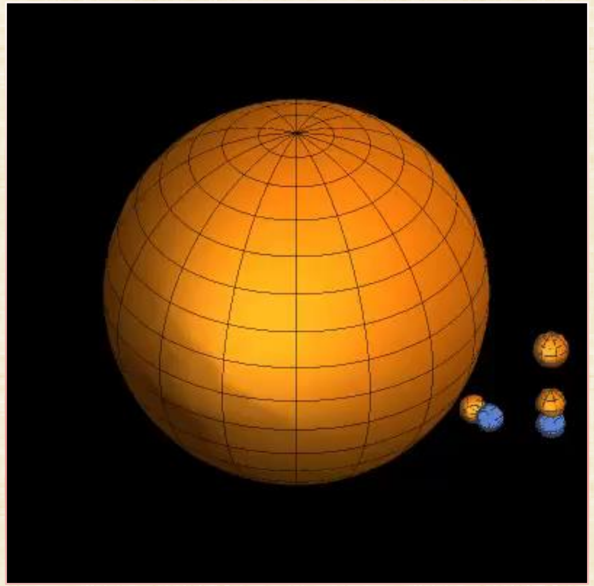


ALGEBRAIC TOPOLOGY

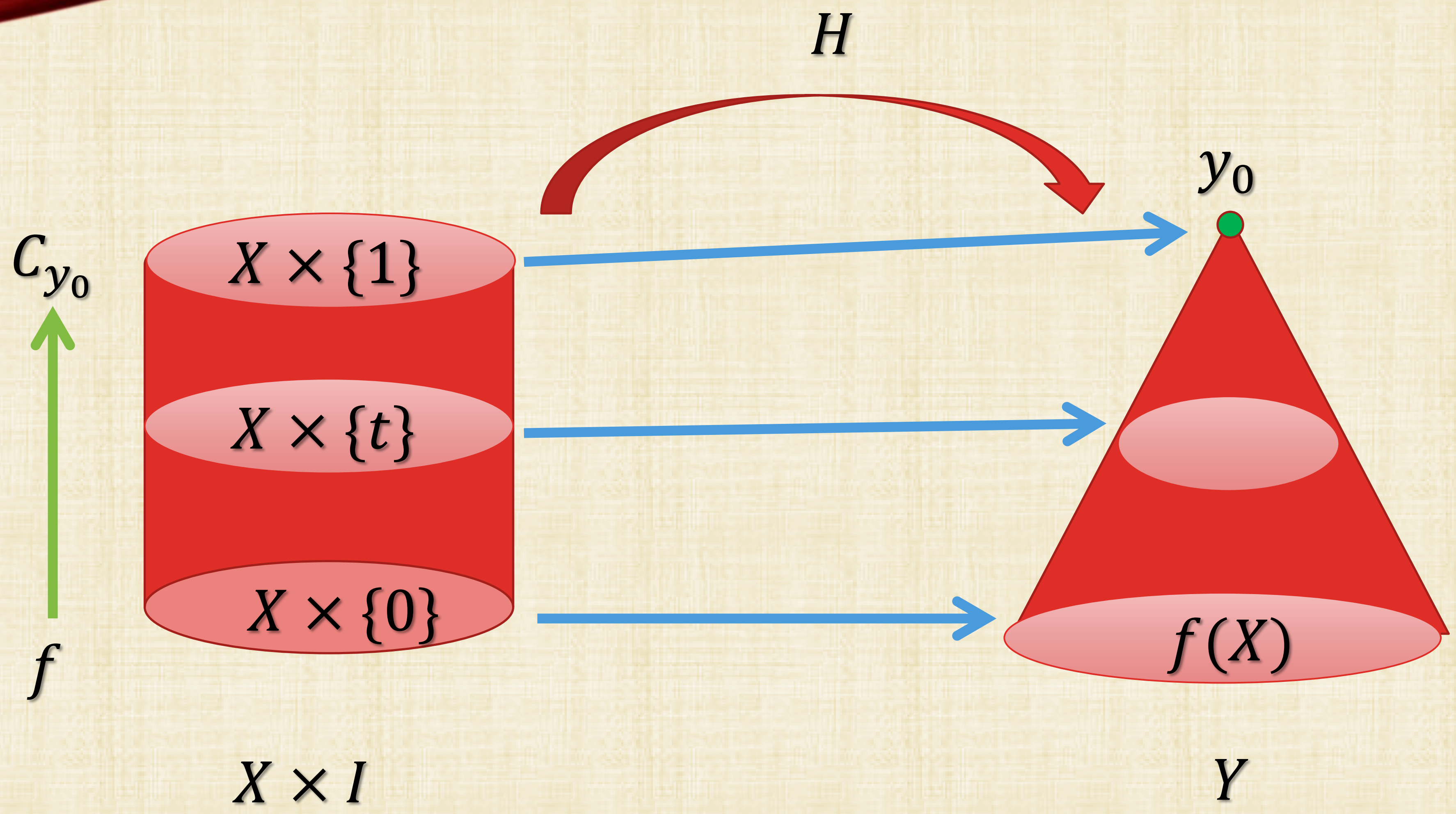
Prof. Dr. Hana' M. Ali

Lecture 3

NULL HOMOTOPIC MAPS

Definition:

A continuous map $f: X \to Y$ from a topological space X into a topological space Y, are said to be null homotopic if, it is homotopic to a constant map $C_{y_0}: X \to Y$, (i.e. $C_{y_0}(x) = y_0$, for all $x \in X$ and for some $y_0 \in Y$ and $f \simeq C_{y_0}$).



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Examples:

1. Every continuous map $f: X \to \mathbb{R}^n$ from a topological space X into the Euclidean space $\mathbb{R}^n$ is null homotopic. In fact, $f \simeq C_{y_0}$, for any $y_0 \in \mathbb{R}^n$ and the homotopy $H: X \to \mathbb{R}^n$ between them can be defined as:

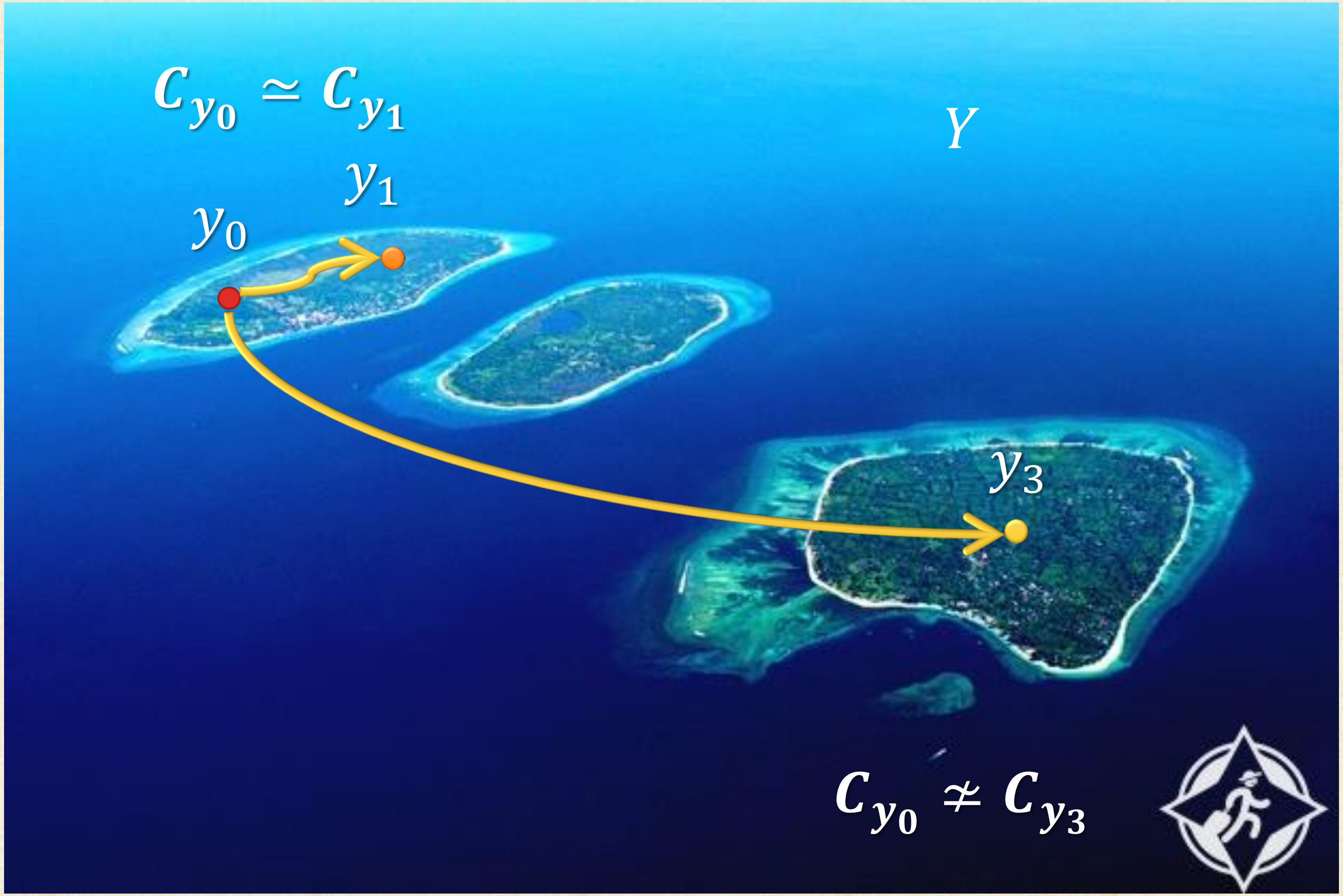
$$H(x,t) = (1-t).f(x) + t.y_0, \text{ for all, } (x,t) \in X \times I.$$

2. If S is a convex subset of $\mathbb{R}^n$, then any continuous map $f: X \to S$ is null homotopic.
3. If Y is indiscrete topological space (i.e. $T = \{\emptyset, X\}$), then any map $f: X \to Y$ is null homotopic.

Remark:

Null homotopic maps need not to be homotopic, indeed any two constant maps $C_{y_0}, C_{y_1}: X \to Y$ from a topological space X into a topological space Y need not to be homotopic.

In fact, if $f \simeq C_{y_0}$ and $g \simeq C_{y_1}$, then $f \simeq g$ if and only if, $C_{y_0} \simeq C_{y_1}$. As we know, the homotopy $H: C_{y_0} \simeq C_{y_1}$ form a path from y_0 into y_1 (show that). Therefore, two null homotopic maps $f, g: X \to Y$ with $f \simeq C_{y_0}$ and $g \simeq C_{y_1}$, are homotopic if and only if, y_0 and y_1 contained in the same path component (show that).



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$f(X)$
y_0

Definition:

- ✓ Let X be a topological space and $p \notin X$ be a point.
- ✓ Let $p\cup(X \times I)$ be the disjoint union of p and the product space $X \times I$, i.e. a subset G is open in $p\cup(X \times I)$ if and only if, $G\cap(X \times I)$ is pen in $X \times I$.
- ✓ Define an equivalence relation $\sim$ on $p\cup(X \times I)$ as:

$$p \sim (x, 1), \text{ for all } x \in X.$$

- ✓ A join pX is the quotient space $p\cup(X \times I)/\sim$, i.e. pX denotes the set of all the equivalence classes that related to $\sim$, with the identification topology, i.e. if $\theta: p\cup(X \times I) \to pX$ be the identification map that defined as:

Definition:

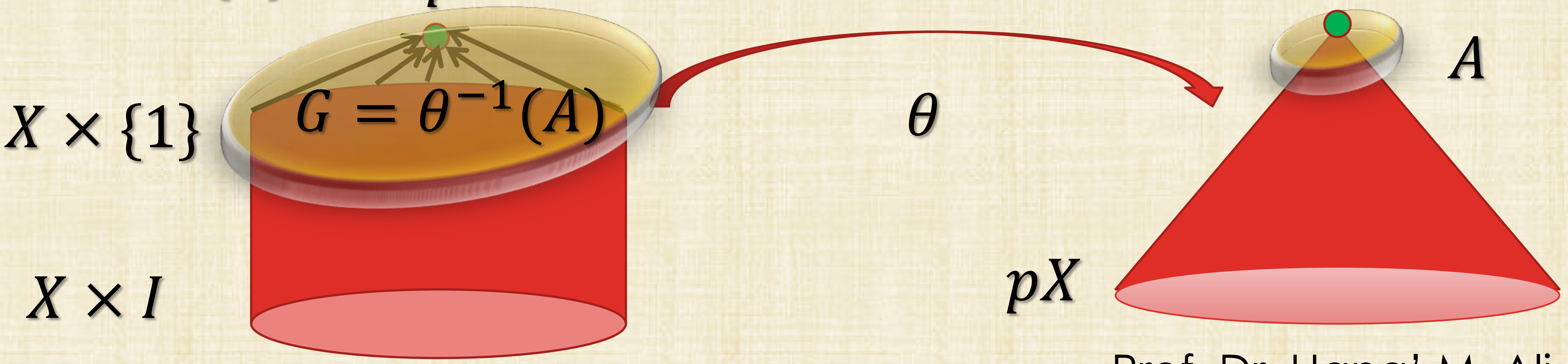
$$\theta(y) = [y], \text{ for all } y \in p\cup(X \times I),$$

Then $A \subseteq pX$ is open in pX, if and only if, $\theta^{-1}(A)$ is open in $p\cup(X \times I)$.

Note:

The join pX, obtain a new topology assigning a topology on $X \times I$, in which any open set G meet $X \times \{1\}$ (i.e. $G\cap(X \times \{1\}) \neq \emptyset$), contains $X \times \{1\}$, since y_0

$[p] = p\cup(X \times \{1\})$.



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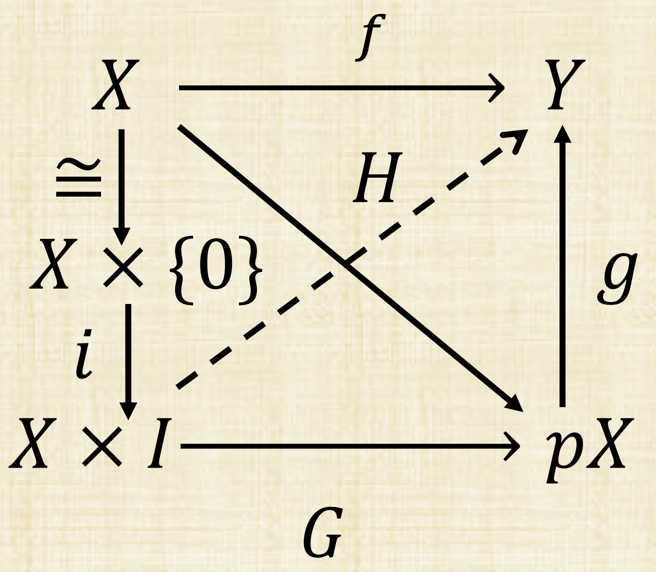
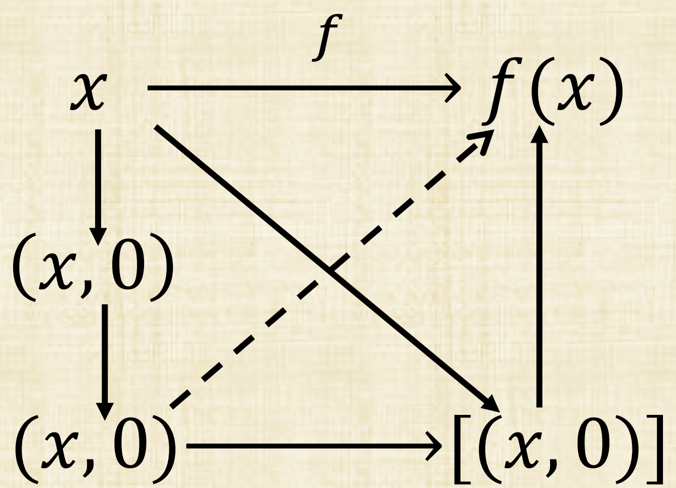
Theorem:

A mapping $f: X \to Y$ is null homotopic, if , and only if, f can be extended to all of a join pX.

Proof: Suppose f can be extended to all of a join pX and $g: pX \to Y$ be an extension map of f, so we have the following commutative diagrams:





i.e. $g_{/\theta(X\times\{0\})} = f$, where $G: X \times I \to pX$ is a continuous map given by:

$G(x, 1) = p$ and $G(x, t) = [(x, t)] = \{(x,t)\}$, for all $(x, t) \in X \times I$.

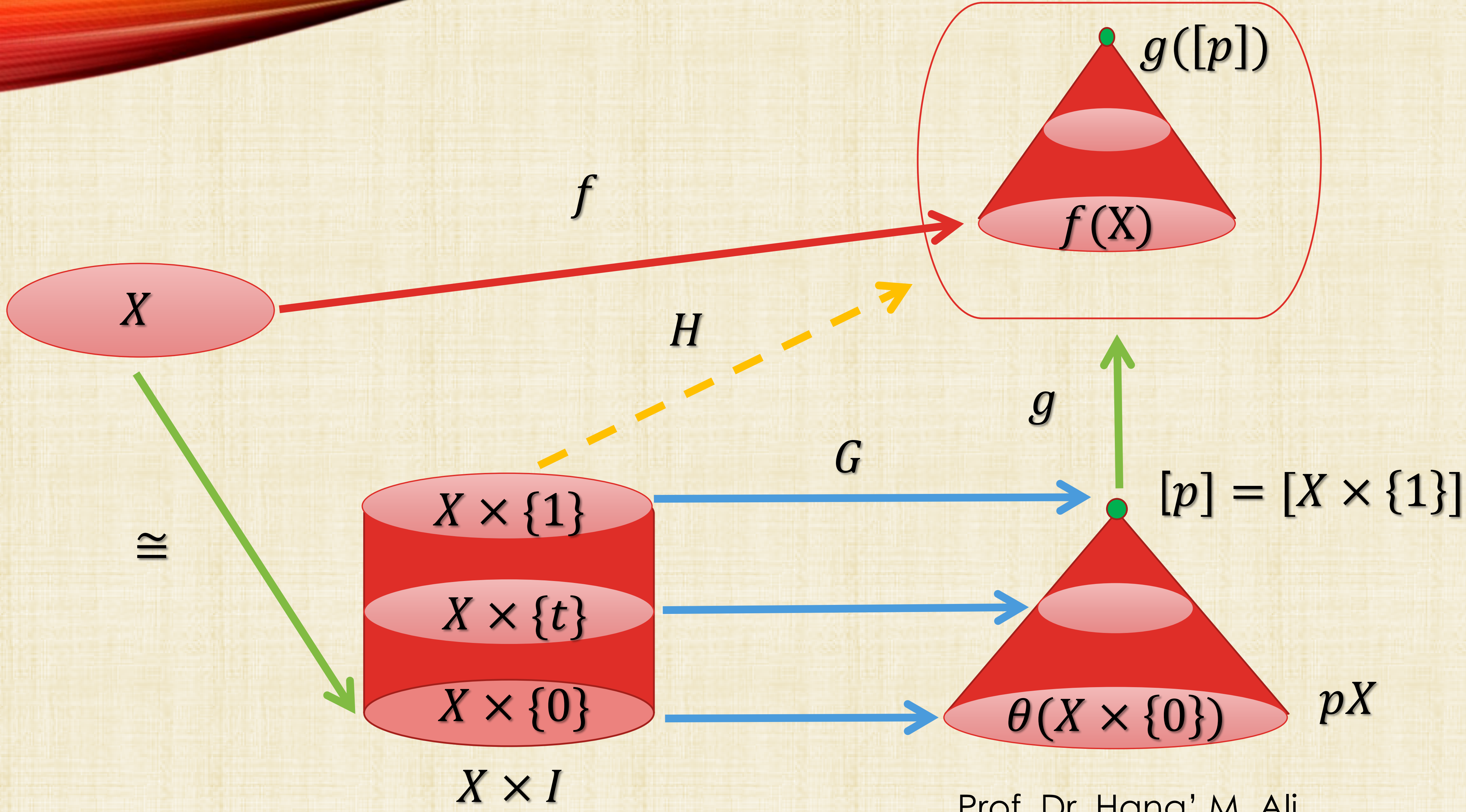
Then we have a continuous map $H = g \circ G: X \times I \to Y$ that satisfied:

$$H(x, 0) = g \circ G(x, 0) = g\big(G(x, 0)\big) = g([(x, 0)]) = f(x), \text{ and;}$$

$$H(x, 1) = g \circ G(x, 1) = g\big(G(x, 1)\big) = g([p]) = C_{g([p])}(x), \text{ for all } x \in X.$$

Thus, H forms a homotopy from f into a constant map $C_{g([p])}$. Therefore, f is null homotopic.

As a homework, prove that if f is null homotopic, then f can be extended to all a join pX.



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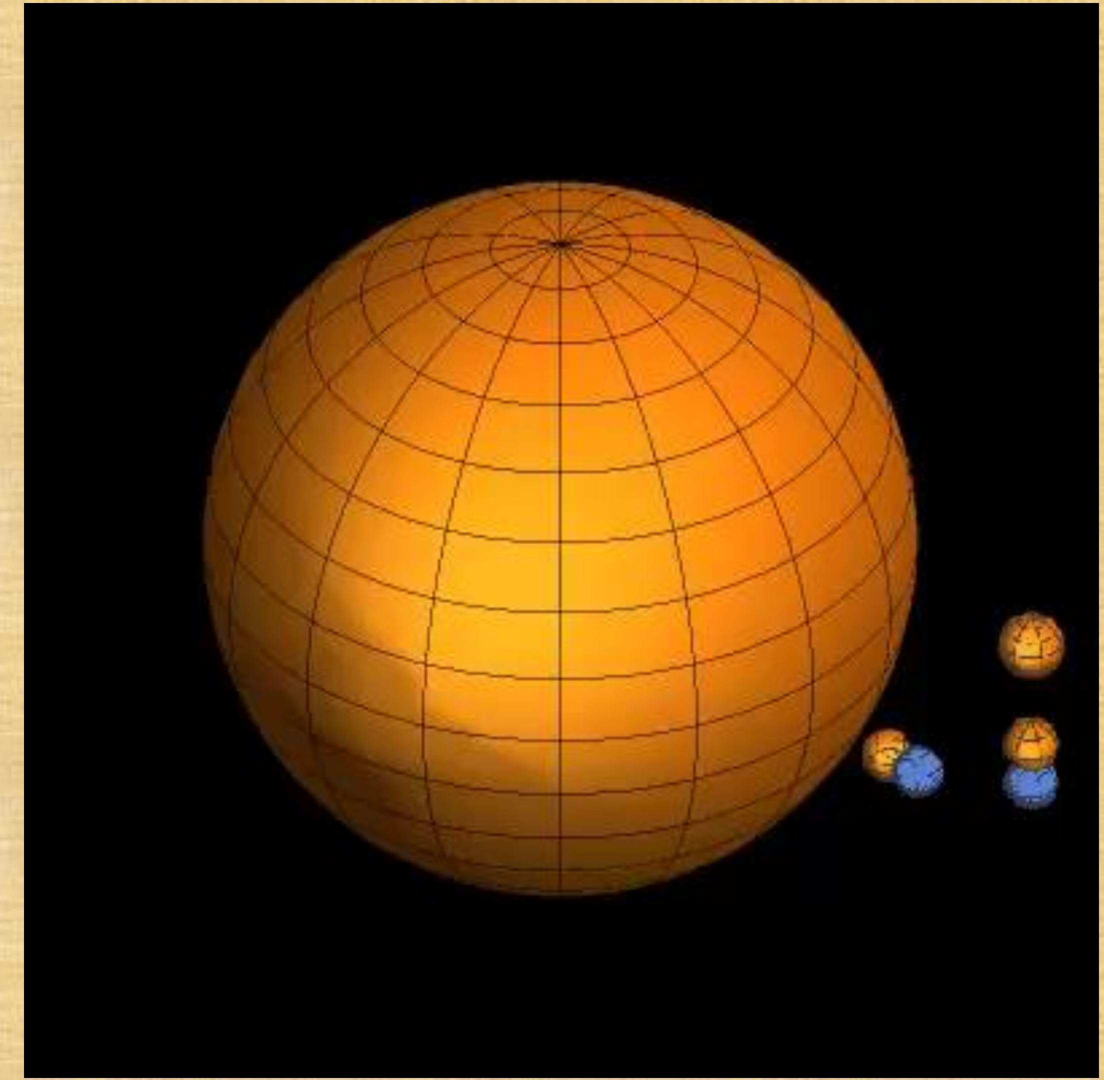
Exercise:

1. Define the notions, topological pair, map of topological pairs and homotopic relative maps.

2. Give an examples.

3. Prove that, the relation (homotopic relative to a set) on the set of all maps of topological pairs, is an equivalence relation.

Thank You Very Much For Lessening



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