THE NULL HOMOTOPIC MAPS

Prof. Dr. Hana' M. Ali

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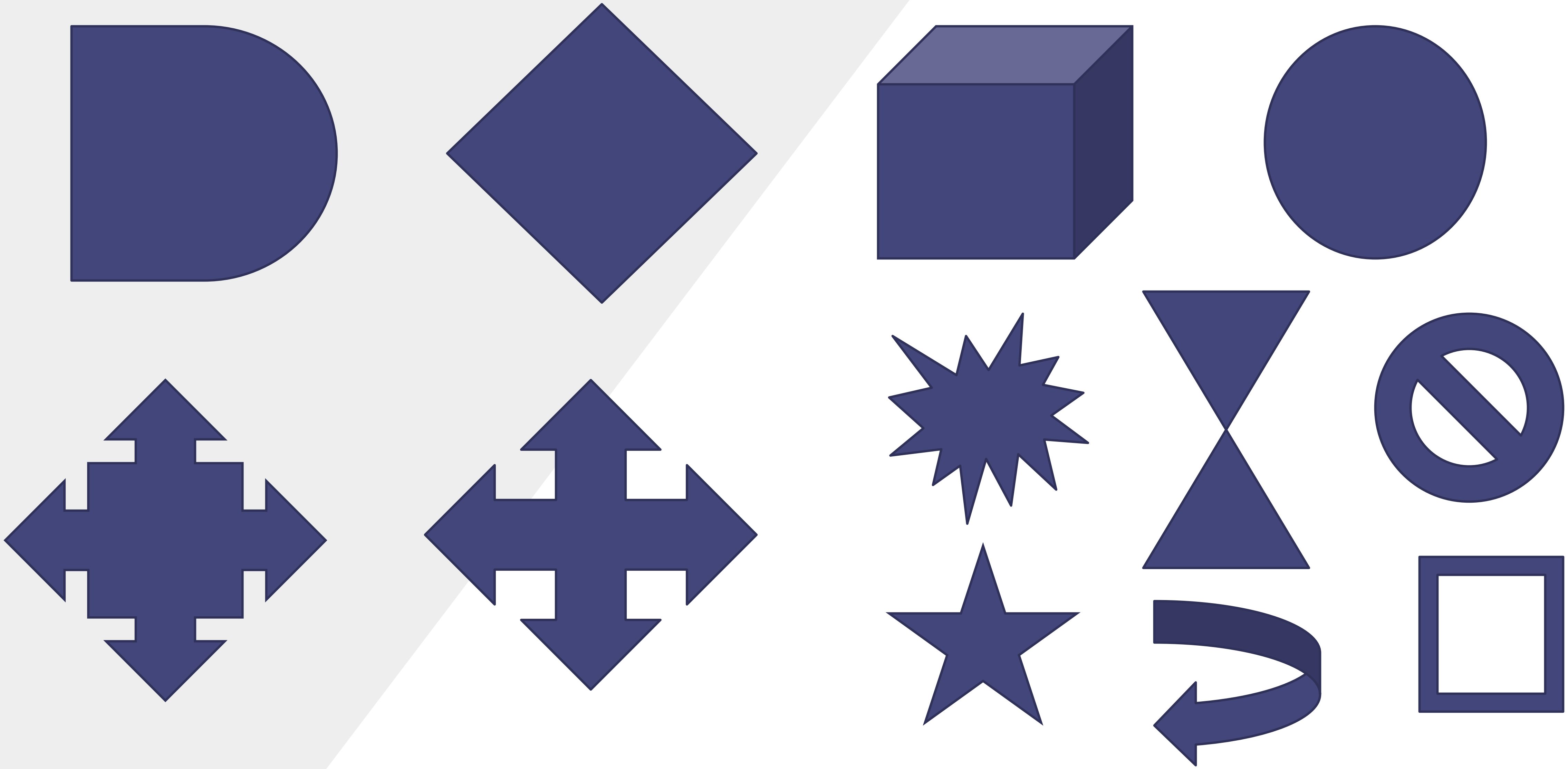
DEFINITION:

Let $f: X \to Y$ be a continuous map between two topological spaces. f is said to be null homotopic if it is homotopic to constant map $C: X \to Y$, (i.e. For some $y \in Y$,$C(x) = y$ for each $x \in X$).

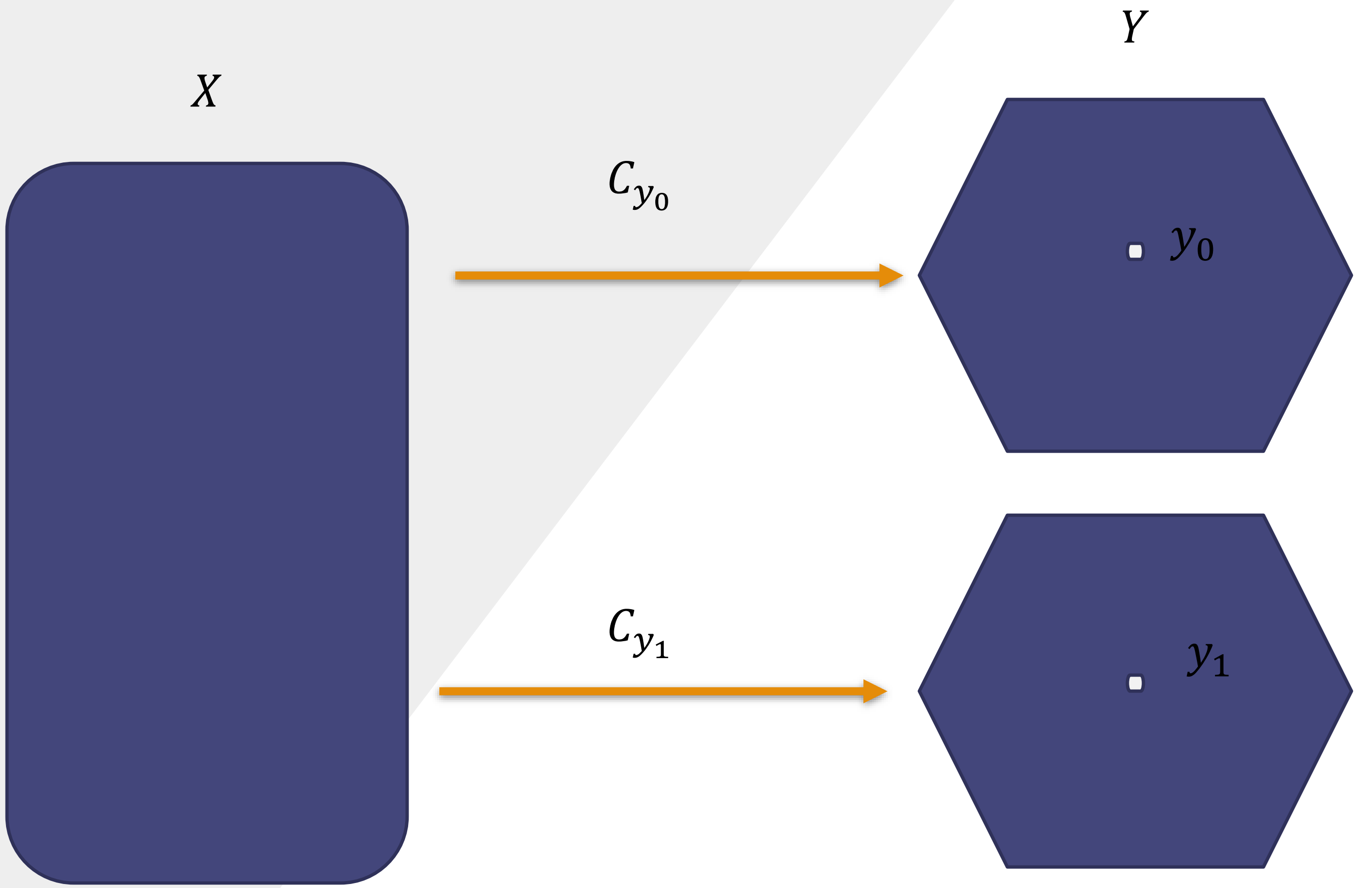
Examples:

1. Any constant map is null homotopic.
2. Any continuous map $f: X \to \mathbb{R}^n$ from any topological space X into the Euclidean space $\mathbb{R}^n$ is null homotopic.
3. Any continuous map $f: X \to Y$ from any topological space X into a convex subset Y of the Euclidean space $\mathbb{R}^n$ is null homotopic. (A subset Y of the Euclidean space $\mathbb{R}^n$ is called convex if $\forall y, y' \in Y$, the line segment $\{(1-t)y + ty' | 0 \leq t \leq 1\} \subseteq Y$).
4. Any continuous map $f: X \to Y$ from any topological space X into a Starlike subset Y of the Euclidean space $\mathbb{R}^n$ is null homotopic . (A subset Y of the Euclidean space $\mathbb{R}^n$ is called Starlike if there exists $y_0 \in Y$, for each $y \in Y$, the line segment $\{(1-t)y + ty_0 | 0 \leq t \leq 1\} \subseteq Y$).

REMARK: every convex subset Y of the Euclidean space $\mathbb{R}^n$ is Starlike, but the converse need not to be true in general as we shown in the following figures:



REMARK: Null homotopic maps need not to be homotopic.



$C_{y_0} \not\simeq C_{y_1}$

A JOIN PX

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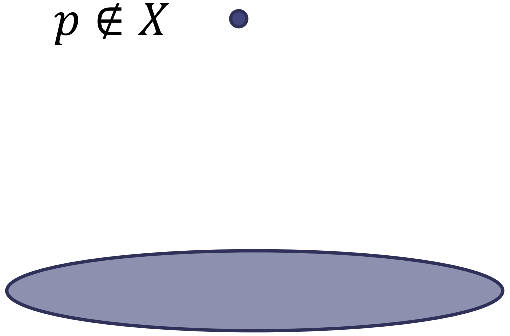
DEFINITION:

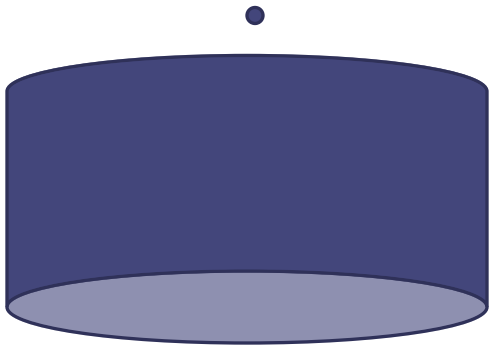
Let X be a topological space and $p \notin X$ be a point. Consider the disjoint union $p \sqcup X \times I$ of the point p and the product space $X \times I$. Define an equivalence relation $\sim$on $p \sqcup X \times I$ as; $p \sim (x, 1)$. Let $pX = p \sqcup X \times I/\sim$ be the quotient space, (that is pX is a topological space with identification topology that induced from the identification map $\theta: p \sqcup X \times I \rightarrow pX$ that defined as;

$$\theta(p) = \theta(x, 1) = [p] \text{ and } \theta(x, t) = [(x, t)] = [x, t], \text{ for all } x \in X \text{ and } 0 \leq t < 1).$$

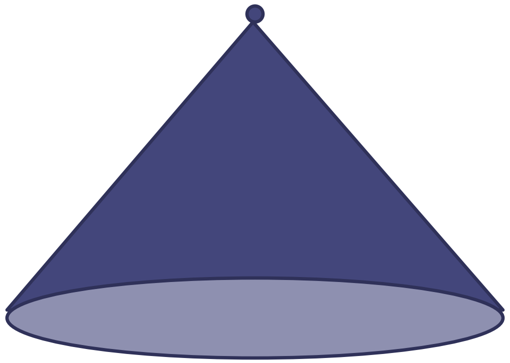
$p \notin X$



X



$p \sqcup X \times I$

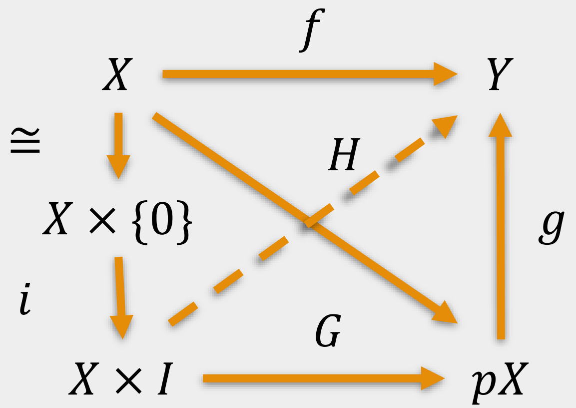


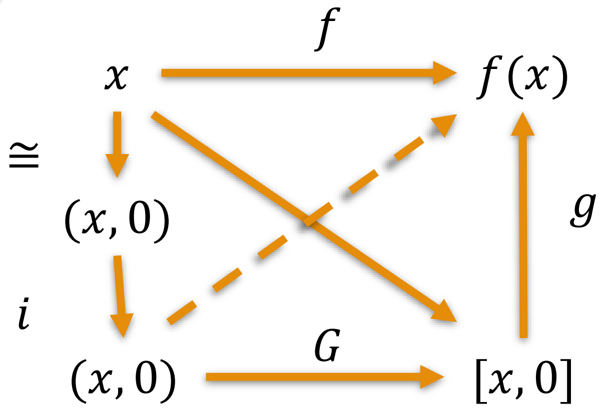
$pX = p \sqcup X \times I/\sim$

THEOREM: A mapping $f: X \to Y$ is null homotopic if an only if, f may extended to all of a join pX.

Proof:

Firstly, suppose $g: pX \to Y$ be an extension map of f, then we have the following commutative diagram:

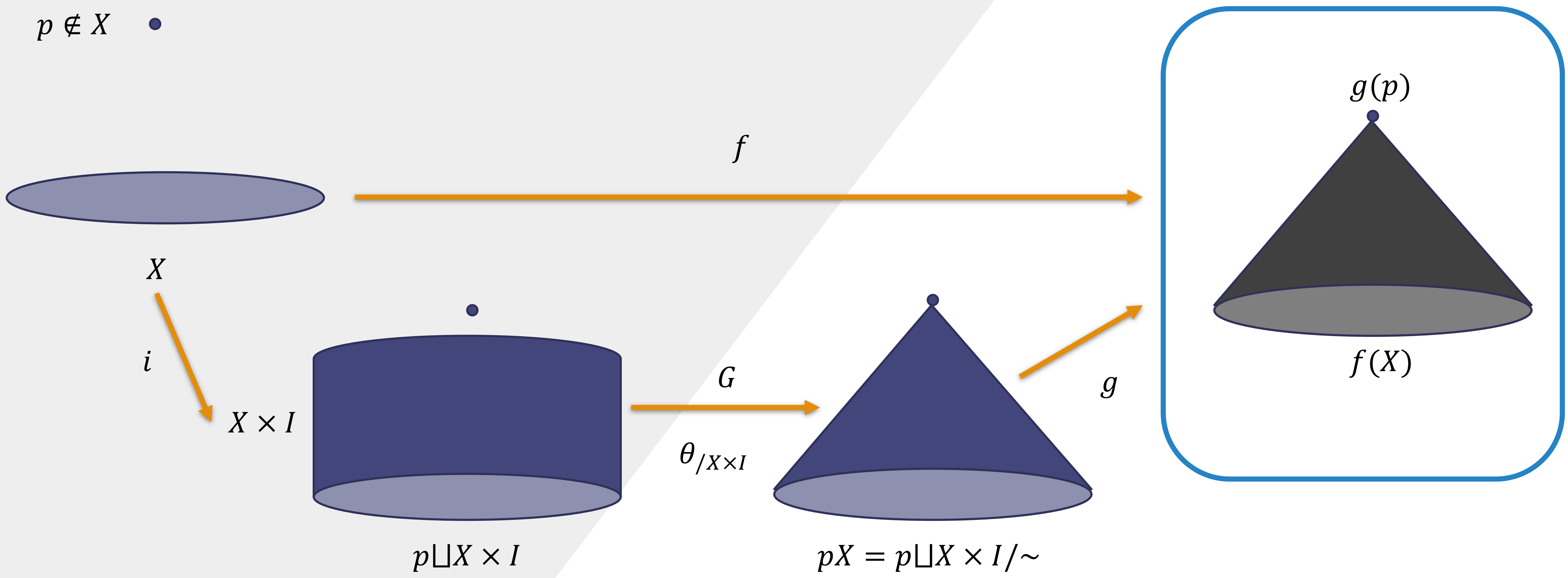




i.e., $g_{/X\times\{0\}} = f$, where $G = \theta_{/X\times I}: X \times I \to pX$ is a continuous map given by, $G(x,t) = [x,t]$, for all $x \in X$ and $0 \leq t < 1$.
Then we have a continuous map $g \circ G: X \times I \to Y$ with:

$$g \circ G\ (x,0) = g\big(G(x,0)\big) = g(x,0) = f(x) \text{ and } g \circ G(x,1) = g\big(G(x,1)\big) = g(p), \text{ for all } x \in X .$$

But $g \circ G(x,1) = C_{g(p)}(x)$ is a constant map. Therefore, $g \circ G$ is a homotopy from f into a constant map. Hence f is null homotopic.



Homework: prove the second direction.

THANK YOU FOR YOUR ATTENTION