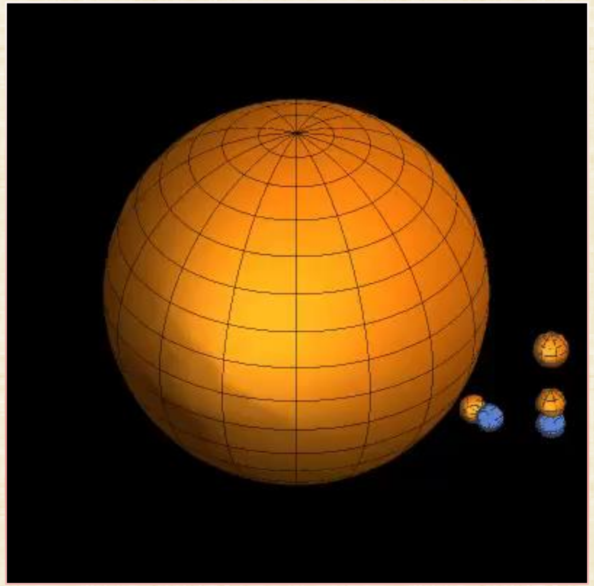


ALGEBRAIC TOPOLOGY

Prof. Dr. Hana' M. Ali

Lecture 1

PATH CONNECTED SPACES

Definition:

By a Path (or arc) in a topological space X, we mean a continuous map $p: I \to X$, where $I = [0,1]$ is the unit interval.

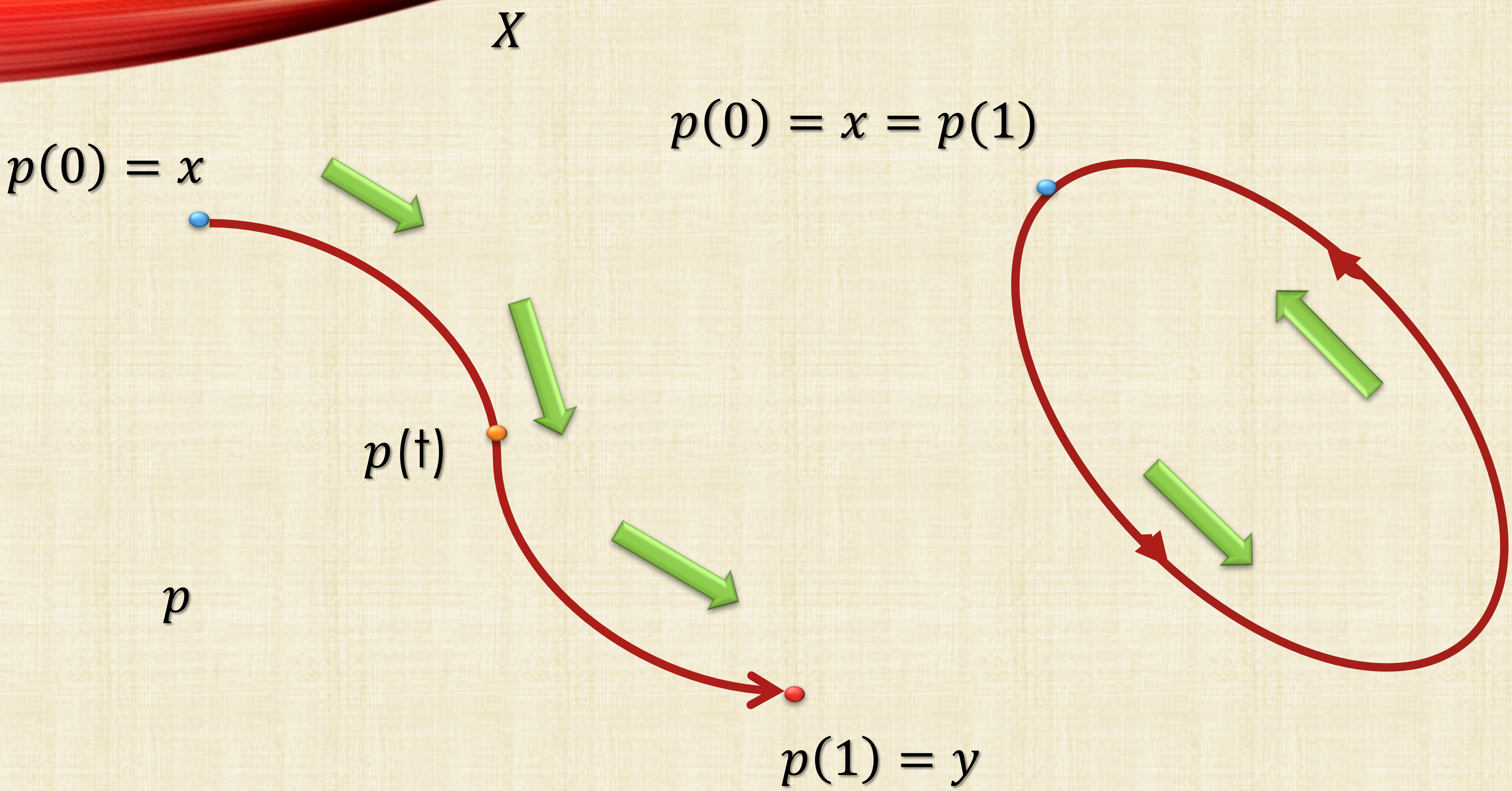
- The point $p(0)$ is called the initial point, and;
- The point $p(1)$ is called the final or terminal point.
- We say that, the path p joins the points $p(0)$ and $p(1)$.

Definition:

- A path p is called closed if $p(0) = p(1)$.

Note:

- ✓ The map p is the path and not the image $p(I)$ which is called a curve in X.
- ✓ We usually think of $t \epsilon I$ as time and I as a one unit of time. So, $p(t)$ represents the position in X at time t.



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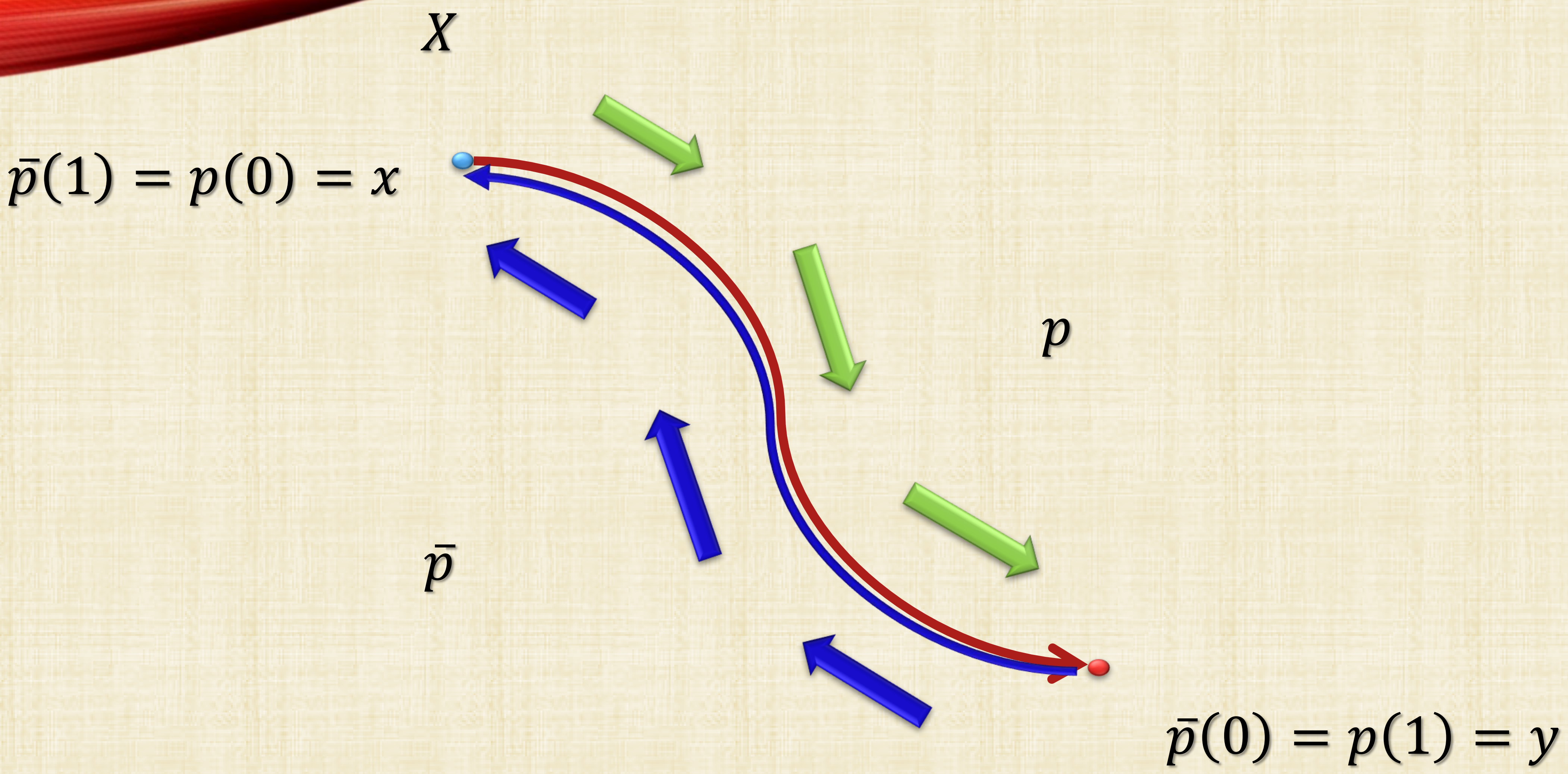
Examples:

1. For some point $x \in X$, by a constant path we mean a constant map $C_x : I \to X$ that defined as; $C_x(t) = x$, for all $t \in I$. In fact, C_x form a closed path since$C_x(0) = C_x(1)$.

2. If $p: I \to X$ is a path in X, then the map $\bar{p}: I \to X$ that defined as;

$$\bar{p}(t) = p(1 - t), \text{ for all } t \in I;$$

Is also a path in X.



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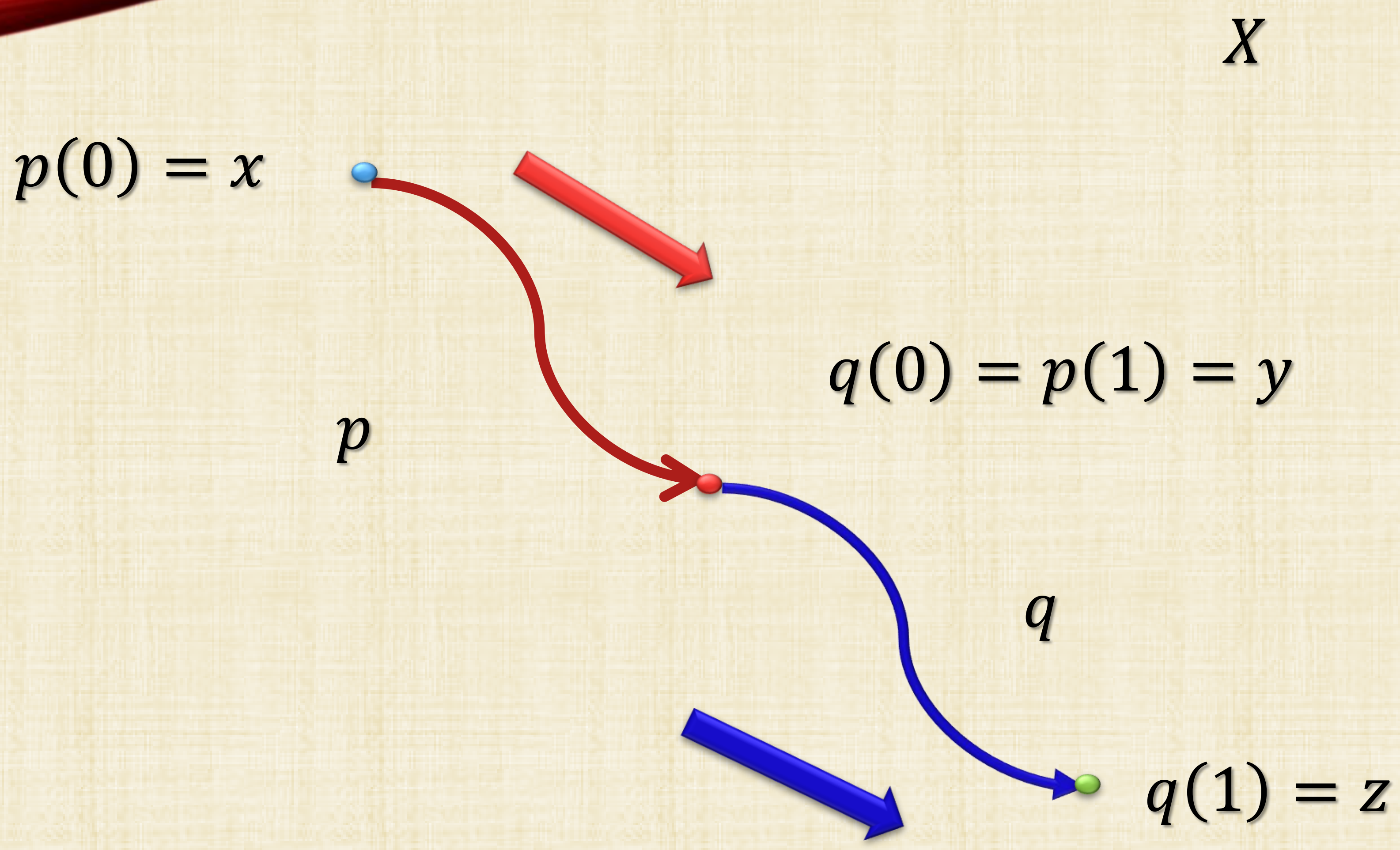
Examples:

3. If $p, q: I \to X$ are two paths in X with $p(1) = q(0)$, then the map

   $p \cdot q: I \to X$ that defined as;

$$p \cdot q(\mathrm{t}) = \begin{cases} p(2t); & 0 \le t \le {}^{1}/_{2} \\ q(2t-1); & {}^{1}/_{2} \le t \le 1 \end{cases};$$

   is also a path in X.



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Definition:

A topological space X is said to be path connected (or arc-wise connected), if for any given two points $x, y \in X$, there is a path $p: I \to X$ such that $p(0) = x$ and $q(1) = y$.

Theorem:

Every path connected space is connected and the converse need not to be true in general.

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Examples:

1. The Euclidean space $\mathbb{R}^n$ is path connected. In fact, for any two points $x, y \in \mathbb{R}^n$, the mapping $p: I \to \mathbb{R}^n$ that defined by:

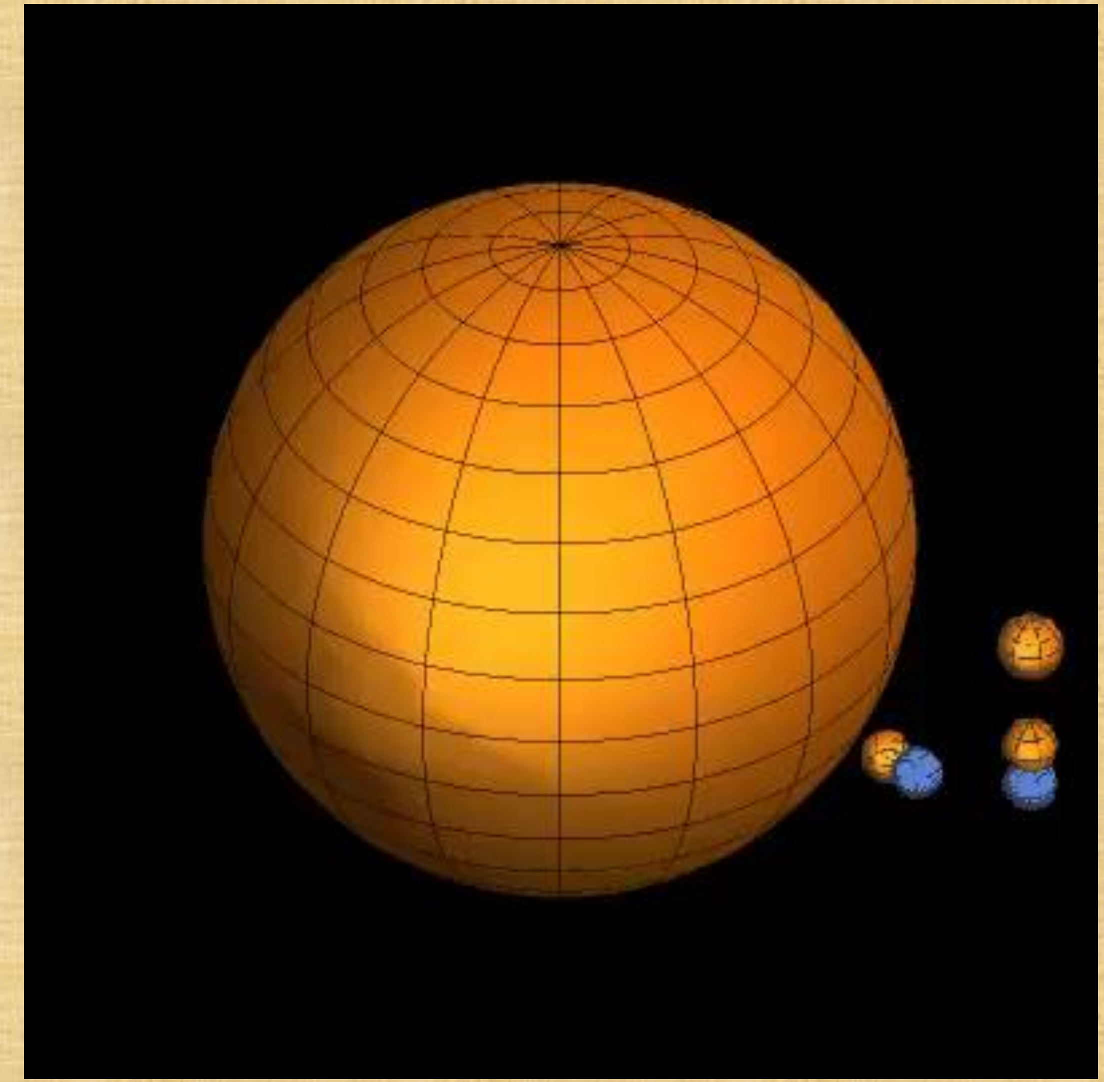
$$p(t) = (1 - t).x + t.y, \text{ for all } t \in I.$$

2. Every convex subset of $\mathbb{R}^n$ is path connected.
3. The circle $S^1 \subseteq \mathbb{R}^2$ is path connected.
4. Every open ball $B(x; r) \subseteq \mathbb{R}^n$ is path connected.

Exercises:

1. If $g: X \rightarrow Y$ be a continuous map between two topological spaces and $p: I \rightarrow X$ be a path in X, then the composition map $g \circ p: I \rightarrow Y$ is a path in Y.
2. If X and Y are homeomorphic topological spaces, then X is path connected if, and only if, then Y is path connected.
3. If X and Y be two topological spaces, then $X \times Y$ is path connected if, and only if, X and Y are path connected.

Thank You Very Much For Lessening



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