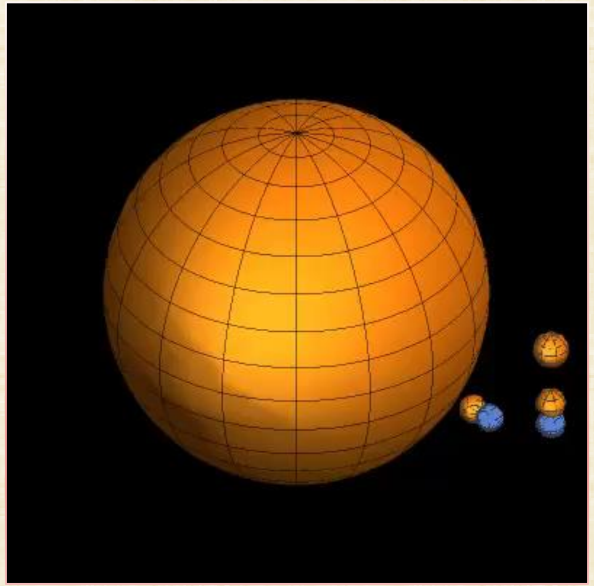

ALGEBRAIC TOPOLOGY

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Lecture 2

HOMOTOPY

Definition:

Two continuous maps $f, g: X \to Y$ from a topological space X into a topological space Y, are said to be homotopic and denoted by $f \simeq g$ if, there exists a continuous map $H: X \times I \to Y$, where $I = [0,1]$ is the unit interval, satisfied for every $x \in X$:

- $H(x, 0) = f(x)$;
- $H(x, 1) = g(x)$.

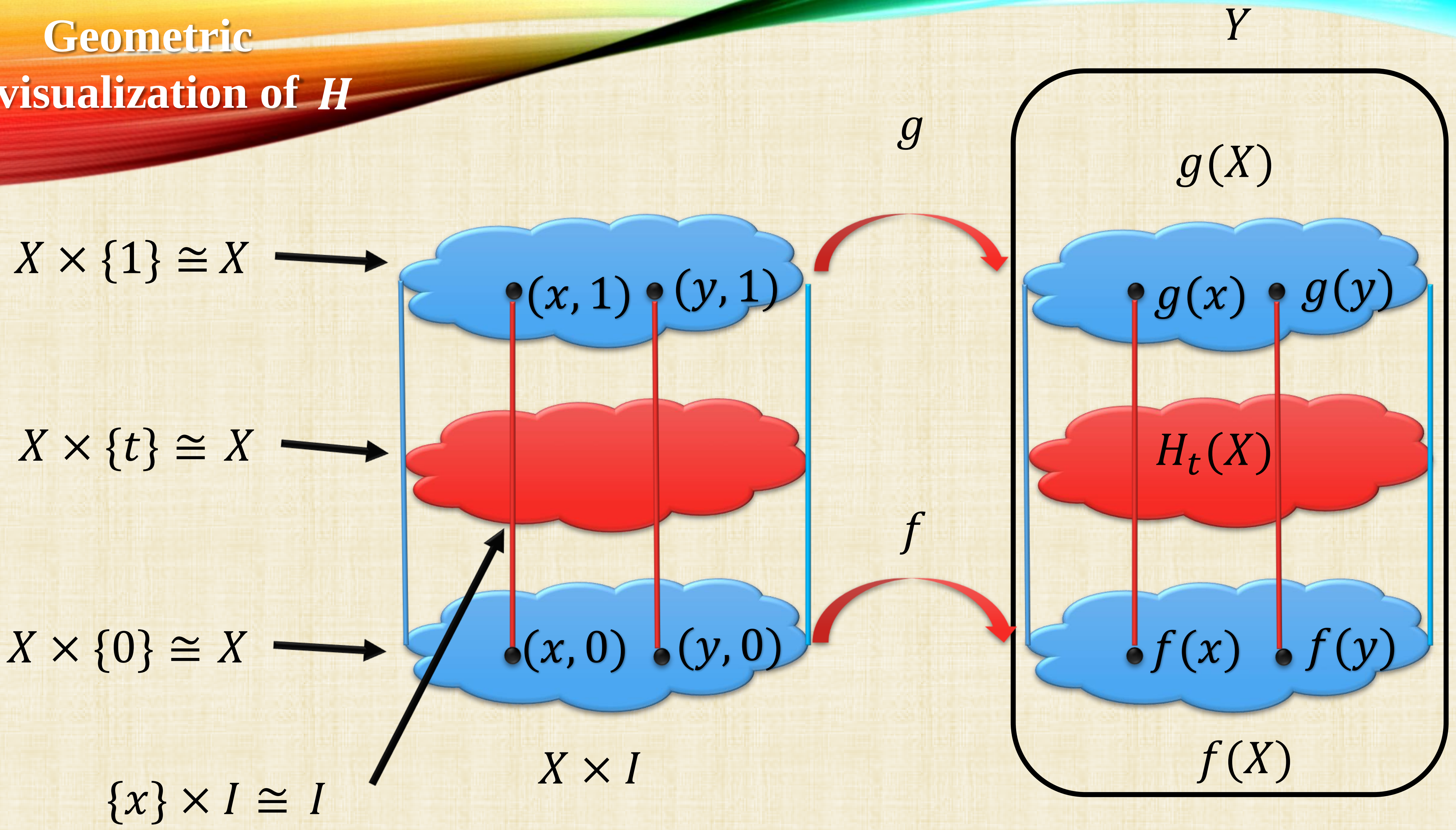
We call *H* a *homotopy* from f to g and denoted by $H: f \simeq g$.

Note:

To better visualize the geometric content of the definition:

- ✓ Write $H_t(x) = H(x,t)$, for any $(x,t) \in X \times I$.
- ✓ Then $H_t: X \to Y$ is a continuous map s.t. $H_0 = f$ and $H_1 = g$.
- ✓ We usually think of the parameter $t\epsilon I$ as time and the homotopy H_t as a continuous deformation of a map.

Geometric visualization of H



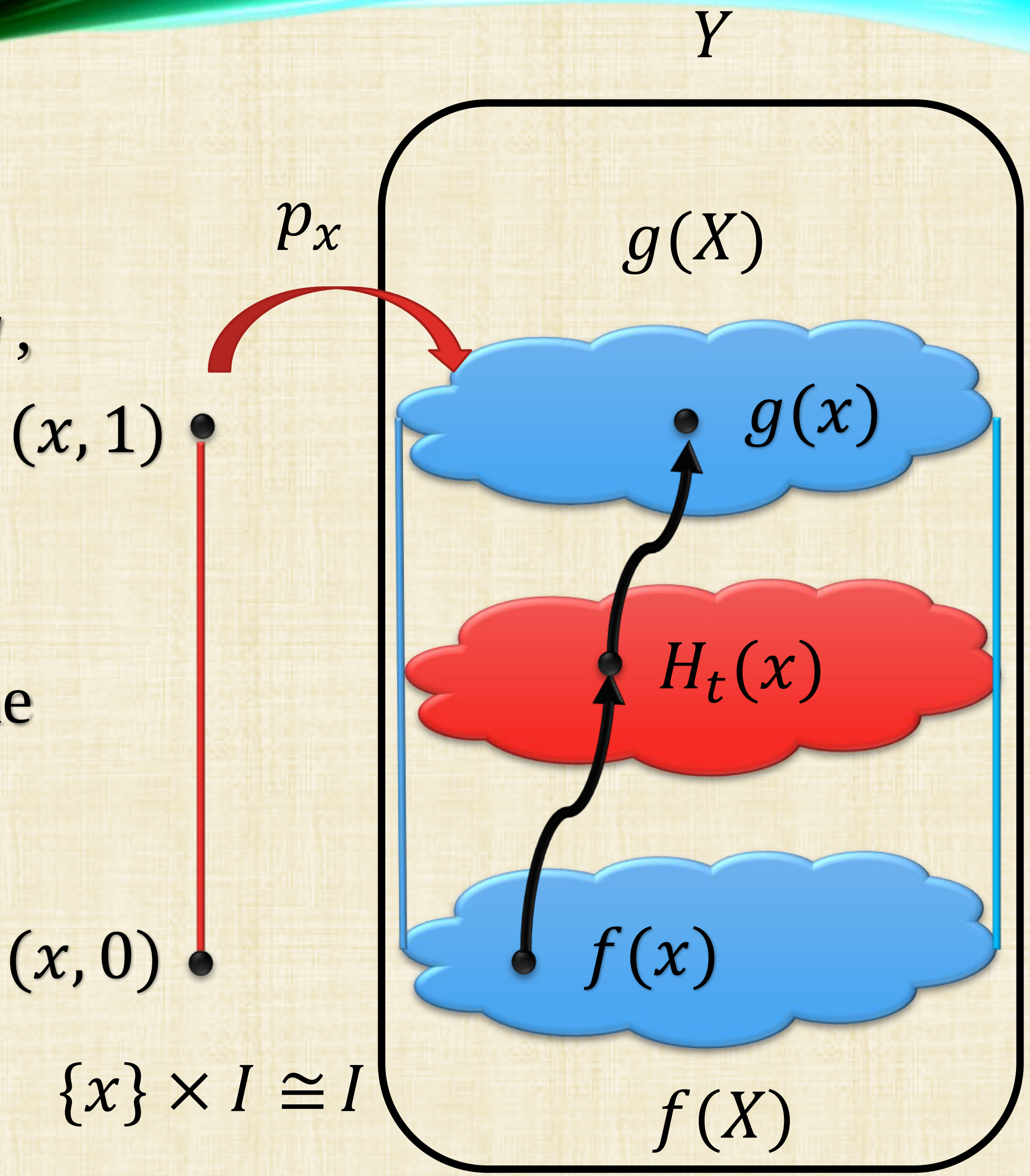
Geometric visualization of H

Fixed $x \in X$, the homotopy $H: f \simeq g$, define a path $p_x: I \to Y$ as;

$$p_x(t) = H(x,t), \text{ for all } t \in I.$$

Notice that, p_x is a continuous map as the following commutative diagram:

$$\begin{array}{ccccc} I & \xrightarrow{\cong} & \{x\} \times I & \xrightarrow{i} & X \times I \\ & & & & \downarrow H \\ & & \overset{p_x}{\searrow} & & Y \end{array}$$



In fact, $p_x(0) = f(x)$ and $p_x(1) = g(x)$.

$f(X)$
$g(X)$


$f(X)$
$g(X)$

Exercise:

The relation $\simeq$, is an equivalence relation on the set of all continuous maps from a topological space X into a topological space Y.

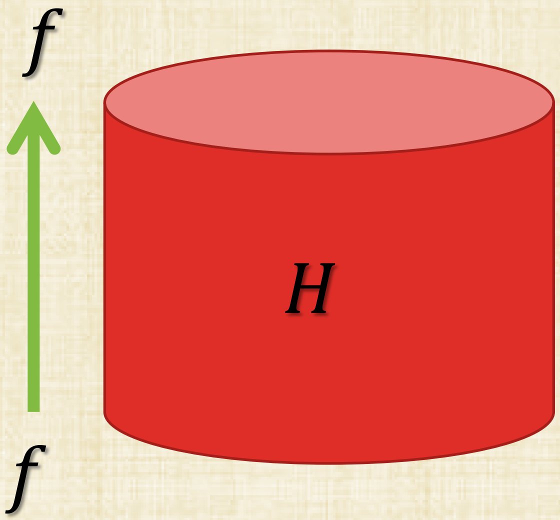
That is;

- ✓ $\simeq$ is reflexive; ($H: f \simeq f$, defined as; $H(x,t) = f(x), \forall\ (x,t) \in X \times I$).
- ✓ $\simeq$ is symmetric; (If $H: f \simeq g$, define $G: g \simeq f$ as; $G(x,t) = H(x, 1-t),\ \forall (x,t) \in X \times I$), and;
- ✓ $\simeq$ is transitive; ($H: f \simeq g$ and $G: g \simeq h$, define $F: f \simeq h$ as;

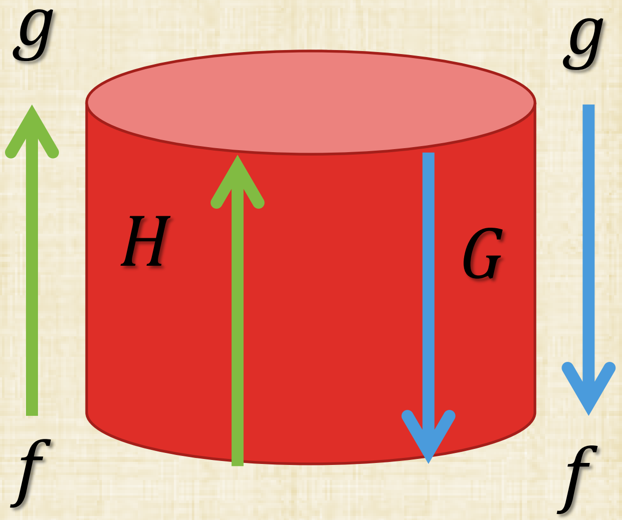
$$F(x,t) = \begin{cases} H(x,2t); & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1); & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Due $\simeq$, the equivalence classes are called homotopy classes.

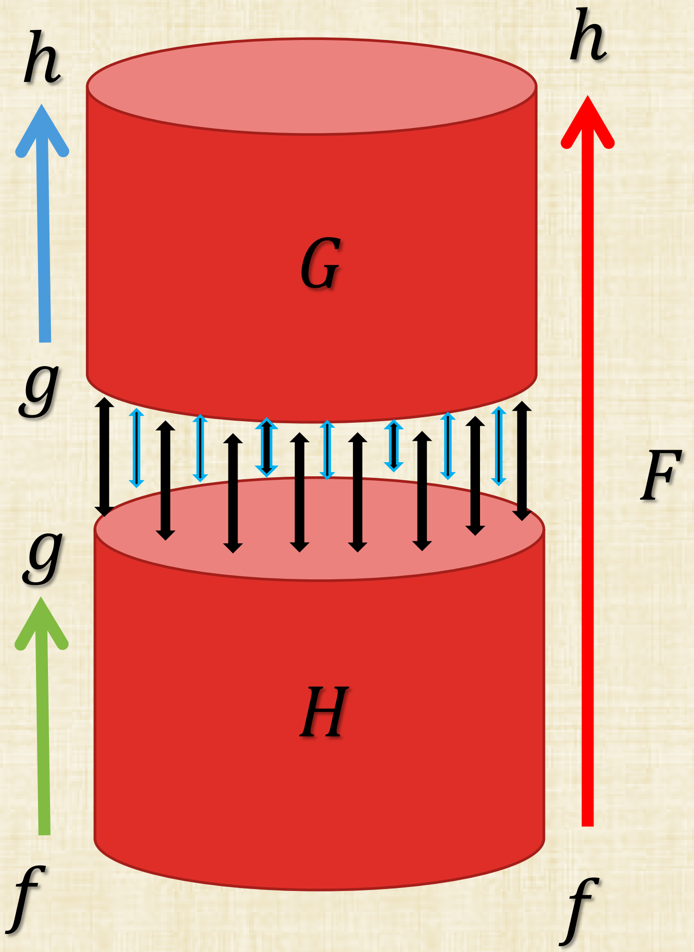
$\simeq$ is reflexive



$\simeq$ is symmetric



$\simeq$ is transitive



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Theorem:

The compositions of homotopic maps are homotopic.

Proof: Suppose we have the following continuous maps $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ such that $H: f_1 \simeq f_2$ and $G: g_1 \simeq g_2$. Wanted,

$$g_1 \circ f_1 \simeq g_2 \circ f_2.$$

Define $F : X \times I \to Z$ as;

$$F(x,t) = \begin{cases} g_1 \circ H(x, 2t); & 0 \leq t \leq \frac{1}{2} \\ G(f_2(x), 2t-1); & \frac{1}{2} \leq t \leq 1 \end{cases};$$

As a homework, show that F is continuous. In fact ;

$$F(x,0) = g_1 \circ H(x,0) = g_1 \circ f_1(x) \text{ and;}$$
$$F(x,1) = G(f_2(x),1) = g_2 \circ f_2(x).$$

Examples:

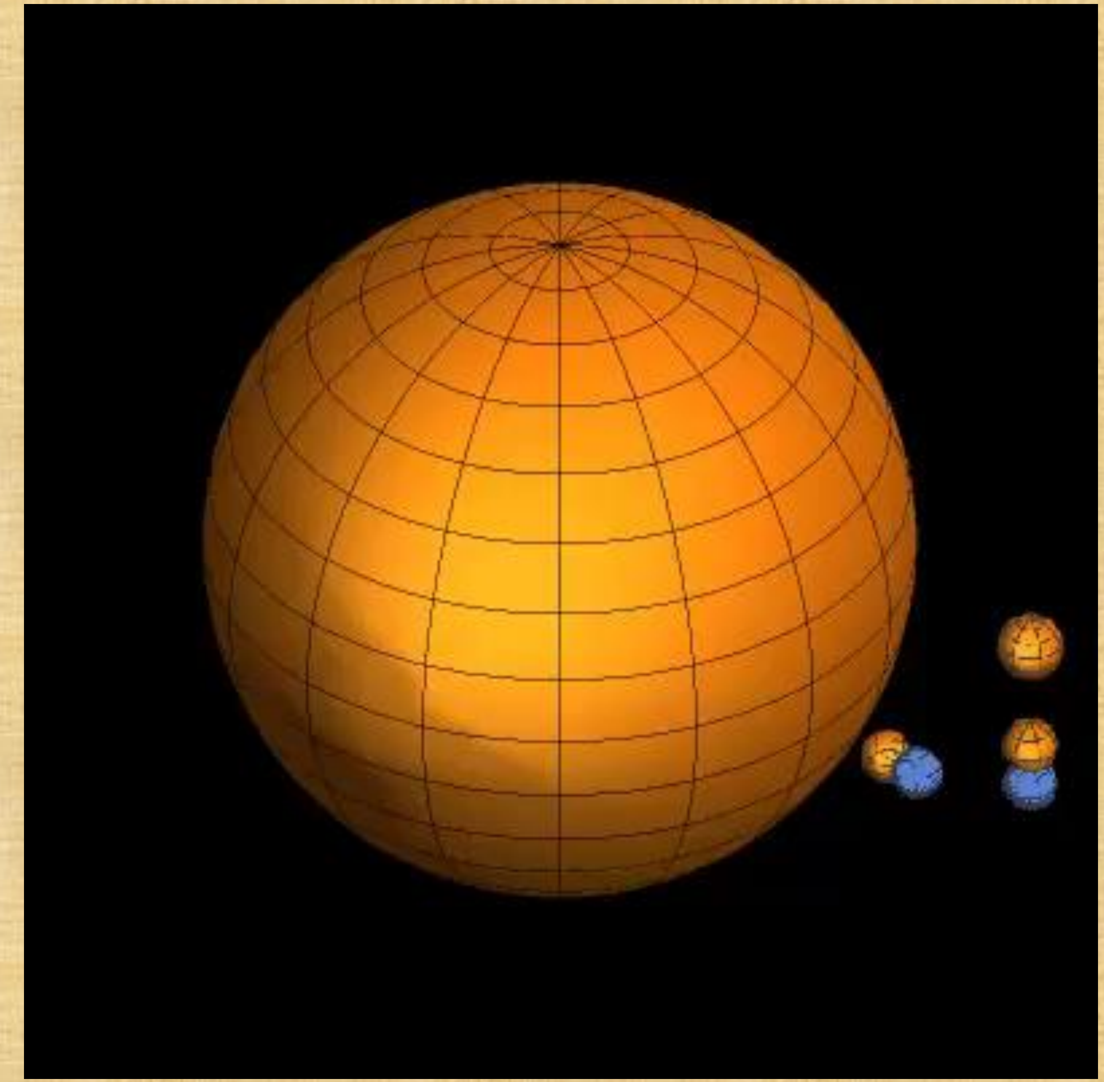
1. Every two continuous maps $f, g: X \to \mathbb{R}^n$ from a topological space X into the Euclidean space $\mathbb{R}^n$ are homotopic. In fact, $H: f \simeq g$ defined as:

$$H(x,t) = (1-t).f(x) + t.g(x), \text{ for all, } (x,t) \in X \times I.$$

2. If S is a convex subset of $\mathbb{R}^n$, then every two continuous maps $f, g: X \to S$ are homotopic.

3. If Y is indiscrete topological space (i.e. $T = \{\emptyset, X\}$), then every two maps $f, g: X \to Y$ are homotopic.

Thank You Very Much For Lessening



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