)_{max}$ at the hole to the nominal stress $\sigma_\circ$ is

$$K = \frac{3\,\sigma_\circ}{\sigma_\circ} = 3$$

Coulomb's Theory of Torsion of a Circular Shaft

L = length of bar

R = radius

J = polar moment of inertia

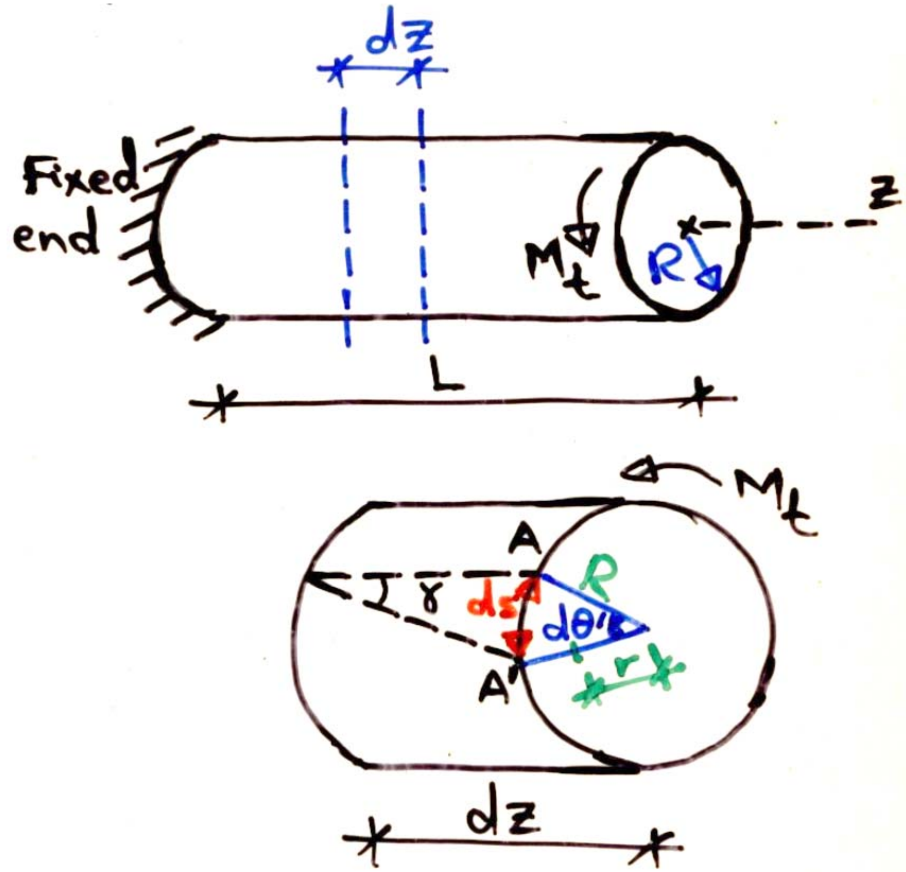
M_t = applied torque

θ' = angle of twist

γ = shearing strain

$d_s = \gamma \cdot d_z = \text{R} \cdot d_{\theta'}$

$\therefore \gamma = \text{R} \cdot \frac{d_{\theta'}}{d_z}$



Torsion

Since $\tau = G \cdot \gamma$

where G = shear modulus of elasticity

$\therefore \tau_R = G \cdot R \cdot \frac{d_{\theta'}}{d_z}$ **(shear stress at distance R from the shaft axis)**

Similarly

$\tau_r = G \cdot r \cdot \frac{d_{\theta'}}{d_z}$ **(shear stress at distance r from the shaft axis)**

$$\therefore \frac{\tau_r}{\tau_R} = \frac{r}{R}$$

Or

$$\tau_r = \frac{r}{R}\ \tau_R$$

The condition of equilibrium between torque M_t and the internal moment is

$$M_t = \int_0^R \tau_r . r . d_A = \int_0^R (G . r . \frac{d_{\theta'}}{d_z}) . \mathrm{r} . (2\pi\mathrm{r} . d_r)$$

$$= 2\pi\mathrm{G} . \frac{d_{\theta'}}{d_z} \int_0^R r^3 . d_r$$

Since

$$\mathrm{J} = 2\pi \int_0^R r^3 . d_r$$

$$= \frac{\pi R^4}{2} \quad \text{(polar moment of inertia)}$$

$$\therefore M_t = \mathrm{G} . \frac{d_{\theta'}}{d_z} . \mathrm{J}$$

Or

$$\frac{M_t}{J} = \mathrm{G} . \frac{d_{\theta'}}{d_z}$$

$\therefore \tau_R = \frac{M_t}{J} . R$ **and** $\tau_r = \frac{M_t}{J} . R$

Also

$$d_{\theta'} = \frac{M_t}{G.J} . d_z$$

$$\therefore \theta' = \int d_{\theta'} = \int_0^L \frac{M_t}{G.J} . d_z = \frac{M_t}{G.J} \int_0^L d_z = \frac{M_t . L}{G.J}$$

$$\therefore \boxed{\frac{M_t}{J} = \frac{G . \theta'}{L} = \frac{\tau_r}{r}}$$

In this theory it is assumed that plane cross sections remain plane after twist .

Torsion

Navier's Theory of Torsion

According to Naviers theory the components of the elastic displacement are

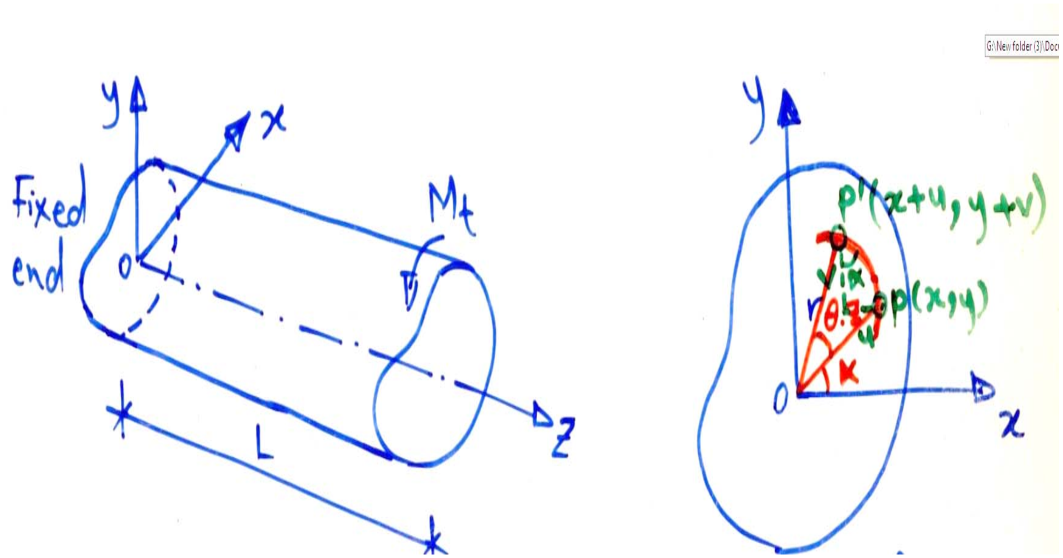
$u = -pp' \sin\alpha = -(\theta\,.\,z\,.\,r)\sin\alpha = -\theta\,.\,z\,.\,(r\sin\alpha)$

$= \theta\,.\,z\,.\,y$ **(in X – direction)**

$v = -pp' \cos\alpha = -(\theta\,.\,z\,.\,r)\cos\alpha = -\theta\,.\,z\,.\,(r\cos\alpha)$

$= \theta\,.\,z\,.\,x$ **(in Y – direction)**

$w = 0$ **(in Z – direction)**



Where the angle of twist at a distance z from the

fixed end is θ . z (θ is the angle of twist per unit length)

The strains will be

Hence $\tau_{rz} = 0$ and $\tau_{\theta z} = G \cdot \theta \cdot r$

By computing the resultant forces and moments on a cross sectional area of the sided:

1) **For the resultant forces**

$\iint_{\Omega} \tau_{xz} \cdot d_x \cdot d_y = -G\theta \iint_{\Omega} y \cdot d_x \cdot d_y = 0$ **(in X - direction)**

$\iint_{\Omega} \tau_{yz} \cdot d_x \cdot d_y = -G\theta \iint_{\Omega} x \cdot d_x \cdot d_y = 0$ **(in Y - direction)**

$\iint_{\Omega} \sigma_{z} \cdot d_x \cdot d_y = 0$ **(in Z - direction)**

2) **For the resultant moments**

$\iint_{\Omega} y \cdot \sigma_{z} \, d_x \cdot d_y = 0$ **(about X - direction)**

$\iint_{\Omega} x \cdot \sigma_{z} \, d_x \cdot d_y = 0$ **(about Y - direction)**

$\iint_{\Omega} (x \cdot \tau_{yz} - y \cdot \tau_{xz}) \, d_x \cdot d_y = G\theta \iint_{\Omega} (x^2 + y^2) \, d_x \cdot d_y = M_t$ **(applied torque)**

Calling J the polar moment of inertia of the cross section where

$$J = \iint_{\Omega} (x^2 + y^2)\, d_x \,.\, d_y$$

Then

$$\theta = \frac{M_t}{G.J}$$

The weak point in Navier's theory lies in the fact that the boundary conditions are not satisfied, i.e.

$$\bar{x} = \sigma_x \,.\, \text{Cos(n x)} + \tau_{xy} \,.\, \text{Cos(n y)} + \tau_{xz} \,.\, \text{Cos(n z)}$$

$$\bar{y} = \tau_{yx} .\, \text{Cos(n x)} + \sigma_y \,.\, \text{Cos(n y)} + \tau_{yz} \,.\, \text{Cos(n z)}$$

$$\bar{z} = \tau_{zx} .\, \text{Cos(n x)} + \tau_{zy} \,.\, \text{Cos(n y)} + \sigma_z \,.\, \text{Cos(n z)}$$

For the present case

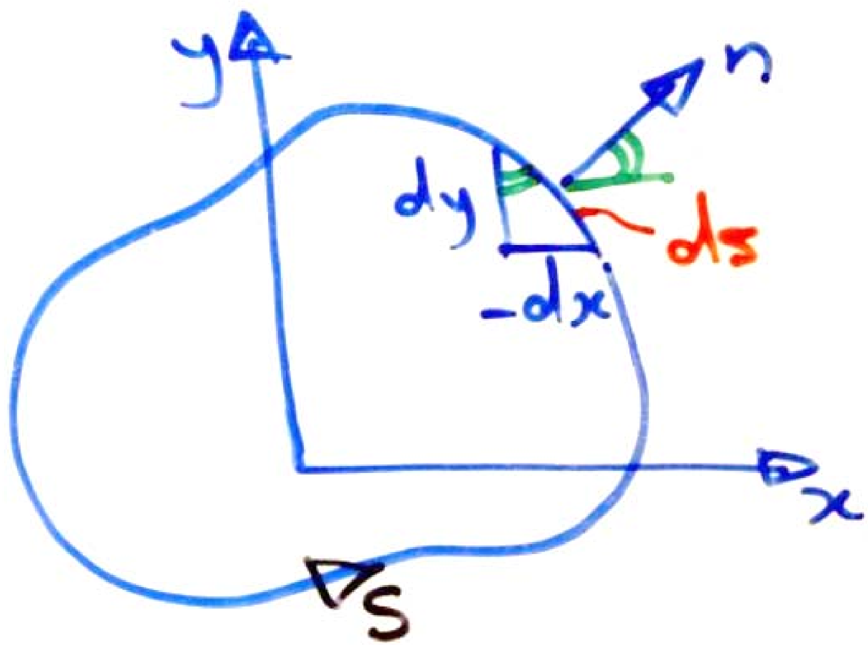
$$\bar{x} = \bar{y} = \bar{z} = \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \cos\text{ (n z)} = 0$$

Hence the first two B.Cs. are identically satisfied, while the third becomes

τ_{xz} . Cos (n x) + τ_{yz} . Cos (n y) = 0

Substituting for τ_{xz} and τ_{yz}

∴ - G . θ . y . Cos (n x) + G . θ . x . Cos (n y) = 0



But

Cos (n x) = $\frac{d_y}{d_s}$ and Cos (n y) = - $\frac{d_x}{d_s}$

$$\therefore -G.\theta.y.\frac{d_y}{d_s} - G.\theta.x.\frac{d_x}{d_s} = 0$$

Or y . d_y + x . d_x = 0

Or $x^2 + y^2$= const.

This equation shows that with the exception of the cases in which the cross section of the bar is a circle or two concentric circles, the boundary conditions are not satisfied by Navier's theory.

Saint – Venant's Semi – Inverse Method

Saint – Venant assumed for the components of the elastic displacement, the following expressions:

$$\left.\begin{aligned} u &= -\theta \, . \, z \, . \, y \\ v &= \theta \, . \, z \, . \, X \\ w &= \theta \, . \, \phi \, (x, y) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where ϕ (x , y) is a function of the variables x and y . It is called the warping function.

The strains will be

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0$$

$$\gamma_{zy} = \theta \left(x + \frac{\partial \phi}{\partial y} \right) \qquad , \gamma_{zx} = \theta \left(-y + \frac{\partial \phi}{\partial x} \right)$$

The components of stress are:

$$\left.\begin{aligned}
&\boldsymbol{\sigma_x} = \boldsymbol{\sigma_y} = \boldsymbol{\sigma_x} = \boldsymbol{\tau_{xy}} = 0\\
&\boldsymbol{\tau_{zy}} = G\,\theta\,\left(x + \frac{\partial \phi}{\partial y} \right)\\
&\boldsymbol{\tau_{zx}} = G\,\theta\,\left(- y + \frac{\partial \phi}{\partial x} \right)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting Eqs.(2) into the equations of equilibrium (the body forces are zero), the first two equations are identically satisfied and the third equation becomes

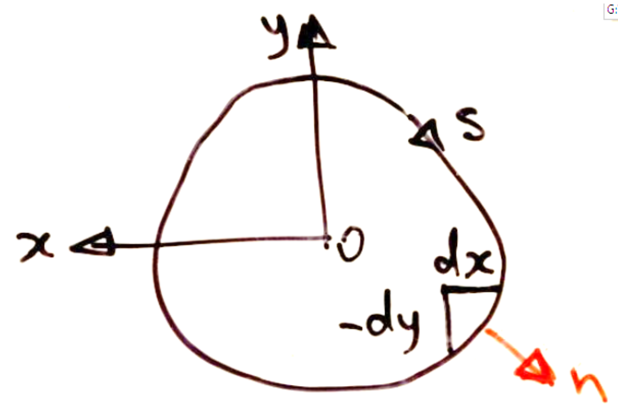
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Substituting Eqs.(2) into the boundary conditions equations (where $\overline{x} = \overline{y} = \overline{z}$ = Cos(n z) =0), the first two equations are identically satisfied, while the third equation becomes

$$\boldsymbol{\tau_{zx}} \cdot \mathrm{Cos}(n\ x) + \boldsymbol{\tau_{zy}} \cdot \mathrm{Cos}(n\ y) = 0 \ \ldots\ldots\ldots(4)$$

Using $\quad \text{Cos(n x)} = -\dfrac{dy}{ds}$

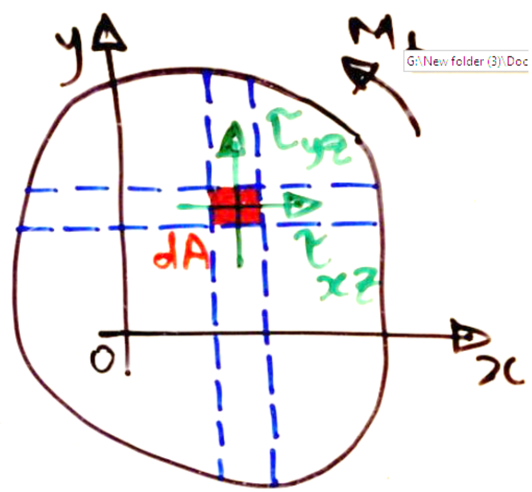
$$\text{Cos(n y)} = \frac{dx}{ds}$$



and substituting for τ_{zx} and τ_{zy} from Eqs. (2), Eq. (4) becomes

$$\left(\frac{\partial\phi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\phi}{\partial y} + x\right)\frac{dx}{ds} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5)$$

According to Saint – Venant's theory, the torsion problem is reduced to the problem of finding the function ϕ (x , y) satisfying Eq.(3) and the B.C equation (Eq.5).



The resultant from twisting moment is obtained from

$$M_t = \iint_\Omega (x.\tau_{yz} - y\,.\tau_{xz})\, d_x\,.\,d_y$$

$$= G\,\theta \iint_\Omega \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right) d_x\,.\,d_y$$

The integral $J = \iint_\Omega \left(x^2 + y^2 + x\dfrac{\partial\phi}{\partial y} - y\dfrac{\partial\phi}{\partial x}\right) d_x\,.\,d_y$ **is called the torsional con**

Example:

For the case where the cross section of the shaft is an ellise which has semi – axes a and b , assume the warping function as

$$\phi(x, y) = \frac{b^2 - a^2}{a^2 + b^2}\ x \cdot y$$

This function satisfies Eq.(3).

Substituting ϕ into Eq.(5), then

$$\left[\frac{b^2 - a^2}{a^2 + b^2} - 1\right] y \cdot \frac{dy}{ds} - \left[\frac{b^2 - a^2}{a^2 + b^2} + 1\right] x \cdot \frac{dx}{ds} = 0$$

$$\therefore \frac{-2\,a^2}{a^2 + b^2} \cdot y \cdot \frac{dy}{ds} - \frac{2\,b^2}{a^2 + b^2} \cdot x \cdot \frac{dx}{ds} = 0$$

$$\therefore a^2 .\ y \cdot \frac{dy}{ds} + b^2 .\ x \cdot \frac{dx}{ds} = 0$$

$$\therefore \frac{a^2}{2}\frac{d}{ds}(y^2) + \frac{b^2}{2}\frac{d}{ds}(x^2) = 0 \qquad \therefore a^2 .\ d(y^2) + b^2 .\ d(x^2) = 0$$

$$\therefore a^2 .\ (y^2) + b^2 .\ (x^2) = C$$

Dividing by ($a^2 \, . \, b^2$) and letting $\frac{c}{a^2 \, . \, b^2} = 1$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the equation of an ellipse .

The torsional constant for the bar is

$$J = \iint_\Omega \left(x^2 + y^2 + x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \right) d_x \, . \, d_y$$

$$= \iint_\Omega \left(x^2 + y^2 + \frac{b^2 - a^2}{a^2 + b^2} \, . \, x^2 - \frac{b^2 - a^2}{a^2 + b^2} \, . \, y^2 \right) d_x \, . \, d_y$$

$$= \frac{2 \, b^2}{a^2 + b^2} I_y + \frac{2 \, a^2}{a^2 + b^2} I_x \quad = \frac{\pi \, a^2 \, b^3}{a^2 + b^2}$$

where $I_x = \frac{\pi \, a \, b^3}{4}$ and $I_y = \frac{\pi \, b \, a^3}{4}$

The angle of twist per unit length is

$$\theta = \frac{M_t}{G.J} = \frac{M_t\,(a^2+b^2)}{G\,.\pi\,.a^3.b^3}$$

The components of elastic displacement are

$$u = -\theta\,.\,z\,.\,y = -\frac{M_t\,(a^2+b^2)}{G\,.\pi\,.a^3.b^3}\;z\,.\,y$$

$$v = \theta\,.\,z\,.\,x = \frac{M_t\,(a^2+b^2)}{G\,.\pi\,.a^3.b^3}\;z\,.\,x$$

$$w = \theta\;.\,\phi = \frac{M_t\,(b^2-a^2)}{G\,.\pi\,.a^3.b^3}\;x\,.\,y$$

The stresses are

$$\tau_{yz} = G\,\theta\,(\,x + \frac{\partial\phi}{\partial y}\,) = \frac{2\,M_t\;.x}{\pi\,.a^3.b}$$

$$\tau_{xz} = G\,\theta\,(\,-\,y + \frac{\partial\phi}{\partial x}\,) = -\,\frac{2\,M_t\;.y}{\pi\,.b^3.a}$$

Prandtl's Theory

The components of the elastic displacements are

$$\left.\begin{aligned} u &= -\theta \, . \, z \, . \, y \\ v &= \theta \, . \, z \, . \, X \\ w &= \theta \, . \, \phi \, (x, y) \end{aligned}\right\} \text{........................ (1)}$$

The components of stress are

$$\left.\begin{aligned} &\sigma_x = \sigma_y = \sigma_x = \tau_{xy} = 0 \\ &\tau_{yz} = G\,\theta \left(x + \frac{\partial \phi}{\partial y} \right) \\ &\tau_{xz} = G\,\theta \left(-y + \frac{\partial \phi}{\partial x} \right) \end{aligned}\right\} \text{................(2)}$$

Substituting Eqs.(2) into the equations of equilibrium, then

$$\left.\begin{aligned} &\frac{\partial \tau_{xz}}{\partial z} = 0 \\ &\frac{\partial \tau_{yz}}{\partial z} = 0 \\ &\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3)$$

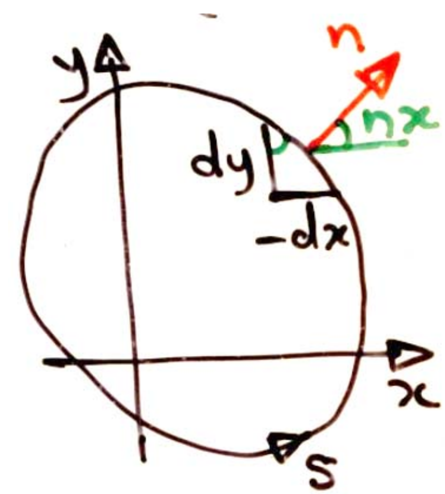
The first two of Eqs(3) are satisfied since τ_{xz} and τ_{yz} are independent of z.

In order to satisfy the third of Eqs.(3), Prandtl introduces a stress function ψ (x ,y) such that

$$\tau_{xz} = \frac{\partial \psi}{\partial y} \quad , \tau_{yz} = - \frac{\partial \psi}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\left.\begin{aligned} &\therefore \tau_{xz} = \frac{\partial \psi}{\partial y} = G\,\theta\left(- y + \frac{\partial \phi}{\partial x} \right) \\ &\tau_{yz} = - \frac{\partial \psi}{\partial x} = G\,\theta\left(x + \frac{\partial \phi}{\partial y} \right) \end{aligned}\right\} \quad \ldots\ldots\ldots (5)$$

Differentiating the first of Eqs. (5) w.r.t.y, the second w.r.t.x, and subtracting the first from the second, then



$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2G.\ \theta \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (6)$$

Considering the B.Cs. equations (as in Saint – Venant theory), then

$$\tau_{xz}\ .\ \mathrm{Cos}\ (n\ x) + \tau_{yz}\ .\ \mathrm{Cos}\ (n\ y) = 0$$

$$\therefore \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds} = 0 \qquad\qquad \therefore \frac{\partial\psi}{\partial s} = 0 \ \ldots\ldots\ (7)$$

Eq.(7) shows that the stress function ψ (x ,y) must be constant along the boundary of the cross section. The value of ψ at the boundary can be chosen arbitrarily. It is usually conveniently made equal to zero, i.e., ψ (x ,y) = 0 on the boundary.

According to Prandtl's theory, the problem is reduced to finding the function ψ (x ,y) which satisfies Eq.(6) and which is zero along the boundary.

The conditions at the end surface are:

Cos (n x) = Cos (n y) = 0 , **Cos (n z) = ± 1**

$\boldsymbol{\sigma_x = \sigma_y = \sigma_x = \tau_{xy} = 0}$

The boundary conditions become

$$\left.\begin{aligned} \bar{x} &= \pm\,\tau_{xz} \\ \bar{y} &= \pm\,\tau_{yz} \\ \bar{z} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

Substituting Eqs.(4) into Eqs.(8) and computing the resultant force, then

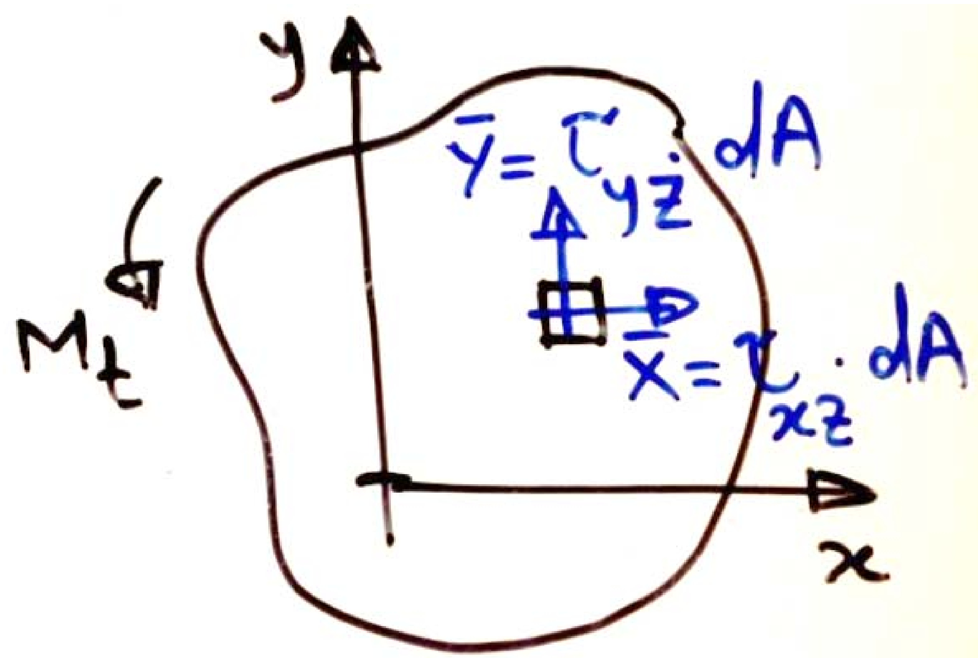
$$\iint_{\Omega} \bar{x}.\,d_x\,.\,d_y = \iint_{\Omega} \tau_{xz.\,d_x\,.\,d_y} = \iint_{\Omega} \frac{\partial\psi}{\partial y}.\,d_x\,.\,d_y$$

$$= \int [\int \frac{\partial\psi}{\partial y}.\,d_y].\,d_x = \int [\psi\,]_{y_1}^{y_2}.\,d_x = 0$$

Since ψ is zero at the boundary of the cross section. Also

$$\iint_{\Omega} \bar{Y}.\, d_x \,.\, d_y = \iint_{\Omega} \tau_{yz} \,.\, d_x \,.\, d_y = -\iint_{\Omega} \frac{\partial\psi}{\partial x}.\, d_x \,.\, d_y$$

$$= -\int [\int \frac{\partial\psi}{\partial x}.\, d_x].\, d_y = -\int [\psi]_{x_1}^{x_2}.\, d_y = 0$$



Calling M_t the couple acting on a free end,

$$M_t = \iint_{\Omega} (\bar{Y}.\, x - \bar{X}\,.\, y)\, d_x \,.\, d_y$$

$$= \iint_{\Omega} x \,.\, \tau_{yz} \,.\, d_x \,.\, d_y - \iint_{\Omega} y \,.\, \tau_{xz} \,.\, d_x \,.\, d_y$$

But $\iint_{\Omega} x \,.\, \tau_{yz} \,.\, d_x \,.\, d_y = -\iint_{\Omega} x \,.\, \frac{\partial\psi}{\partial x}.\, d_x \,.\, d_y$

$$= -\int [x.\psi]_{x_1}^{x_2} - \int_{x_1}^{x_2} \psi \,.\, d_x]d_y = \iint_{\Omega} \psi \,.\, d_x \,.\, d_y$$

Similarly

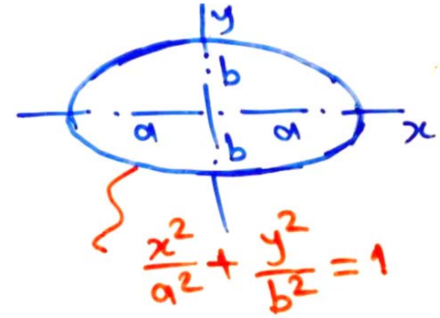
$$-\iint_{\Omega} y \,.\, \tau_{xz} \,.\, d_x \,.\, d_y = -\iint_{\Omega} y \,.\, \frac{\partial \psi}{\partial y} \,.\, d_x \,.\, d_y$$

$$= - \int [\, y \,.\, \psi \,]_{y_1}^{y_2} - \int_{y_1}^{y_2} \psi \,.\, d_y \,] d_x = \iint_{\Omega} \psi \,.\, d_x \,.\, d_y$$

$$\therefore M_t = 2\iint_{\Omega} \psi\,(x, y) \,.\, d_x \,.\, d_y$$

Example

Torsion of a bar with an elliptical cross section. Assume the stress function



$$\psi\,(x, y) = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

where m is a constant to be determined. The function ψ satisfies the B.C. $\psi = 0$. Substituting ψ into Eq.(6)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2G\,\theta$$

$$\therefore 2m\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -2G\,\theta$$

$$\therefore\ m = \frac{a^2 .\ b^2}{2\,(a^2+b^2)}\ F \qquad \text{where } F = -2G\,\theta$$

The magnitude of the constant F is determined by

$$M_t = 2\iint_\Omega \psi\ .\ d_x \,.\, d_y$$

$$= \frac{F\,.\,a^2 .\ b^2}{a^2+b^2}\left[\ \frac{1}{a^2}\iint_\Omega x^2 .\ d_x \,.\, d_y + \frac{1}{b^2}\iint_\Omega y^2 .\ d_x \,.\, d_y - \iint_\Omega\ d_x \,.\, d_y\right.$$

But

$$\iint_{\Omega} x^2.\ \ d_x\,.\,d_y = I_y = \frac{\pi\, b\, a^3}{4}$$

$$\iint_{\Omega} y^2.\ \ d_x\,.\,d_y = I_x = \frac{\pi\, a\, b^3}{4}$$

$$\iint_{\Omega}\ \ d_x\,.\,d_y = \text{Area} = \pi\ .\ a\ .b$$

$$\therefore M_t = -\frac{\pi\,.\,a^3.\ b^3.F}{2\,(\,a^2+b^2)}$$

$$\therefore F = -\frac{2\,M_t(\,a^2+b^2)}{\pi\,.\,a^3.\ b^3}$$

$$\therefore \psi = -\frac{M_t}{\pi\,.a.b}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

The stress components are

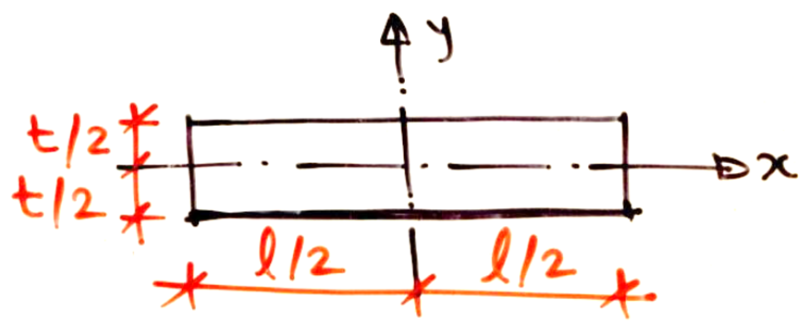
$$\tau_{xz} = \frac{\partial\psi}{\partial y} = -\frac{2\,M_t\,y}{\pi\,.a\,.\,b^3} \qquad \tau_{yz} = -\frac{\partial\psi}{\partial x} = \frac{2\,M_t\,x}{\pi\,.b\,.\,a^3}$$

The angle of twist per unit length is

$$\theta = \frac{-1}{2\,G}\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] = M_t \cdot \frac{a^2+b^2}{\pi\,.\,a^3.\;b^3\,.G}$$

Example

Narrow rectangular section. Since the thickness t is small, then the only shear stress in the section will be τ_{xz} . That is τ_{yz} is nearly zero (or taken zero here)

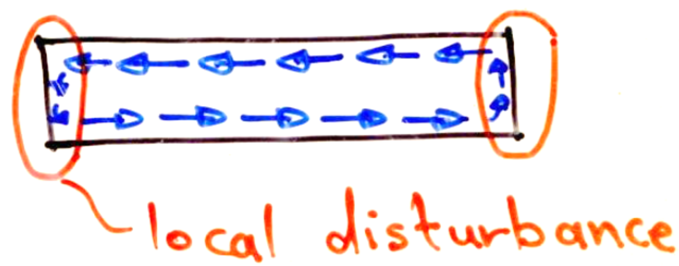


but $\tau_{yz} = -\frac{\partial\psi}{\partial x} = 0$

Hence $\frac{\partial\psi}{\partial x} = 0$

Thus $\psi = \psi\,(y)$

Trying a parabolic expression



$\psi = \{(\frac{t}{2})^2 - y^2\}\,.\,m$ **where m is a constant**

This function give $\psi = 0$ on the boundary $y = \pm\,\frac{t}{2}$

Substituting into $\nabla^2 \psi = -2G\theta$ **, then**

$-2m = -2G\theta$

$\therefore m = G\theta$

Hence $\psi = G\theta\{(\frac{t}{2})^2 - y^2\}$

If M_t **is the applied torque, then**

$$M_t = 2\iint_{\Omega} \psi \,.\, d_x \,.\, d_y$$

$$= 2\int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} G\theta\{(\tfrac{t}{2})^2 - y^2\} \,.\, d_x \,.\, d_y$$

$$= 2G\theta\left[(\tfrac{t}{2})^2 \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} d_x \,.\, d_y - \int_{-\frac{t}{2}}^{\frac{t}{2}} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} y^2 \,.\, d_x \,.\, d_y\right]$$

$$= 2\, G\, \theta\, [\, (\tfrac{t}{2})^2 \,.\, A - I_x \,] = 2\, G\, \theta\, [\, (\tfrac{t}{2})^2 \,.\ell\, .t - \frac{\ell\, t^3}{12}\,]$$

$$= G\, \theta\, .\, \frac{\ell\, t^3}{3}$$

The shear stress $\tau_{xz} = \frac{\partial \psi}{\partial y} = G\, \theta\, (-2\, y) = -2\, G.\, \theta\, .\; Y$

But

$$G\, \theta = \frac{3 M_t}{\ell\, t^3}$$

$$\therefore \tau_{xz} = -6\, \frac{M_t}{\ell\, t^3}\, .\, y$$

Hence $(\tau_{xz})_{max} = -6\, \frac{M_t}{\ell\, t^3}\, (\pm\, \frac{t}{2}) = \pm\, \frac{3 M_t}{\ell\, t^2}$

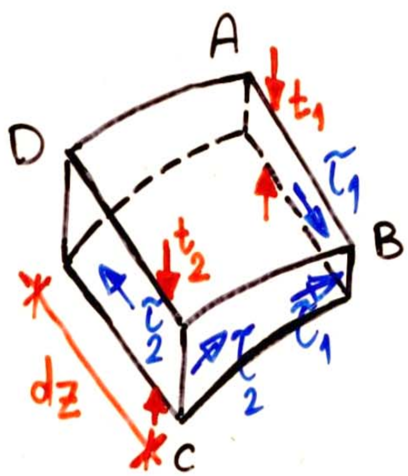
The torsional stiffness $k = \frac{M_t}{\theta} = \frac{G\, .\ell . t^3}{3}$

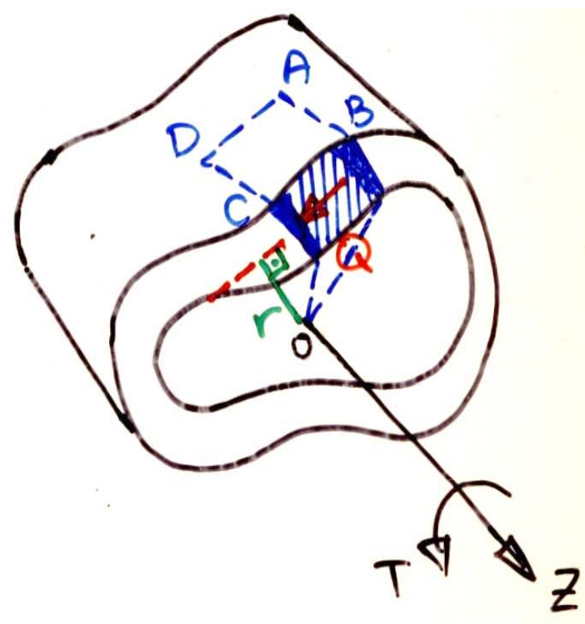
Torsion of Thin – Walled Tubes

Bredt – Batho Theory

Consider the thin – walled closed tube subjected to a torque T about the Z – axis

If τ_1 is the shear stress at B and τ_2 is the shear stress at C (where the thickness has increased to t_2) then from the equilibrium of the complementary shears on the sides AB and CD of the element





$$\tau_1 \,.\, t_1 \,.\, d_z = \tau_2 \,.\, t_2 \,.\, d_z$$

$$\therefore \tau_1 \,.\, t_1 = \tau_2 \,.\, t_2$$

i.e., the product of the shear stress and the thickness is constant at all points on the periphery of the tube. This constant is termed the shear flow and denoted by the symbol q (shear force per unit length). Thus

q = τ . t = constant …………… (1)

At any point, the shear force Q on an element of length d_s is Q = q . d_s and the shear stress is $\frac{q}{t}$.

The moment of force Q about o is d_T

d_T = Q . r

∴ d_T = q . d_s . r

∴ the moment, or torque, for the whole section T

T = $\int d_T = \int q \,.\, r.\, d_s$ = q $\int r.\, d_s$

But the area COB = $\frac{1}{2}$ * base * height = $\frac{1}{2}$ r . d_s

i.e., $d_A = \frac{1}{2}$ r . d_s

Or

$2\ d_A = r\ .\ d_s$

$\therefore T = q \int 2\ .\ d_A$

$\therefore T = 2\ q\ A$ …………… (2)

Where A is the area enclosed within the median line of the wall thickness.

Since $q = \tau\ .\ t$

$\therefore T = 2\ \tau\ .\ t\ .\ A$

Or $\tau = \frac{T}{2\ A\ .\ t}$ ……....(3)

Where t is the thickness at the point in question.

Consider an axial strip of the tube, of length ℓ, along which the thickness and hence the shear stress is constant. The strain energy per unit volume is

$$U' = \frac{1}{2}\,\tau\,.\,\gamma = \frac{\tau^2}{2\,G}$$

Total energy in the tube

$$U = \int U'.\,d_V = \int \frac{\tau^2}{2\,G}.\,d_V = \int \frac{\tau^2}{2\,G}.\,t\,.\,\ell.\,d_S = \int \left(\frac{T}{2\,A\,.\,t}\right)^2 .\,\frac{t\,.\,\ell}{2\,G}.\,d_S$$

$$= \frac{T^2\,\ell}{8\,A^2\,.G}\int \frac{d_S}{t}$$

Where the integration is along the perimeter of the cell.

The external work done by T is

$$W = \frac{1}{2}\,.\,T\,.\,\theta' \qquad \text{where } \theta' \text{ is the angle of twist}$$

Hence

$$\frac{1}{2}\,.\,T\,.\,\theta' = \frac{T^2\,\ell}{8\,A^2\,.G}\int \frac{d_S}{t}$$

$$\therefore\ \theta' = \frac{\boldsymbol{T\,\ell}}{\boldsymbol{4\,A^2\,.G}} \int \frac{\boldsymbol{d_S}}{\boldsymbol{t}}$$

Or

$$\theta = \frac{\theta'}{\ell} = \frac{\boldsymbol{T}}{\boldsymbol{4\,A^2\,.G}} \int \frac{\boldsymbol{d_S}}{\boldsymbol{t}} \quad \text{.................. } \mathbf{(4)}$$

Where θ is the angle of twist per unit length, or

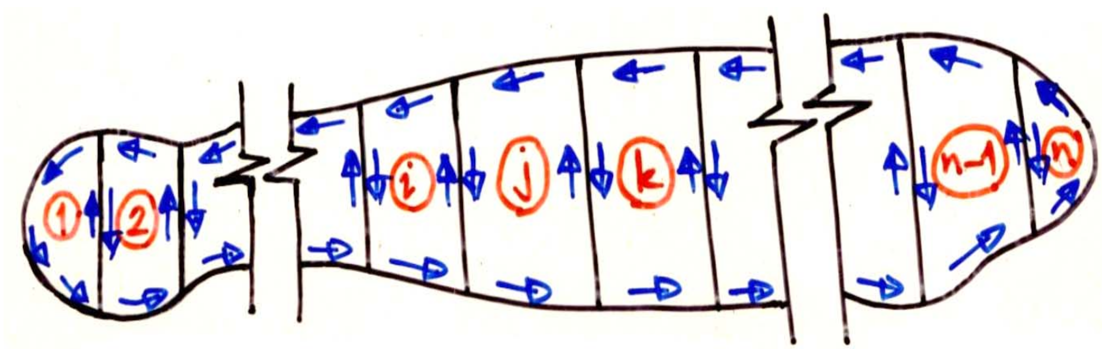
$$\theta = \frac{\boldsymbol{q}}{\boldsymbol{2A\,.G}} \int \frac{\boldsymbol{d_S}}{\boldsymbol{t}}$$

Thin Walled Cellular Sections

Consider the three typical cells i , j , and k .

$$\therefore \theta_j = \frac{q_j}{2A.G} \int_{s_j} \frac{ds}{t} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Where s_j indicates integration around cell j.



If shear flow q_i and q_k are now introduced in cells i and k, the flows in web s_{jk} are reduced to (q_j - q_i) and (q_j - q_k) respectively. Then

$$\theta_j = \frac{1}{2GA_j} [q_j \int_{s_j} \frac{ds}{t} - \sum_{r=1}^{m}(q_r \int_{s_r} \frac{ds}{t})] \ldots\ldots \text{(2)}$$

In our case, cell j is bounded by two cells, i.e., m=2 and

$$\theta_j = \frac{1}{2GA_j}\left(q_j \int_{s_j} \frac{d_s}{t} - q_i \int_{s_{ji}} \frac{d_s}{d_t} - q_k \int_{s_{jk}} \frac{d_s}{t}\right) \text{..............} (3)$$

One such equation (Eq.2 or 3) may be written for each of the n cells, and the resulting n linearly independent equations plus the equation

$$T = 2\sum_{j=1}^{n} q_j \, . A_j = T_1 + T_2 + \text{..........} + T_n$$

provide (n + 1) equations in the n+1 unknowns, q_1 , q_2 , q_3 , ……. , q_n , and θ .

Multiplying both sides of Eq.(3) [or 2] by 2 A_j , this equation foe cell j becomes

$$s_{ji} \, . \, q_i + s_{jj} \, . \, q_j + s_{jk} \, . \, q_k - 2\,A_j \, . \, \theta = 0 \text{ ....} (4)$$

Where

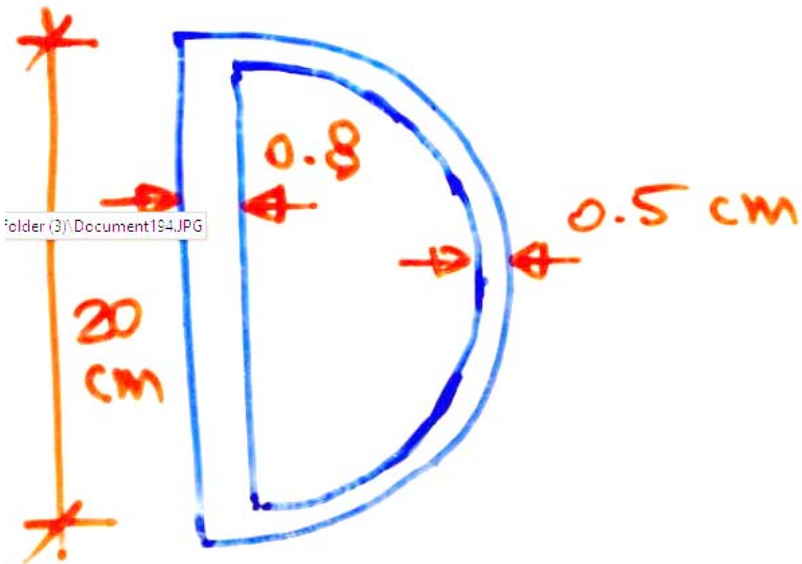
$$s_{ji} = -\frac{1}{G}\int_{s_{ji}} \frac{d_s}{t} \quad , s_{jk} = -\frac{1}{G}\int_{s_{jk}} \frac{d_s}{t} \quad , s_{jj} = \frac{1}{G}\int_{s_j} \frac{d_s}{t}$$

Note that θ is used instead of θ_j , since $\theta_1 = \theta_2 = \ldots\ldots = \theta_j = \theta_n$

Example:

The semi – circular tube is under a torque of 2 t.m. calculate the shear stresses in the walls and also the angle of twist per unite length. G = 800 t / cm^2.

Solution:



The constant shear flow in the wall is

$$q = \frac{T}{2A} = \frac{200}{2*\frac{\pi}{8}*(20)^2} = 0.64 \ \text{t/cm}$$

In the vertical wall the shear stress is

$$\tau = \frac{q}{t} = \frac{0.64}{0.8} = 0.8 \text{ t / } cm^2$$

In the curved wall

$$\tau = \frac{q}{t} = \frac{0.64}{0.5} = 1.28 \text{ t / } cm^2$$

The angle of twist per unit length is

$$\theta = \frac{T}{4\,A^2\,.G}\int \frac{d_S}{t}$$

$$= \frac{200}{4\,\{\frac{\pi}{8}*(20)^2\}^2.800}\,\{\frac{20}{0.8} + \frac{\frac{\pi}{8}*20}{0.5}\}$$

= 0.224 * 10 $^{-3}$ rad / cm = 1.27 o/cm

Example :

If the section is subjected to a torque of 320 N.m, determine the angle of twist per unit length and the maximum shear stress set up. G = 30 * 10 ^{9} N / m^2

Solution:

$$T = 2\, q_1 \,.\, A_1 + 2\, q_2 \,.\, A_2$$

$$320 * 10^3 = 2\, q_1 * 20 * 40 + 2\, q_2 * 50 * 40 \;\ldots\ldots\ldots\; (1)$$

$$s_{11} = \frac{1}{G}\left[\frac{40+2*20}{2} + \frac{40}{3}\right] = \frac{160}{3\,G}$$

$$s_{22} = \frac{1}{G}\left[\frac{40}{3} + \frac{2*50+40}{1.5}\right] = \frac{320}{3\,G}$$

$$s_{12} = s_{21} = -\frac{1}{G}\left[\frac{40}{3}\right] = -\frac{40}{3\,G}$$

Using the following equation for each cell

$$s_{ji} \,.\, q_i + s_{jj} \,.\, q_j + s_{jk} \,.\, q_k - 2\, A_j \,.\, \theta = 0$$

$\therefore \frac{160}{3G} q_1 - \frac{40}{3G} q_2$ - 2θ (20*40) = 0 ………………. (2)

And

· $\frac{40}{3G} q_1 + \frac{320}{3G} q_2$ - 2θ (50*40) = 0……………….(3)

Simplifying

$\therefore$ 160 q_1 **- 40** q_2 **- 4800 G.θ = 0 …………………(4)**

And

- 40 q_1 **- 320** q_2 **- 12000 G.θ = 0 ………………(5)**

Solving Eqs.(4) and (5)

$\therefore$ q_1 **= 43.86 G.θ**

q_2 **= 44.35 G.θ**

Substituting into Eq.(1)

$$\therefore 320 * 10^{3} = 2\,(43.86\ G.\theta) * 800 + 2\,(44.35\ G.\theta) * 2000$$

$$\therefore \theta = 0.000043\ \text{rad}/\text{mm} = 0.043\ \text{rad}/\text{m}$$

$$\therefore q_1 = 43.86 * 30 * 10^{3} * 0.000043 = 56.58\ \text{N}/\text{mm}$$

$$q_2 = 44.35 * 30 * 10^{3} * 0.000043 = 57.21\ \text{N}/\text{mm}$$

$$\tau_1 = \frac{q_1}{t_1} = \frac{56.58}{2} = 28.29\ \text{Mpa}$$

$$\tau_2 = \frac{q_2}{t_2} = \frac{57.21}{1.5} = 38.14\ \text{Mpa}$$

$$\tau_3 = \frac{q_2 - q_1}{t_3} = \frac{0.63}{3} = 0.21\ \text{Mpa}$$

$\therefore$ **maximum shear stress** $= \tau_2 = 38.14$ **Mpa**