HOMOTOPY EQUIVALENT SPACES

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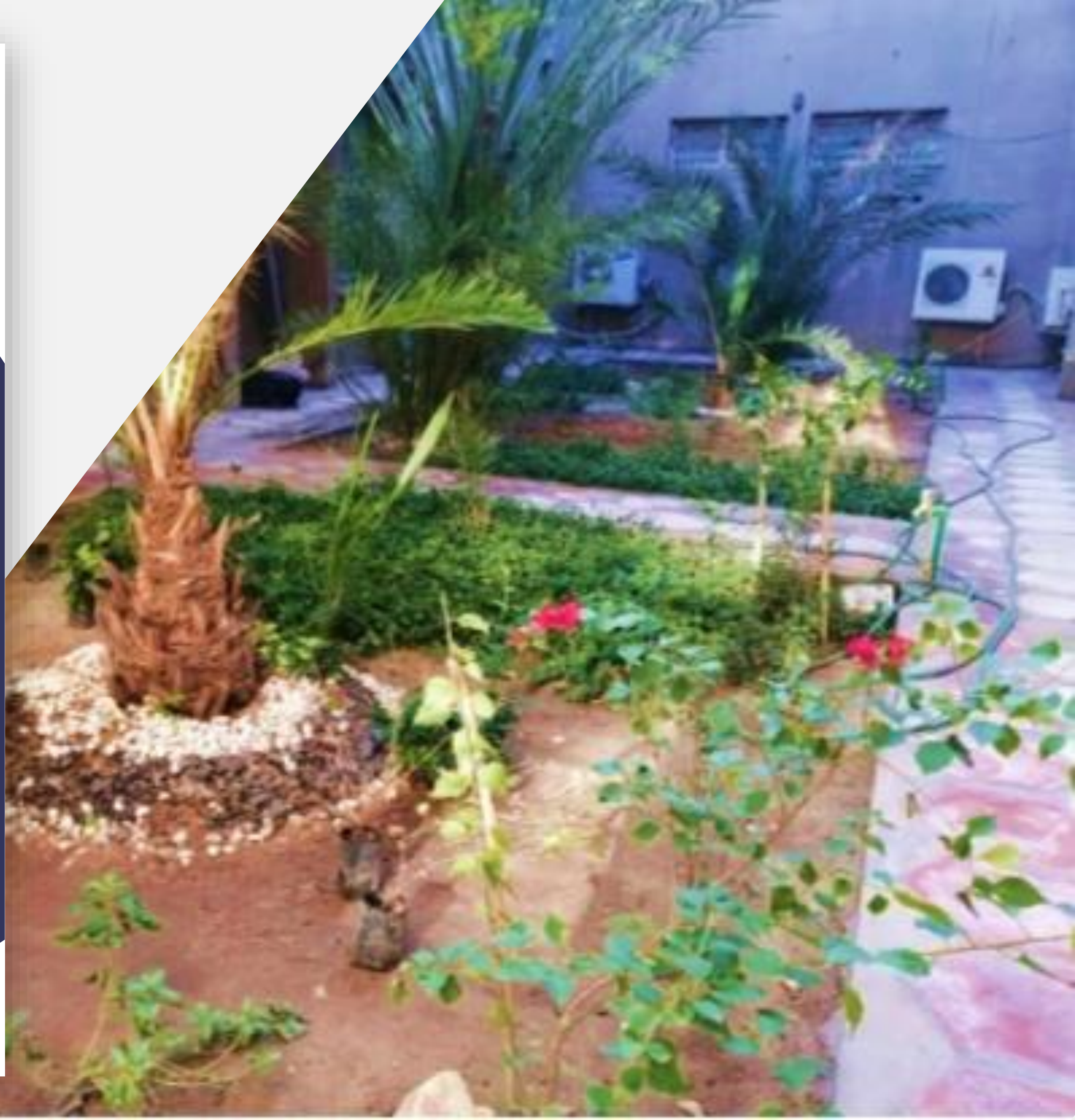

OF THE SAME HOMOTOPY TYPE TOPOLOGICAL SPACES

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DEFINITION:

Two topological spaces X and Y are said to be of the same homotopy type (or homotopy equivalent) and denoted by $X \simeq Y$, if there exist two continuous maps $f: X \to Y$ and $g: Y \to X$ such that:

$$g \circ f \simeq \mathcal{J}_X : X \to X \text{ and } f \circ g \simeq \mathcal{J}_Y : Y \to Y.$$

The maps f and g are called homotopy equivalences.

Note:

Recall, two topological spaces X and Y are said to be homeomorphic and denoted by $X \equiv Y$, if there exist two continuous maps $f: X \to Y$ and $g: Y \to X$ such that:

$$g \circ f = \mathcal{J}_X : X \to X \text{ and } f \circ g = \mathcal{J}_Y : Y \to Y.$$

That is, the homeomorphic spaces are of same homotopy type, but the converse need not to be true in general.

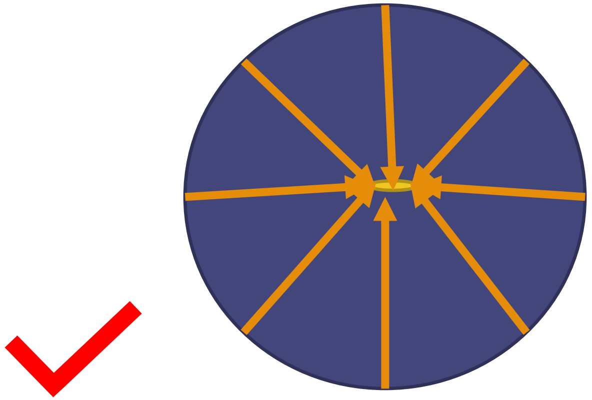
Example: Let $D^n = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$ is the n-disk of the Euclidean space $\mathbb{R}^n$ and assume that $x_0 \in D^n$. We can deduce that $D^n \simeq \{x_0\}$ but they are not homeomorphic.

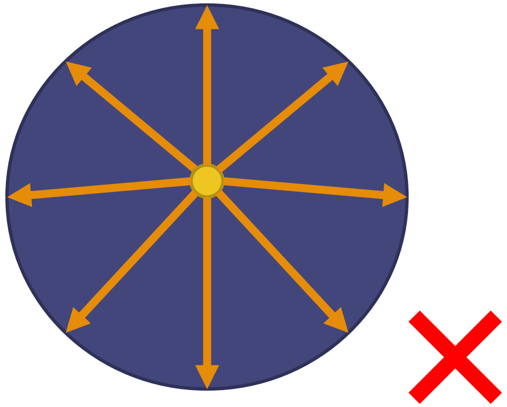
To explain that the constant map $C: D^n \to \{x_0\}$ and the inclusion map $i: \{x_0\} \to D^n$ form homotopy equivalences between the. In fact, each one of C and i is a continuous map and they satisfied:

$$C \circ i = \mathcal{J}_{\{x_0\}}: \{x_0\} \to \{x_0\} \text{ and } i \circ C \simeq \mathcal{J}_{D^n}: D^n \to D^n.$$

Since $C \circ i(x_0) = x_0$ and the continuous map $H: D^n \times I \to D^n$ that defined as; $H(x,t) = (1-t)x_0 + tx$, for all $x \in D^n$, play as a homotopy from $i \circ C$ into $\mathcal{J}_{D^n}$.

More generally, $\mathbb{R}^n \simeq \{x_0\}$.

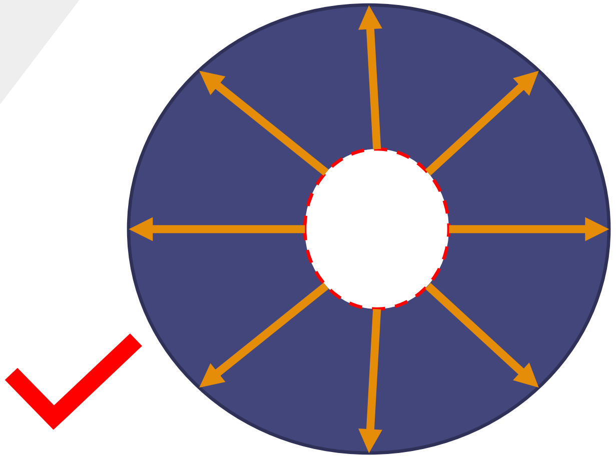




Example: The unit $(n-1)-S^{n-1}=\{x\in\mathbb{R}^n|\|x\|=1\}$ is of the same type of $D^n-\{0\}$ (the closed n -disk of the Euclidean space $\mathbb{R}^n$ minus the origin.

To show that the inclusion map $i: S^{n-1}\to D^n-\{0\}$ and the map r: $D^n-\{0\}\to S^{n-1}$that defined as; $r(x)=\frac{x}{\|x\|}$, for all $x\in D^n-\{0\}$; form homotopy equivalences between S^{n-1} and$\in D^n-\{0\}$. In fact, each one of r and i is a continuous map and they satisfied: $r\circ i=\mathcal{I}_{S^{n-1}}$and $i\circ r\simeq\mathcal{I}_{D^n-\{0\}}$.

More generally, $\mathbb{R}^n-\{0\}\simeq S^{n-1}$.



THE REDUCED MAPPING CYLINDER

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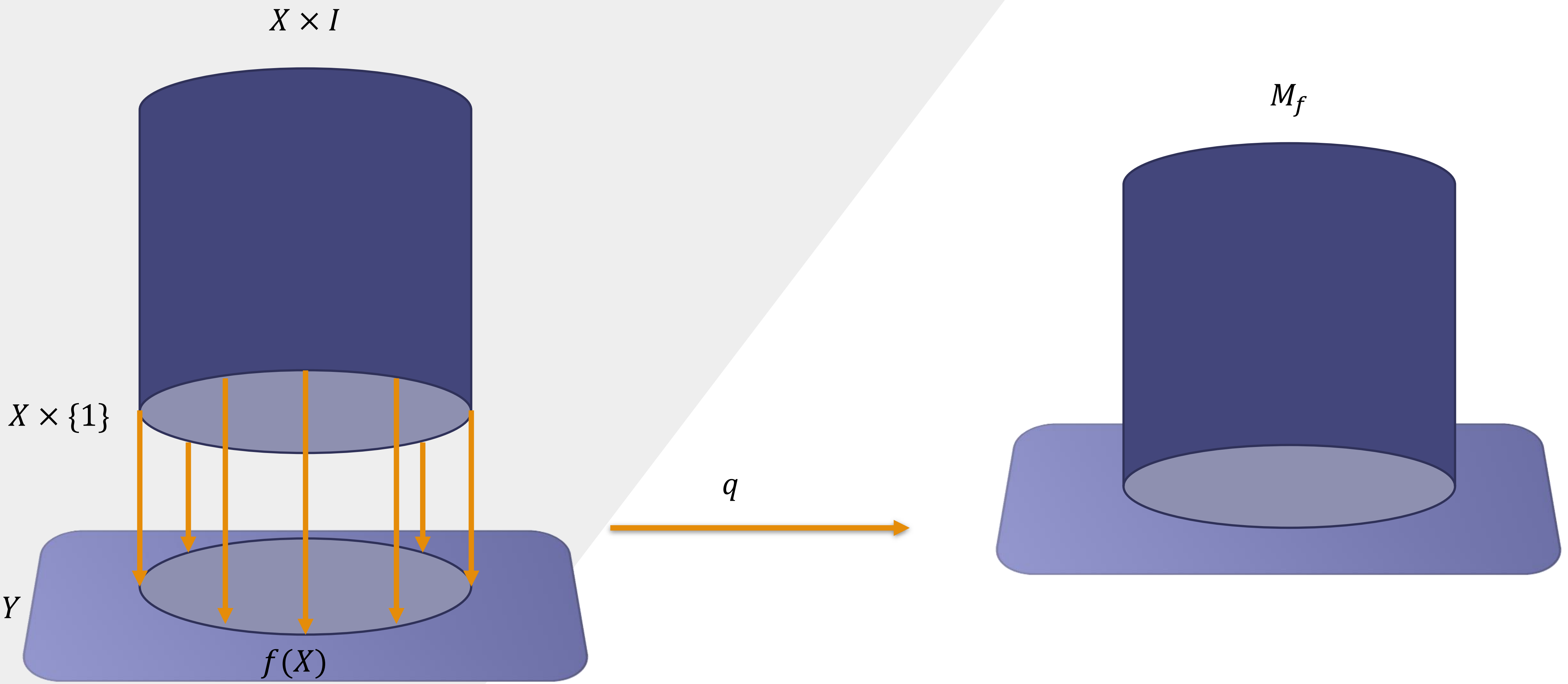
DEFINITION:

Given any continuous map $f:(X,x_0)\to(Y,y_0)$, (i.e. $f(x_0)=y_0$). The reduced mapping cylinder M_f of f is the space obtained from the disjoint union $(X\times I)\sqcup Y$ by identification $(x,1)\in X\times I$ with $f(x)\in Y$, for every $x\in X$, (i.e. we will define an equivalence relation $\sim$ on $(X\times I)\sqcup Y$ as; $(x,1)\sim f(x)$ and letting; $M_f=(X\times I)\sqcup Y/\sim$ be the identification topological space that its topology induced from the identification map $q:(X\times I)\sqcup Y\to M_f$ that defined as;

$$q(x,t)=[(x,t)]=[x,t]\ ;$$

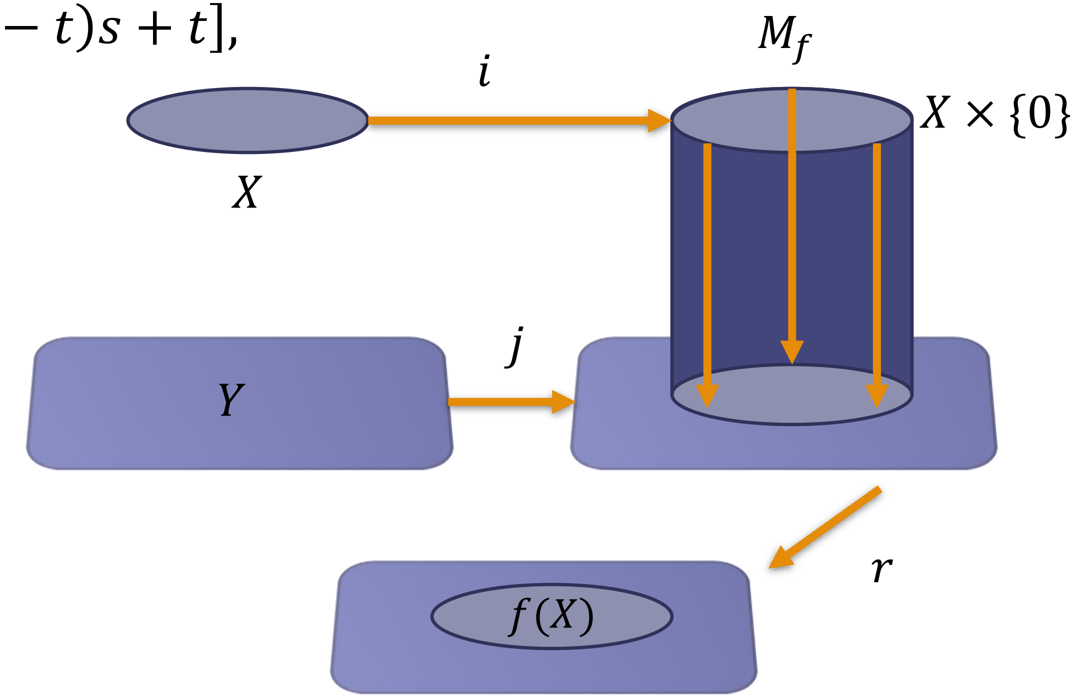
$$q(x,1)=[(x,1)]=[x,1]=[f(x)];$$

$$q(y)=[y], \text{ for all } (x,t)\in X\times I \text{ and } y\in Y.$$


$X \times I$
$X \times \{1\}$
Y
$f(X)$
q
M_f

For all$(x,s) \in X \times I$, $y \in Y$ and $t \in I$, we can define the following maps:

1. $i:(X,x_0) \to (M_f,[y_0])$ as; $i(x) = [x,0]$.
2. $j:(Y,y_0) \to (M_f,[y_0])$ as; $i(y) = [y]$.
3. $r:(M_f,[y_0]) \to (Y,y_0)$ as; $r(y) = [y]$ and $r[x,t] = f(x)$.
4. $H: M_f \times I \to M_f$ as; $H([y],t) = [y]$ and $H([x,s],t) = [x,(1-t)s+t]$,



THEOREM: $(Y,y_0) \simeq M_f$.

As a homework deduce that, r is a homotopy equivalence.

THANK YOU FOR YOUR ATTENTION