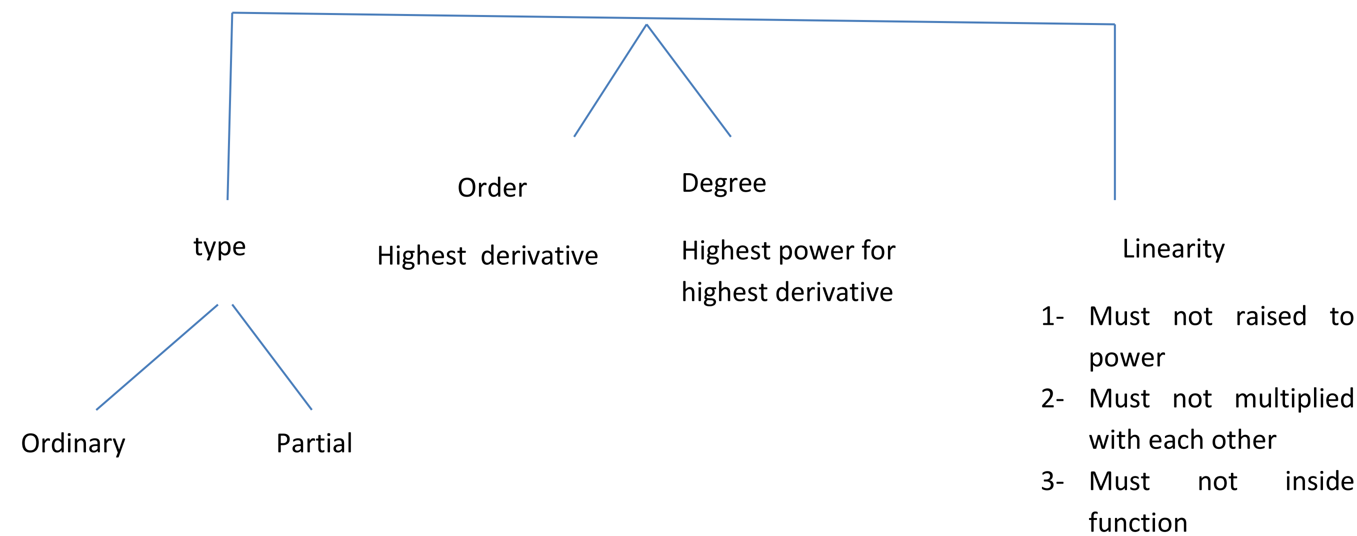
Differential equation

Classification and verification

Definition :- differential equation is an equation that contains derivatives of an unknown variable.

$$y' = x + y \ , \quad y + x + 4 = 0$$

Classification of DE



Verification

Verify that the given function is a solution to the DE.

$$y'' - 4y' + 3y = 0 \ , \quad y = e^{3x}$$

1- ***Solution of 1st order DE.***

A- Separable DE

Separable equation

If $y' = \frac{dx}{dy} = f(x).f(y)$ $\quad 0r \;\; y' = \frac{f(x)}{f(y)}$ $\;\; then\ the\ DE\ is\ separable$

Steps of solution

1- $y' = f(x).g(y)$
2- $\frac{dx}{dy} = f(x).g(y)$
3- $\frac{dy}{g(y)} = f(x)dx$
4- $\int \frac{dy}{g9x)} = \int f(x)dx + c$

Ex. Find the general solution of $y' = -2xy^2$

Sol.

$$\frac{dy}{dx} = -2xy^2$$

$$\frac{dy}{y^2} = -2xdx, \quad \int y^{-2}dy = \int -2xdx$$

$$-y^{-1} = -x^2 + c \quad\quad implicit\ solution$$

$$y = \frac{1}{x^2 - c} \quad explicit\ solution$$

Ex

$$\frac{dy}{dx} = \frac{4y}{x} \quad\quad sol.\, y = x^4.\, c_1$$

Ex

$$\frac{dy}{dx} = \left(\frac{2y + 3}{4x + 5}\right)^2$$

$$\frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2}$$

$$\int \frac{dy}{(2y+3)^2} = \int \frac{dx}{(4x+5)^2}$$

$$\frac{(2y+3)^{-1}}{-2} = \frac{(4x+5)^{-1}}{-4} + c$$

Ex. $y' = xe^{x+y}$

Sol.

$$e^{-y} = -xe^{x} + e^{x} - c\,, \quad y = -\ln(xe^{x} + e^{x} - c)$$

2- Homogenous DE

Steps

1- $\frac{dy}{dx}$

2- Replaced any x = xt, y=yt the equation not change

3- Suppose that y=ux or x=uy

Ex.

Solve the DE

$$(x^3 + y^3)dx - 3xy^2dy = 0$$

Sol

$$\frac{dy}{dx} = \frac{x^3 + y^3}{3xy^2}\,, \qquad \frac{dy}{dx} = \frac{t^3x^3 + t^3y^3}{t.t^2 3xy^2} = \frac{x^3 + y^3}{3xy^2}$$

Use y=ux

dy= xdu+udx sub. in DE

$$(x^3 + u^3x^3)dx - 3x(u^2y^2)(xdu + udx) = 0$$

$$x^3dx + u^3x^3dx - 3x^4u^2du - 3x^3u^3dx) = 0$$

$$x^3dx - 2u^3x^3dx = 3x^4u^2du$$

$$x^3(1 - 2u^3)dx = 3x^4u^2du$$

$$(1 - 2u^3)dx = 3xu^2du$$

$$\frac{dx}{x} = \frac{u^2du}{(1 - 2u^3)}$$

$$\int \frac{dx}{x} = \int \frac{u^2du}{(1 - 2u^3)}$$

$$lnx = -\frac{1}{2}ln|1 - 2u^3| + c$$

$$lnx = \frac{1}{2}ln\left|1 - 2\left(\frac{y}{x}\right)^3\right| + c$$

4- <u>Exact DE</u>

$$M(x,y)dx + N(x,y)dy = 0$$

Must be

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial X}$$

Methods of solution

1- Sum of equation limits
2- $d(u(x,y)) = 0$

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Ex.

Solve the DE

$$(2x - y + 1)dx + (2y - x - 1)dy = 0$$

Sol.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial X}\ , -1 = -1$$

1- Sum

$$2xdx - ydx + dx + 2ydy - xdy - dy = 0$$
$$d(x^2) + d(y^2) + dx - dy = ydx + xdy$$
$$d(x^2 + y^2 + x - y) = d(xy)$$

by integration

$$x^2 + y^2 + x - y = xy + c$$
$$x^2 + y^2 - xy + x - y = c$$

2- $M = \frac{\partial u}{\partial x}$

By integration

$$u = x^2 - yx + x + c_y$$

$$N = \frac{\partial u}{\partial y} \rightarrow 2y - x - 1 = -x + c'_y$$

$$\therefore c'_y = 2y - 1 \quad int. \int dy$$

$$c_y = y^2 - y + c$$

$$\therefore = x^2 + y^2 - xy + x - y + c$$

Ex. Solve the DE.

$$\frac{y}{x}dx + (y^2 + ln|x|)dy = 0$$

Sol.

$$\frac{y}{x}dx + y^2dy + ln|x|dy = 0$$

$$d\left(\frac{y^3}{3}\right) = -\frac{y}{x}dx - ln|x|dy \rightarrow d\left(\frac{y^3}{3}\right) = d(-yln|x|) \rightarrow \int$$

$$\frac{y^3}{3} = -yln|x| + c$$

H.W.e

$$(2\sin(x)\cos(x) + \tan(y))dx + xsec^2(x)dy = 0$$

5- Non-Exact DE

$$M(x, y)dx + N(x, y)dy = 0$$

If

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial X}\ , this\ Non - excat\ DE$$

And find μ call the integration factor (IF)

Three cases for IF

1- $f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] \rightarrow \quad \mu = e^{\int f(x)dx}$
2- $f(y) = \frac{1}{y}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] \rightarrow \quad \mu = e^{\int f(y)dy}$
3- $f(x, y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{Nf_x - Mf_y} \rightarrow \quad \mu = e^{\int f(z)dz}, \quad z=f(x,y)$

Ex. Solve the following DE.

$$(x^2 + y^2 + x)dx + xydy = 0$$

Sol.

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = y, \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\ , Non - exact\ DE$$

$$f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{xy}(2y - y) = \frac{1}{x} \rightarrow \quad \mu = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx}$$
$$= e^{lnx} = x = IF$$

Mult. DE by IF

$$(x^3 + xy^2 + x^2)dx + x^2ydy = 0$$

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = 2xy, \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$x^3dx + xy^2dx + x^2dx + x^2ydy = 0$$

$$d\left(\frac{x^4}{4}\right) + d\left(\frac{x^3}{3}\right) = -(\,xy^2dx + x^2ydy) = -d(x^2y^2)$$

$$\frac{x^4}{4} + \frac{x^3}{3} = -\frac{1}{2}x^2y^2 + c$$

$$(2xy^2 - y)dx + (y^2 + x + y)dy = 0$$

Sol.

$$\frac{\partial M}{\partial y} = 4xy - 1, \qquad \frac{\partial N}{\partial x} = 1, \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$f(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{2xy^2 - y}(1 + 1 - 4xy) = \frac{-2(2xy-1)}{y(2xy-1)} = \frac{-2}{y}$$

$$\rightarrow \quad \mu = e^{\int f(y)dy} = e^{\int \frac{-2}{y}dy} = e^{lny^{-2}} = y^{-2} = IF$$

$$\left(2x - \frac{1}{y}\right)dx + (1 + \frac{x}{y^2} + \frac{1}{y})dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{y^2}, \qquad \frac{\partial N}{\partial x} = \frac{1}{y^2}, \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$u = \int \left(2x + \frac{1}{y}\right)dx = x^2 - \frac{x}{y} + c_y$$

$$\frac{\partial u}{\partial y} = N = \frac{x}{y^2} + c'_y = 1 + \frac{x}{y^2} + \frac{1}{y} \rightarrow c'_y = 1 + \frac{1}{y}$$

$$c_y = \int 1 + \frac{1}{y}dy = y + lny + c$$

$$u = x^2 - \frac{x}{y} + y + lny + c$$

$$2xdx - \frac{1}{y}dx + dy + \frac{x}{y^2}dy + \frac{1}{y}dy = 0$$

$$2xdx + dy + \frac{1}{y}dy = \frac{1}{y}dx - \frac{x}{y^2}dy$$

$$d(x^2 + y + lny) = d\left(\frac{x}{y}\right)$$

$$x^2 + y + lny = \frac{x}{y} + c$$

6- Linear Differential Equations

The solution method for linear equations is based on writing the equation as

$$y' \mp p(x)y = q(x)$$

SOLUTION METHOD:

Step 1. Identify and write the equation in the form (1).

Step 2. Calculat

$$IF = e^{\int p(x)dx} \ \ or \ \ h(x) = \int p(x)dx$$

Step 3. Multiply the equation by IF to obtain

$$IFy' + IFp(x)y = IFq(x)$$

$$d(IFy) = IFq(x)$$

Step 4. The equation in Step 3 implies that

$$IFy = \int IFq(x)dx + C$$

Example 1. Find the general solution of

$$y' + 2x\,y = x.$$

$$h(x) = \int 2x\,dx = x^2 \quad \text{and} \quad e^{h(x)} = e^{x^2}.$$

$$
\begin{aligned}
e^{x^2} y' + 2x\, e^{x^2} y &= x\, e^{x^2} \\
\left[e^{x^2} y\right]' &= x\, e^{x^2}
\end{aligned}
$$

$$e^{x^2} y = \int x\, e^{x^2}\, dx = \tfrac{1}{2} e^{x^2} + C$$

$$y = e^{-x^2}\left[\frac{1}{2} e^{x^2} + C\right] = \tfrac{1}{2} + C\, e^{-x^2}.$$

Example 2. Find the solution of the initial-value problem

$$x^2 y' - x\, y = x^4 \cos 2x, \quad y(\pi) = 2\pi.$$

$$y' - \frac{1}{x} y = x^2 \cos 2x,$$

$$h(x) = \int (-1/x)\, dx = -\ln x = \ln x^{-1}. \text{ Then } \; e^{h(x)} = e^{\ln x^{-1}} = x^{-1}.$$

$$x^{-1} y' - x^{-2} y = x \cos 2x \quad \text{which is the same as} \quad \left[x^{-1} y\right]' = x \cos 2x.$$

$$x^{-1} y = \int x \cos 2x\, dx + C = \tfrac{1}{2} x \sin 2x + \tfrac{1}{4} \cos 2x + C$$

$$y = \tfrac{1}{2} x^2 \sin 2x + \tfrac{1}{4} x \cos 2x + Cx.$$

$$y(\pi) = 2\pi \quad \text{implies} \quad \tfrac{1}{2} \pi^2 \sin 2\pi + \tfrac{1}{4} \pi \cos 2\pi + C\pi = 2\pi$$

$$
\begin{aligned}
\tfrac{1}{4}\pi + C\pi &= 2\pi \\
C &= \tfrac{7}{4}
\end{aligned}
$$

$$y = \tfrac{1}{2} x^2 \sin 2x + \tfrac{1}{4} x \cos 2x + \tfrac{7}{4} x$$

7- Bernoulli Equations: The differential equation

$$y' + p(x)y = q(x)y^n, \quad n \neq 0,\ n \neq 1$$

To solve multiply the equation by y^{-n} to obtain

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

The substitution $w, v = y^{1-n}, w' = (1-n)y^{-n}y'$

$$\frac{1}{1-n}w' + p(x)w = q(x) or\ w' + (1-n)p(x)w = (1-n)q(x)$$

A linear equation in *w* and *x*

Ex. solve the DE

$$2xyy' + y^2 = 2x^2$$

Sol

Divide the eq. by $2xy$

$$y' + \frac{y}{2x} = xy^{-1}$$

$$yy' + \frac{y^2}{2x} = x$$

$$w = y^{1-n} = y^{1-(-1)} = y^2, \qquad w' = 2yy'$$

$$\therefore \frac{1}{2}w' + \frac{1}{2x}w = x, \qquad w' + \frac{1}{x}w = 2x$$

$$IF = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = x$$

$$xw' + w = 2x^2, \qquad d(xw) = 2x^2$$

$$xw = \frac{2}{3}x^3 + c$$

Second order of DE.

DE from second and high order

no constant coefficients

constant coefficients

Two methods for solve

Homogenous nonhomogeneous

Reduction order general formula

1- Constant coefficient (homogenous)

When right side equal to 0 and left side include y and y’

For solve let's say $y = e^{mx}$

We convert the equation into a general equation

Ex.

Solve the DE

$$y'' + 5y' - 6y = 0$$

Sol.

$$let\ y = e^{mx}, y' = me^{mx}, \qquad y'' = m^2 e^{mx}$$

$$m^2 e^{mx} + 5me^{mx} - 6e^{mx} = 0$$

$$e^{mx}(m^2 + 5m - 6) = 0$$

$$e^{mx}(m + 6)(m - 1) = 0, m = -6\ or\ m = 1$$

$$y = c_1 e^{-6x} + c_2 e^{x}$$

Ex. Solve $\frac{d^2y}{dx^2} + 4y = 0$

Sol.

$$let\ y = e^{mx}, y' = me^{mx}, \qquad y'' = m^2e^{mx}$$

$$m^2e^{mx} + 4e^{mx} = 0$$

$$e^{mx}(m^2 + 4) = 0$$

$$m = \mp 2i$$

$$m = 0 \mp 2i$$

$$y = e^{0x}(c_1 sin2x + c_2 cos2x)$$

$$y = (c_1 sin2x + c_2 cos2x)$$

Ex. Solve $y'' + 6y' + 9y = 0$

Sol.

$$let\ y = e^{mx}, y' = me^{mx}, \qquad y'' = m^2e^{mx}$$

$$m^2e^{mx} + 6me^{mx} + 9e^{mx} = 0$$

$$e^{mx}(m^2 + 6m + 9) = 0$$

$$e^{mx}(m + 3)(m + 3) = 0, m = -3$$

$$y = c_1e^{-3x} + c_2xe^{-3x}$$

Ex. Find solution for DE

$$y''' + 2y'' + 5y' = 0$$

Sol.

$$let\ y = e^{mx}, y' = me^{mx}, \qquad y'' = m^2e^{mx}, y''' = m^3e^{mx}$$

$$m^3 + 2m^2 + 5m = 0$$

$$m(m^2 + 2m + 5) = 0$$

$m = 0, m = -1 \mp 2i$

$$y = c_1 e^{0x} + e^{-1x}(c_2 sin2x + c_3 cos2x)$$

$$y = c_1 + e^{-x}(c_2 sin2x + c_3 cos2x)$$

2- Nonhomogeneous

Variation of parameters | undetermined coefficient

a- Variation of parameters

$$y^{(n)} + y^{(n-1)} + \cdots + y = p(x)$$

Solution

$$y = y_c + y_p$$

y_c is the solution for the homogeneous second order DE

$$y^{(n)} + y^{(n-1)} + \cdots + y = 0$$

$$y_c = c_1 y_1 + c_2 y_2$$

$$y_p = u_1 y_1 + u_2 y_2$$

Ex. Solve the DE by use variation of parameters

$$y'' + y = secx$$

Sol.

$$y'' + y = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1, m = \pm i,$$

$$y_c = c_1 sinx + c_2 cosx, \;\; y_1 = sinx, y_2 = cosx$$

$$y'' + y = secx$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix} = -sin^2 x - cos^2 x = -1$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ p(x) & y'_2 \end{vmatrix} = \begin{vmatrix} 0 & cosx \\ secx & -cosx \end{vmatrix} = -1$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & p(x) \end{vmatrix} = \begin{vmatrix} sinx & 0 \\ cosx & secx \end{vmatrix} = tanx$$

$$u_1 = \int \frac{w_1}{w} dx = x$$

$$u_2 = \int \frac{w_2}{w} dx = \int \frac{tanx}{-1} = -lncosx$$

$$y_p = xsinx - cosxlncosx$$

$$y = y_c + y_p = c_1 sinx + c_2 cosx + xsinx - cosxlncosx$$

Ex. Find general solution for

$$y'' + y = sin6x$$

Sol.

$$y'' + y = 0$$

$$m^2 + 1 = 0$$

$$m^2 = 1, m = \pm 1,$$

$$y_c = c_1 e^x + c_2 e^{-x}, \;\; y_1 = e^x, y_2 = e^{-x}$$

$$y'' + y = sin6x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$w=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2$$

$$w_1=\begin{vmatrix} 0 & y_2 \\ p(x) & y'_2 \end{vmatrix}=\begin{vmatrix} 0 & e^{-x} \\ sin6x & -e^{-x} \end{vmatrix}=-e^{-x}sin6x$$

$$w_2=\begin{vmatrix} y_1 & 0 \\ y'_1 & p(x) \end{vmatrix}=\begin{vmatrix} e^x & 0 \\ e^x & sin6x \end{vmatrix}=e^{x}sin6x$$

$$u_1=\int\frac{w_1}{w}dx=\frac{1}{2}\left[-\frac{1}{37}e^{-x}(sin6x+6cos6x)\right]$$

$$u_2=\int\frac{w_2}{w}dx=-\frac{1}{2}\left[-\frac{1}{37}e^{x}(sin6x-6cos6x)\right]$$

$$y_p=\frac{1}{74}(sin6x+6cos6x)+\frac{-1}{74}(sin6x-6cos6x)=\frac{6}{37}cos6x$$

$$y=c_1e^x+c_2e^{-x}+\frac{6}{37}cos6x$$

b- Undetermined coefficient

$$y^{(n)}+y^{(n-1)}+\cdots+y=p(x)$$

Solution

$$y=y_c+y_p$$

$y_c$is the solution for the homogeneous second order DE

$$y^{(n)}+y^{(n-1)}+\cdots+y=0$$

$$y_c=c_1y_1+c_2y_2$$

$y_p$is the special solution depended on *p(x)*

Ex.

$$y''+2y'-3y=5e^{2x}$$

Sol.

$$m^2 + m - 3 = 0, (m+3)(m-1) = 0, \therefore m = -3, m = 1$$

$$y_c = c_1 e^{-3x} + c_2 e^x$$

$$y'' + 2y' - 3y = 5e^{2x}$$

$$y_p = Ae^{2x}, y'_p = 2Ae^{2x}, y''_p = 4Ae^{2x}$$

$$4Ae^{2x} + 4Ae^{2x} - 3Ae^{2x} = 5e^{2x}$$

$$A = 1$$

$$\therefore y_p = e^{2x}$$

$$y = c_1 e^{-3x} + c_2 e^x + e^{2x}$$

Ex. Find y_p for the DE

$$\frac{y''}{x} + \frac{4y}{x} - \frac{y'}{x} = e^{-2x}$$

Sol.

$$y'' - y' + 4y = xe^{-2x}$$

Find y_c

$$m^2 - m + 4 = 0, \qquad m = \frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$

$$y_c = e^{\frac{1}{2}x} \left[c_1 sin \frac{\sqrt{15}}{2} x + c_2 cos \frac{\sqrt{15}}{2} x \right]$$

$$y'' - y' + 4y = xe^{-2x}, \qquad f(x) = xe^{-2x}$$

$$y_p = (A + Bx)e^{-2x}$$

Ex. Find y_p for the DE

$$y'' - y = e^x + sinx$$

Sol.

$$y'' - y = e^x + sinx, \qquad f(x) = e^x + sinx$$

$$m^2 - 1 = 0, \qquad m_{1,2} = \pm 1$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = Axe^x + (Bsinx + Lcosx)$$

Ex. Find y_p for the DE

$$y^{(4)} - y = e^x - xe^{-x} + 2sinx + x^3 cosx$$

Sol.

$$m^4 - 1 = 0, \qquad m^2 = \pm 1, m = \pm 1, \pm i$$

$$y_c = c_1 e^x + c_2 e^{-x} + [c_3 sinx + c_4 cosx]$$

$$y_1 = Axe^x$$

$$y_2 = (B + Cx)xe^{-x}$$

$$y_3 = (Dx^3 + Ex^2 + Fx + G)x(Hsinx + Icosx)$$

$$y_3 = (Dx^4 + Ex^3 + Fx^2 + Gx)(Hsinx + Icosx)$$

$$y_p = y_1 + y_2 + y_3$$

EX.

$$y'' - 2y' - 3y = e^{2t}$$

Sol.

$y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$

$y_c = C_1 e^{-t} + C_2 e^{3t}$

$$let, y_p = Ae^{2t}, y'_p = 2Ae^{2t}, y''_p = 4Ae^{2t}$$

$$(4A\,e^{2t}) - 2(2A\,e^{2t}) - 3(A\,e^{2t}) = e^{2t}$$

$$-3A e^{2t} = e^{2t}$$

$$A = -1/3$$

$$y_p = -1/3e^{2t}$$

$$y = y_c + y_p$$

Ex.

$$y'' - 2y' - 3y = 3t^2 + 4t - 5$$

Sol.

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = (r+1)(r-3) = 0$$

$$y_c = C_1 e^{-t} + C_2 e^{3t}$$

$$g(t) = 3t^2 + 4t - 5$$

$$y_p = At^2 + Bt + C, y'_p = 2At + B, y''_p = 2A$$

$$(2A) - 2(2At + B) - 3(At^2 + Bt + C) = 3t^2 + 4t - 5$$

$$-3At^2 + (-4A - 3B)t + (2A - 2B - 3C) = 3t^2 + 4t - 5$$

t^2 :	$3 = -3A$		$A = -1$
t :	$4 = -4A - 3B$	$\rightarrow$	$B = 0$
1 :	$-5 = 2A - 2B - 3C$		$C = 1$

$$y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - t^2 + 1$$

Ex.

$$y'' - 2y' - 3y = 5\cos(2t)$$

Sol.

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = (r+1)(r-3) = 0$$

$$y_c = C_1 e^{-t} + C_2 e^{3t}$$

$$y_p = A\cos(2t) + B\sin(2t), y'_p = -2A\sin(2t) + 2B\cos(2t), y''_p = -4A\cos(2t) - 4B\sin(2t)$$

$$(-4A\cos(2t) - 4B\sin(2t)) - 2(-2A\sin(2t) + 2B\cos(2t)) - 3(A\cos(2t) + B\sin(2t)) = 5\cos(2t)$$

$$(-4A - 4B - 3A)\cos(2t) + (-4B + 4A - 3B)\sin(2t) = 5\cos(2t)$$

$$(-7A - 4B)\cos(2t) + (4A - 7B)\sin(2t) = 5\cos(2t) + 0\sin(2t)$$

$\cos(2t)$:	$5 = -7A - 4B$	$\rightarrow$	$A = -7/13$
$\sin(2t)$:	$0 = 4A - 7B$	$\rightarrow$	$B = -4/13$

$$y = C_1 e^{-t} + C_2 e^{3t} - \frac{7}{13}\cos(2t) - \frac{4}{13}\sin(2t)$$

Cauchy-Euler Equations

Definition and Solution Method

1. A second order Cauchy-Euler equation is of the form
$$a_2x^2y'' + a_1xy' + a_0y = g(x)$$
If g(x) = 0, then the equation is called homogeneous.
2. To solve a homogeneous Cauchy-Euler equation we set $y = x^m$ and solve for *m*.
3. The idea is similar to that for homogeneous linear differential equations with constant coefficients.

Ex. Solve the Initial Value Problem

$$2x^2y'' + xy' - y = 0, \qquad y(1) = 1, y'(1) = 2$$

Sol.

Let.

$$y = x, \;\; y' = mx^{m-1}, xy' = mx^m, \;\; y'' = m(m-1)x^{m-2}, x^2y'' = m(m-1)x^m$$

Sub.

$$2m(m-1)x^m + mx^m - x^m = 0$$

$$2m^2 - m - 1 = 0, (2m+1)(m-1) = 0$$

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+8}}{4} = 1, -1/2$$

$$y = c_1x^1 + c_2x^{-1/2} = c_1x + \frac{c_2}{\sqrt{x}}$$

Initial value

$$1 = c_1 + c_2$$

$$y' = c_1 - \frac{1}{2}c_2x^{-\frac{3}{2}}, 2 = c_1 - \frac{1}{2}c_2$$

$$c_1 = \frac{5}{3}, c_2 = -\frac{2}{3}$$

4- Inhomogeneous Cauchy-Euler equations are solved with Variation of Parameters.

Ex. solve the flowing equation

$$xy'' - 2y' + \frac{2}{x}y = x$$

Sol.

$$x^2y'' - 2xy' + 2y = x^2$$

Let,

$y = x^m, xy' = mx^m, x^2y'' = m(m-1)x^m$

Sub.

$$m(m-1)x^m - 2mx^m + 2x^m = 0$$

$$m^2 - m - 2m + 2 = 0, m^2 - 3m + 2 = 0, m_{1,2} = 1,2$$

$$y_c = c_1x + c_2x^2$$

$$y_p, f(x) = 1, y_1 = x, y_2 = x^2$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ p(x) & y'_2 \end{vmatrix} = -x^2$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & p(x) \end{vmatrix} = x$$

$$u_1 = \int \frac{w_1}{w} dx = \int \frac{-x^2}{x^2} dx = -x$$

$$u_2 = \int \frac{w_2}{w} dx = lnx$$

$$y = y_c + y_p = c_1 x + c_2 x^2 + xlnx - x^2$$

<u>Second–Order Differential Equations Reducible to the First Order</u>

Case I: F(x, y', y") = 0 –– y does not appear explicitly

[Example] y" = y' tanh x

[Solution] Set y' = z and $y'' = \frac{dz}{dx}$

Thus, the differential equation becomes first order

z' = z tanh x

which can be solved by the method of separation of variables

$$\frac{dz}{z} = tanhx\, dx = \frac{sinhx}{coshx} dx$$

$$ln|z| = ln|coshx| + c$$

$$\therefore z = c_1 coshx, \;\; or\ y' = c_1 coshx$$

$$\frac{dy}{dx} = c_1 coshx$$

$$y = c_1 sinhx + c_2$$

Example: Solve $y'' + 2y' = 4x$.

$$z = f(x) = \frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dz}{dx} = \frac{d^2y}{dx^2}.$$

$$\therefore \quad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 4x \quad \text{is reduced to}$$

$$\frac{dz}{dx} + 2z = 4x, \quad \text{(Linear 1}^{\text{st}}\text{ order DE w.r.t } z\text{)}$$

where, $P(x) = 2$ and $Q(x) = 4x$.

$$\mu = e^{\int P(x)dx} \quad \Rightarrow \quad \mu = e^{\int 2dx} = e^{2x}.$$

$$\mu.z = \int \mu.Q dx + C \qquad \Rightarrow \qquad e^{2x}.z = \int e^{2x}.(4x)dx + C_1,$$

$$\mu.z = \int \mu.Q dx + C \qquad \Rightarrow \qquad e^{2x}.z = \int e^{2x}.(4x)dx + C_1,$$

$$\Rightarrow \quad ze^{2x} = 4\int xe^{2x}dx + C_1 \quad \Rightarrow \quad ze^{2x} = 4[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}] + C_1,$$

$$\Rightarrow \quad ze^{2x} = 2xe^{2x} - e^{2x} + C_1 \quad \Rightarrow \quad z = 2x - 1 + C_1e^{-2x},$$

$$\text{but} \quad z = \frac{dy}{dx} \quad \Rightarrow \quad \therefore \frac{dy}{dx} = 2x - 1 + C_1e^{-2x}, \quad \text{(Separable DE)}$$

$$\Rightarrow \quad y = x^2 - x - \frac{1}{2}C_1e^{-2x} + C_2. \qquad \text{(G.S)}$$

Case II: F(y, y', y") = 0 – x does not appear explicitly

[Example] $y'' + y'^3 \cos y = 0$

[Solution]

set

z = y' = dy/dx thus, $y'' = \frac{dz}{dy} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy}y' = \frac{dz}{dy}z$

Thus, the above equation becomes a first–order differential equation of z (dependent variable) with respect to y (independent variable):

$$\frac{dz}{dy}z + z^3 cosy = 0$$

which can be solved by separation of variables:

$$\frac{dz}{z^2} = cosydy$$

$$z^{-1} = siny + c$$

$$z = y' = \frac{dy}{dx}$$

$$\therefore (siny + c)dy = dx$$

$$-cosy + cy + c_1 = x$$

Example: Solve $4y(y')^2 y'' = (y')^4 + 3$.

but $z = \frac{dy}{dx} \quad \Rightarrow \quad \therefore \quad z.\frac{dz}{dy} = \frac{d^2y}{dx^2}$.

$\therefore \; 4y(y')^2 y'' = (y')^4 + 3$ is reduced to

$4yz^2(z.\frac{dz}{dy}) = z^4 + 3$. (Separable DE)

$$\therefore \quad \frac{4z^3 dz}{z^4 + 3} = \frac{dy}{y} \quad \Rightarrow \quad \ln(z^4 + 3) = \ln y + C \quad \Rightarrow \quad \ln(\frac{z^4 + 3}{y}) = C,$$

$$\Rightarrow \quad \frac{z^4 + 3}{y} = C_1 \quad [\text{where } C_1 = e^C] \quad \Rightarrow \quad z^4 = C_1 y - 3,$$

$$\Rightarrow \quad z = (C_1 y - 3)^{1/4}. \qquad \text{But } z = \frac{dy}{dx},$$

$$\therefore \quad \frac{dy}{dx} = (C_1 y - 3)^{1/4}, \quad \text{(Separable DE)}$$

$$\therefore \quad \frac{dy}{(C_1 y - 3)^{1/4}} = dx \quad \Rightarrow \quad (C_1 y - 3)^{-1/4} dy = dx,$$

$$\Rightarrow \quad \frac{4}{3C_1}(C_1 y - 3)^{3/4} = x + C_2,$$

or $$(C_1 y - 3)^{3/4} = \frac{3C_1}{4}(x + C_2). \qquad \text{(G.S)}$$

y'' = dz dx = dz dy dy dx = dz dy y' = dz dy z

Linear systems of DE:-

1- Elimination method

$$\frac{d}{dt} = D,$$

$$\frac{dy}{dt} = Dy, \frac{dx}{dt} = Dx,$$

$$\frac{d^2y}{dt^2} = D^2y, \frac{d^2x}{dt^2} = D^2x$$

Then solve the equations like the linear systems equations

Ex. Find solve for the following system

$$\frac{dx}{dt} = 3x - 2y \dots\dots 1$$

$$\frac{dy}{dt} = 2x - 2y \dots\dots 2$$

Sol.

$$Dx - 3x + 2y = 0$$

$$Dy + 2y - 2x = 0$$

$$(D-3)x + 2y = 0, \;\; 2(D-3)x + 4y = 0$$

$$-2x + (D+2)y = 0, -2(D-3)x + (D^2 - D - 6)y = 0$$

$$y'' + y' - 2y = 0$$

Let $y = e^{mt}$

$$m^2 - m - 2 = 0, \;\; m_1 = -1, m_2 = 2$$

$$y = c_1 e^{-t} + c_2 e^{2t}$$

Sub. y and y' in 2

$$y' = -c_1 e^{-t} + 2c_2 e^{2t}$$

$$-c_1e^{-t} + 2c_2e^{2t} = 2x - 2c_1e^{-t} - 2c_2e^{2t}$$

$$x = \frac{1}{2}c_1e^{-t} + 2c_2e^{2t}$$

Quiz

Solve

$$cos(x + y)dx + (3y^2 + 2y + cos(x + y)dy = 0$$

Ex.

Mixing problems occur quite frequently in chemical industry. We explain here how to solve the basic model involving a single tank. The tank in Fig. 1 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal min. Find the amount of salt in the tank at any time t.

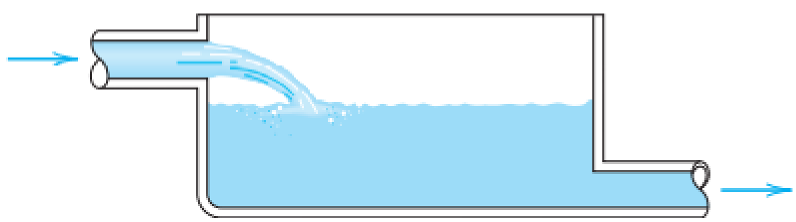


Fig. 1

Solve.

Step 1. Setting up a model. Let denote the amount of salt in the tank at time t. Its time rate of change is

$$y' = salt\ inflow\ rate - salt\ outflowrate$$

5 lb times 10 gal gives an inflow of 50 lb of salt. Now, the outflow is 10 gal of brine. This is $\frac{10}{1000} = 0.01$ $(= 1\%)$of the total brine content in the tank, hence 0.01 of the salt content , that is, 0.01 . Thus the model is the ODE

$$y' = 50 - 0.01y = -0.01(y - 5000)$$

$$\frac{dy}{dt} = -0.01(y - 5000)$$

$$\frac{dy}{y - 5000} = -0.01dt, \qquad \ln(y - 5000) = -.01t + c$$

$$y = c_1 e^{-0.01t} + 5000$$

Ex.

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 2). You are asked to find the height of the water in the tank at any time if the tank has diameter 2 m, the hole has diameter 1 cm, and the initial height of the water when the hole is opened is 2.25 m. When will the tank be empty?

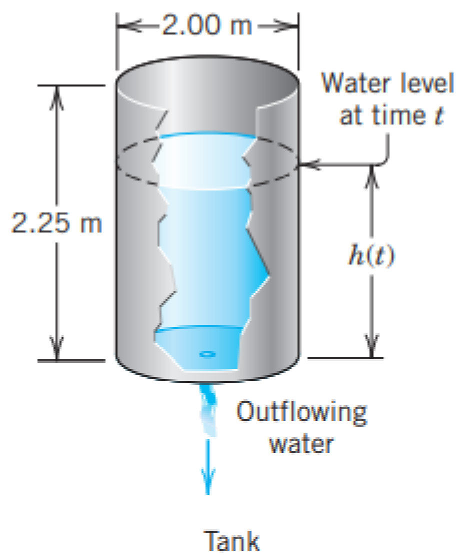


Fig. 2

Physical information. Under the influence of gravity the outflowing water has velocity

$$v(t) = 0.6\sqrt{2gh(t)} \qquad Torricelli'slaw$$

Step 1. Setting up the model. To get an equation, we relate the decrease in water level to the outflow. The volume of the outflow during a short timeΔt is

$$\Delta V = Av\Delta t, \quad A = area\ of\ the\ hole$$

ΔV must equal the change of the volume of the water in the tank. Now

$$\Delta V = -B\Delta h, \quad B = Crss - sectional\ area\ of\ the\ tank$$

$$-B\Delta h = Av\Delta t$$

We now express v according to Torricelli's law and then let (the length of the time interval considered) approach 0—this is a standard way of obtaining an ODE as a model. That is, we have

$$\frac{\Delta h}{\Delta t} = -\frac{A}{B}v = -\frac{A}{B}0.6\sqrt{2gh(t)}$$

and by letting$\Delta t \to 0$ we obtain the ODE

$$\frac{dh}{dt} = -26.56\frac{A}{B}\sqrt{h}$$

A/B is constant

$$\frac{dh}{\sqrt{h}} = -26.56\frac{A}{B}dt$$

$$2\sqrt{h} = c_1 - 26.56\frac{A}{B}t, h(t) = (c - 0.000332t)^2$$

Laplace Transforms

Step 1. The given ODE is transformed into an algebraic equation, called the subsidiary equation.

Step 2. The subsidiary equation is solved by purely algebraic manipulations. Step 3. The solution in Step 2 is transformed back, resulting in the solution of the given problem.

$$F(s) = \mathcal{L}f(t) = \int_0^{\infty} e^{-st} f(t)dt$$

inverse transform of and is denoted by ; that is, we shall write

$$f(t) = \mathcal{L}^{-1}(F)$$

Ex.

Let y=t when $t \geq 0$ find the $F(s)$

Sol.

$$F(s) = \mathcal{L}f(t) = \int_0^{\infty} e^{-st} f(t)dt$$

$$F(s) = \mathcal{L}f(t) = \int_0^{\infty} e^{-st} dt = \frac{-1}{s}(e^{-s\infty} - e^{-s*0}) = \frac{1}{s}$$

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

Transforms of Derivatives and Integrals. ODEs:-

Laplace Transform of Derivatives The transforms of the first and second derivatives of satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0).$$

Differential Equations, Initial Value Problems

$$y'' + ay' + by = r(t), \qquad y(0) = K_0, \qquad y'(0) = K_1$$

Step 1. Setting up the subsidiary equation. This is an algebraic equation for the transform $Y = \mathcal{L}(y)$

$$[s^2Y - sy(0) - y'(0)] + a[sY - y(0)] + bY = R(s)$$

where $R(s) = \mathcal{L}(r)$. Collecting the Y-terms, we have the subsidiary equation

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s).$$

Step 2. Solution of the subsidiary equation by algebra. We divide by $s^2 + s + b$ and use the so-called transfer function

$$Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + b - \frac{1}{4}a^2}.$$

This gives the solution

$$Y(s) = [(s + a)y(0) + y'(0)]Q(s) + R(s)Q(s).$$

If $y(0) = 0, and\ y'(0) = 0$, this is simply $Y = RQ$;hence

$$Q = \frac{Y}{R} = \frac{\mathcal{L}(\text{output})}{\mathcal{L}(\text{input})}$$

Step 3. Inversion of Y to obtain $y = \mathcal{L}^{-1}(Y)$

Ex.

Solve $y'' - y = t, \ \ y(0) = 1, y'(0) = 1$

Sol.

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$s^2y - sy(0) - 1y'(0) - y = \frac{1}{s^2}$$

$$s^2y - s - 1 - y = \frac{1}{s^2}, y = \frac{(s+1)}{s^2-1} + \frac{1}{s^2(s^2-1)}$$

$$y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$

$$\therefore y(t) = \mathcal{L}^{-1}(y) = e^t + \sinh(t) - t$$

First shifting theorem (property).

If the given function as $f(t) = e^{at}g(t)$ the laplace transform for $f(t)$ is

$$\mathcal{L}\left(e^{at}g(t)\right) = G(s-a)$$

Note: $\mathcal{L}\left(g(t)\right) = G(s)$

Ex.

$$\mathcal{L}\{e^{2t}cos2t\} = G(s-2)$$

Sol.

$$a = 2, g(t) = cos2t$$

Laplace transform

$$G(s) = \frac{s}{s^2+4}$$

Shift s to s-2

$$G(s-2) = \frac{s-2}{(s-2)^2+4}$$

Ex. Determine Laplace transform for

$$f(t) = e^{2t}sin4t$$

Sol.

$$g(t) = sin4t$$

$$G(s) = \frac{4}{s^2+16}$$

$$\mathcal{L}\{e^{2t} sin4t\} = G(s-2) = \frac{4}{(s-2)^2 + 16}$$

Ex. Determine Laplace transform for

$$f(t) = (t^2 + \frac{1}{6}t^3)e^{-2t}$$

Sol.

$$F(s) = \mathcal{L}\left\{(t^2 + \frac{1}{6}t^3)e^{-2t}\right\}$$

$$g(t) = \left(t^2 + \frac{1}{6}t^3\right), G(s) = \frac{2!}{s^3} + \frac{1}{6}.\frac{3!}{s^4}$$

$$G(s-a) = G\big(s-(-2)\big) = G(s+2)$$

$$\therefore F(s) = \frac{2!}{(s+2)^3} + \frac{1}{6}.\frac{3!}{(s+2)^4} = \frac{2}{(s+2)^3} + \frac{3}{(s+2)^4}$$

(H.W). Solve the initial value problem

$$y'' + y' + 9y = 0$$

Mathematical Modeling in Chemical Engineering

The mathematical model is an expression that represent a phenomenon or an operation. When deriving the model we make use of the basic theoretical principles and the validity of the model is, then, tested experimentally.

The main problems to be solved re.

1- storage tanks.

2- mixing tanks.

3- chemical reaction vessels.

4- heat transfer problems.

5- mass transfer problems.

6- Process control systems.

7- Another problems.

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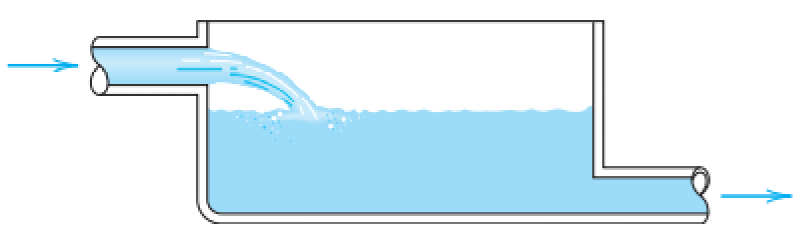

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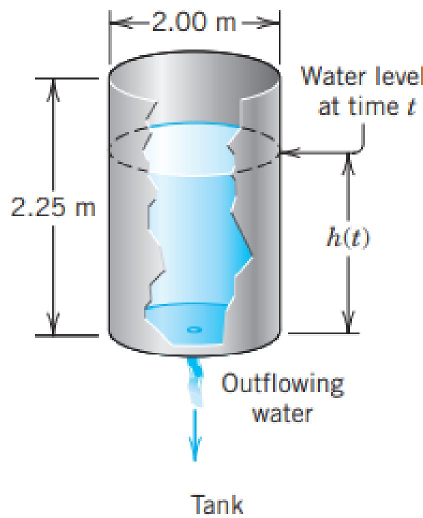

Fig. 2

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Step 1. Setting up the model. To get an equation, we relate the decrease in water level to the outflow. The volume of the outflow during a short timeΔt is

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$$2\sqrt{h} = c_1 - 26.56\frac{A}{B}t, h(t) = (c - 0.000332t)^2$$

Newton's law of cooling

The rate of change of body temperature with respect to time is directly proportional to the difference between body temperature and ambient temperature

Ex

A cup of tea has a temperature of 80 in a room that has a temperature of 30 and after one minute the temperature of the cup has become 70, how long does it take for the temperature of the tea to become 40

H.O.W

Ex.

A tank contains 400 gal of brine in which 100 lb of salt are dissolved. Fresh water runs into the tank at a rate of The mixture, kept practically uniform by stirring, runs out at the same rate. How much salt will there be in the tank at the end of 1 hour?

Ex

If a wet sheet in a dryer loses its moisture at a rate proportional to its moisture content, and if it loses half of its moisture during the first 10 min of drying, when will it be practically dry, say, when will it have lost 99% of its moisture? First guess, then calculate.

Fourier series

PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* T or to be *periodic* with period T if for all x, $f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the least period or simply the period of $f(x)$.

Basics

$$sinx.siny = \frac{1}{2}[cos(x-y) - cos(x+y)]$$

$$cosx.cosy = \frac{1}{2}[cos(x-y) + cos(x+y)]$$

$$sinx.cosy = \frac{1}{2}[sin(x-y) + sin(x-y)]$$

Fourier transforms table

$f(t)$	$F(\omega)$	$f(t)$	$F(\omega)$
e^{-at} $t \geq 0$ te^{-at}	$\frac{1}{i\omega + a}$ $\frac{1}{(i\omega + a)^2}$	$1 - \frac{\lvert t \rvert}{T}$ $\lvert t \rvert < T$	$\frac{sin^2\left(\frac{\omega T}{2}\right)}{\omega^2 T/2}$
1 $\delta(t)$	$\delta(\omega)$ 1	$e^{-\frac{t^2}{2}}$	$e^{-\frac{\omega^2}{2}}$
$\frac{a}{t^2 + a^2}$ $e^{-a\lvert t \rvert}$	$e^{-a\lvert\omega\rvert}$ $\frac{a}{\omega^2 + a^2}$	$\sin(at)$ $t > 0$ $\cos(at)$	$\frac{a}{(i\omega)^2 + a^2}$ $\frac{i\omega}{(i\omega)^2 + a^2}$
$U(t)$ $\lvert t \rvert < T$	$\frac{\sin(\omega T)}{\omega}$	$\sin(at)$ $t \in (-\infty, \infty)$ $\cos(at)$ $t \in (-\infty, \infty)$	$\frac{1}{2i}[\delta(\omega - a)$ $- \delta(\omega + a)]$ $\frac{1}{2}[\delta(\omega - a)$ $+ \delta(\omega + a)]$

Fourier transforms laws

1- Time shift

If $f(t) \xrightarrow{F(\omega)} F(\omega)$

Then $f(t-a)u(t-a) \rightarrow e^{-ia\omega}F(\omega)$

2- Frequency shift (modulation)

If $f(t) \xrightarrow{F(\omega)} F(\omega)$

Then $f(t).e^{\mp iat} \rightarrow F(\omega \pm a)$

$f(t).\cos(at) \rightarrow \frac{1}{2}[F(\omega - a) + F(\omega + a)]$, $\cos(at) = \frac{e^{iat}+e^{-iat}}{2}$

$f(t).\sin(at) \rightarrow \frac{1}{2i}[\mathrm{F}(\omega-a)-\mathrm{F}(\omega+a)]$, $\sin(at)=\frac{e^{iat}-e^{-iat}}{2i}$

Ex. Find F.T. for $f(t)=U(t).\sin(5t)$.

Sol.

1- $U(t) \xrightarrow{\mathrm{F}(\omega)} \mathrm{F}(\omega)=\frac{\sin(\omega T)}{\omega}$

2- $U(t).\sin(5t) \xrightarrow{\mathrm{F}(\omega)} \frac{1}{2i}\left[\frac{sin((\omega-5)T)}{\omega-5}-\frac{sin((\omega+5)T)}{\omega+5}\right]$

Ex. Find F.T. for $f(t)=\left(1-\frac{|t|}{2}\right).\sin^2(10t)$

Sol.

1- $1-\frac{|t|}{2} \xrightarrow{\mathrm{F}(\omega)} \mathrm{F}(\omega)=\frac{sin^2\left(\frac{\omega T}{2}\right)}{\omega^2 T/2}$

2- $\sin^2(\emptyset)=\frac{1}{2}-\frac{1}{2}\cos(2\emptyset)$

3- $\therefore f(t)=\left(1-\frac{|t|}{2}\right).\left(\frac{1}{2}-\frac{1}{2}cos(20t)\right)=\frac{1}{2}\left(1-\frac{|t|}{2}\right)-\frac{1}{2}\left(1-\frac{|t|}{2}\right).cos(20t)$

4- $f(t) \xrightarrow{\mathrm{F}(\omega)} \frac{1}{2}.\frac{sin^2(\omega)}{\omega^2}-\frac{1}{4}\left[\frac{sin^2(\omega-20)}{(\omega-20)^2}-\frac{sin^2(\omega+20)}{(\omega+20)^2}\right]$

<u>3- Derivative in time domain</u>

If $f(t) \xrightarrow{\mathrm{F}(\omega)} F(\omega)$

$$\therefore \frac{d^n f(t)}{dt^n} \xrightarrow{\mathrm{F}(\omega)} (i\omega)^n F(\omega)$$

<u>4- Integration in time domain</u>

If $f(t) \xrightarrow{\mathrm{F}(\omega)} F(\omega)$

$$\therefore \int_{-\infty}^{t} f(t)dt \xrightarrow{\mathrm{F}(\omega)} \frac{F(\omega)}{i\omega}$$

Ex. Find F. T. for $f(t) = \frac{d^2}{dt^2}(te^{-4t})$

Sol.

$$\therefore\ te^{-4t} \xrightarrow{\mathrm{F}(\omega)} \frac{1}{(iw+4)^2}$$

$$\frac{d^2}{dt^2}(te^{-4t}) \xrightarrow{\mathrm{F}(\omega)} (i\omega)^2 . \frac{1}{(iw+4)^2}$$

Ex. Find F. T. for $f(t) = \int_{-\infty}^{t} e^{-4|t|})$

Sol.

$$1 - \quad e^{-4|t|} \xrightarrow{\mathrm{F}(\omega)} \frac{a}{\omega^2 + a^2} = \frac{4}{\omega^2 + 16}$$

2- $f(t) \xrightarrow{\mathrm{F}(\omega)} \frac{1}{i\omega} . \left(\frac{4}{\omega^2+16}\right)$.

Ex. Find F.T. for $\cosh(t) \,.\, u(t-2)$

Sol.

$$\cosh(t) = \frac{e^t + e^{-t}}{2}$$

$$\therefore f(t) = \left(\frac{e^t + e^{-t}}{2}\right). u(t-2) = \frac{1}{2}[e^t . u(t-2) + e^{-t} . u(t-2)]$$

$$= \frac{1}{2}\left[e^{\left((t-2)+2\right)} . u(t-2) + e^{-\left((t-2)+2\right)} . u(t-2)\right] =$$

$$\frac{1}{2}\left[e^2 e^{t-2} . u(t-2) + e^{-2} e^{-(t-2)} . u(t-2)\right]$$

$$f(t) \xrightarrow{\mathrm{F}(\omega)} \frac{e^2}{2}\left(\frac{e^{-2i\omega}}{i\omega - 1}\right) + \frac{e^{-2}}{2}\left(\frac{e^{-2i\omega}}{i\omega + 1}\right)$$

Ex. Find F.T. $f(t) = e^{ibt}.\sinh(at)$

Sol.

$$\sinh(\mathrm{at}) = \frac{e^{at} - e^{-at}}{2}$$

$$\therefore f(t) = e^{ibt}(\frac{e^{at} - e^{-at}}{2})$$

$$f(t) \xrightarrow{\mathrm{F}(\omega)} \frac{1}{2}[\frac{1}{i(\omega - b) - a} - \frac{1}{i(\omega - b) + a}]$$

Ex. Find F.T. $f(t) = e^{-3|t|}.\sin^2(t)$

Sol.

$$f(t) = e^{-3|t|}.\left[\frac{1}{2} - \frac{1}{2}\cos(2t)\right]$$

$$f(t) \xrightarrow{\mathrm{F}(\omega)} \frac{1}{2}\left[\frac{3}{\omega^2 + 9} - \frac{1}{2}\left(\frac{3}{(\omega - 2)^2 + 9} + \frac{3}{(\omega + 2)^2 + 9}\right)\right]$$

Ex. Find F.T. $\frac{d}{dt}f(t) - f(t) = \delta(t)$.

Sol.

$$i\omega F(\omega) - F(\omega) = 1 \rightarrow F(\omega)(i\omega - 1) = 1$$

$$\therefore F(\omega) = \frac{1}{i\omega - 1}$$

Ex. Find F.T. $\int_{-\infty}^{t} e^{-|t-2|}u(t-2)dt$.

Sol.

1- $F(\omega) for\ u(t-2) = e^{-i2\omega}$

2-$F(\omega) for \int_{-\infty}^{t} e^{-|t|}dt = \frac{1}{i\omega}.\frac{1}{\omega^2+1}$

3- $F(\omega) for f(t) = \frac{e^{-i2\omega}}{i\omega}.\frac{1}{\omega^2+1}$

Ex.

Find F.T. for $f(t) = \frac{e^{-it}u(t+3)}{(t+3)^2+1}$

Sol.

$$e^{-it} = e^{-i(t+3-3)} = e^{3i}.e^{-i(t+3)}$$

$$f(t) = \frac{e^{3i}.e^{-i(t+3)}u(t+3)}{(t+3)^2+1}$$

1- $F(\omega) for\ u(t+3) = e^{i3\omega}$

2-$F(\omega) for \frac{e^{3i}.e^{-i(t)}}{(t)^2+1} = e^{3i}e^{-|\omega+1|}$

3- $F(\omega) for f(t) = e^{i3\omega}\left(e^{3i}e^{-|\omega+1|}\right)$

Inverse Fourier

Principle

Type function

1- Neither even nor odd

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

2- Even function

$$f(t) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} F(\omega) \cos(\omega t) d\omega$$

3- Odd function

$$f(t) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} F(\omega) \sin(\omega t) d\omega$$

Steps solution

1- table of furrier transform

2- partial fraction

3- laws

Ex. Find inverse F.T for $F(\omega) = \delta(\omega) + 3e^{-4|\omega|}$

Sol.

By use table

$$f(t) = 1 + 3\left(\frac{4}{t^2 + 16}\right)$$

Ex. Find inverse F.T for $F(\omega)=\frac{1+3(i\omega)}{(i\omega)^2+i5\omega+6}$

Sol.

$$F(\omega)=\frac{1+3(i\omega)}{(i\omega)^2+i5\omega+6}=\frac{A}{i\omega+3}+\frac{B}{i\omega+2}$$

$$f(t)=Ae^{-3t}+Be^{-2t}$$

Ex. Find inverse F.T for $F(\omega)=\frac{2+i\omega}{(1+\omega^2)(1+i\omega)}$

Sol.

$$F(\omega)=\frac{2+i\omega}{(1+\omega^2)(1+i\omega)}$$

$$\omega^2=\left(\frac{i\omega}{i}\right)^2=-(iw)^2$$

$$\therefore\ F(\omega)=\frac{2+i\omega}{(1-(i\omega)^2)(1+i\omega)}=\frac{2+i\omega}{(1-i\omega)(1+i\omega)(1+i\omega)}$$
$$=\frac{2+i\omega}{(1+i\omega)^2(1-i\omega)}$$

$$F(\omega)=\frac{2+i\omega}{(1+i\omega)^2(1-i\omega)}=\frac{A}{1-i\omega}+\frac{B}{1+i\omega}+\frac{C}{(1+i\omega)^2}$$

$$F(\omega)=\frac{-A}{i\omega-1}+\frac{B}{1+i\omega}+\frac{C}{(1+i\omega)^2}$$

$$f(t)=-Ae^{t}+Be^{-t}+Cte^{-t}$$

Laws

If $f(t)\rightarrow F(\omega)$

1- time shift

$$f(t-a).u(t-a)\rightarrow e^{-ia\omega}F(\omega)$$

2- Frequency shift

a-

$$f(t)e^{\pm iat} \rightarrow F(\omega \mp a)$$

b-

$$f(t).\cos(at) \rightarrow \frac{1}{2}[F(\omega - a) + F(\omega + a)]$$

c-

$$f(t).\sin(at) \rightarrow \frac{1}{2i}[F(\omega - a) - F(\omega + a)]$$

3-

$$\frac{d^n}{dt^n} f(t) \rightarrow (i\omega)^n F(\omega)$$

4-

$$\int_{-\infty}^{\infty} f(t)dt \rightarrow \frac{F(\omega)}{i\omega}$$

Ex. Find inverse F.T for $F(\omega) = (i\omega)^2 e^{-|\omega|}$

Sol.

$$n = 2$$

$$f(t) = \frac{d^2}{dt^2}\left(\frac{1}{t^2 + 1}\right)$$

Ex. Find inverse F.T for $F(\omega) = \frac{e^{-a|\omega|}}{i\omega}$

Sol.

$$f(t) = \int_{-\infty}^{t} \frac{a}{t^2 + a^2}$$

Ex. Find inverse F.T for $F(\omega) = \frac{\sin\omega}{\omega^2}$

Sol.

$$F(\omega) = \frac{\sin\omega}{\omega^2} = \frac{i}{i\omega} \frac{\sin\omega}{\omega}$$

$$f(t) = i \int_{-\infty}^{t} U(t)dt \quad , \qquad |t| < 1$$

Ex. Find inverse F.T for $F(\omega) = \frac{e^{-3i\omega}}{\omega^2+1}$

Sol.

$$F(\omega) = \frac{e^{-3i\omega}}{\omega^2 + 1} = e^{-3i\omega} \left[\frac{1}{\omega^2 + 1}\right]$$

2- $e^{-3i\omega} \to u(t-3)$

3- $\frac{1}{\omega^2+1} \to e^{-|t|}$

4-

$$f(t) = e^{-|t-3|}.u(t-3)$$

Ex. Find inverse F.T for $F(\omega) = e^{-ia\omega}\left[\frac{sin^2\omega}{\omega^2}\right]$

Sol.

1-

$$e^{-ai\omega} \to u(t-a)$$

2-

$$\left[\frac{sin^2\omega}{\omega^2}\right] \to \frac{sin^2(\frac{\omega T}{2})}{\omega^2 T/2} \to 1-\frac{|t|}{T} = 1-\frac{|t|}{2},\ \ |t|<2$$

3-

$$\frac{f}{t} = 1-\frac{|t-a|}{2}.u(t-a),\ \ |t-a|<2$$

Ex. Find inverse F.T for $F(\omega) = e^{-\frac{(\omega-c)^2}{2}}$

Sol.

1- Modulation

$$\therefore\ e^{ict}$$

2-

$$F(\omega) = e^{\frac{-\omega^2}{2}}, f(t) = e^{\frac{-t^2}{2}}$$

3-

$$f(t) = e^{\frac{-t^2}{2}}.e^{ict}$$

Ex. Find inverse F.T for $F(\omega) = \frac{\sin(\omega-3)}{3\omega-9}$

Sol.

1-

$$F(\omega) = \frac{\sin(\omega - 3)}{3\omega - 9} = \frac{sin(\omega - 3)}{3(\omega - 3)} = \frac{1}{3}\frac{sin(\omega - 3)}{(\omega - 3)}$$

2- modulation

$$e^{i3t}$$

3-

$$F(\omega) = \frac{1}{3}.\frac{\sin\omega}{\omega}, \therefore f(t) = \frac{1}{3}.U(t) \ \ , |t| < 1$$

4-

$$f(t) = \frac{1}{3}.U(t).e^{i3t}, |t| < 1$$

Finite difference method

The finite-difference method is way of obtaining a numerical solution to differential equations. It does not give a symbolic solution.

Start with the grid

1- type of finite differences

A- Forward differences

If $y = f(x)$

$$x_0 \quad , \qquad y_0 = f(x_0) = f_0$$

$$x_1 = x_0 + h \quad , \qquad y_1 = f(x_1) = f_1$$

$$x_2 = x_1 + h \quad , \qquad y_2 = f(x_2) = f_2$$

$$x_n = x_{n-1} + h \quad , \qquad y_n = f(x_n) = f_n$$

$$h = \frac{x_n - x_0}{n}$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_2 = y_3 - y_2$$

$$\therefore \quad \Delta y_i = y_{i+1} - y_i, \qquad i = 1,2,,3,\ldots.,\quad n-1$$

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 = y_2 - y_1 - y_1 + y_0$$
$$= y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1 = y_3 - y_2 - y_2 + y_1$$
$$= y_3 - 2y_2 + y_1$$

$$\Delta^2 y_i = \Delta(\Delta y_0) = \Delta y_{i+1} - \Delta y_i = y_{i+2} - 2y_{i+1} + y_i, \qquad i = 0,1,2; \ \dots, n-2$$

$$\Delta^3 y_i = \Delta^2 y_{i+1} - \Delta^2 y_i \qquad i = 0,1,2; \ \dots, n-3$$

$$\Delta^4 y_i = \Delta^3 y_{i+1} - \Delta^3 y_i \qquad i = 0,1,2; \ \dots, n-4$$

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

2- Backward Differences

$$\nabla f(x) = f(x) - f(x+h)$$

$$\therefore \ \nabla y_i = y_i - y_{i-1}, \qquad i = n, \dots, 1$$

$$\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1}, \qquad i = n, \dots .2$$

$$= y_i - y_{i-1} - y_{i-1} + y_{i-2}$$

$$= y_i - 2y_{i-1} + y_{i-2}$$

$$\nabla^3 y_i = \nabla(\nabla^2 y_i) = \nabla(y_i - 2y_{i-1} + y_{i-2}) = \nabla y_i - 2\nabla y_{i-1} + \nabla y_{i-2}$$

$$= y_i - y_{i-1} - 2(y_{i-1} - y_{i-2}) + y_{i-2} - y_{i-3}$$

$$= y_i - 3y_{i-1} + 3y_{i-2} + y_{i-3}$$

Or

$$\nabla^3 y_i = \nabla^2(\nabla y_i) = \nabla^2 y_i + \nabla^2 y_{i-1}$$

$$\nabla^4 y_i = \nabla^3(\nabla y_i) = \nabla^3 y_i \nabla^2 y_{i-1}$$

Find Approximate expressions for the derivatives

$$y'(x) \approx \frac{y(x+h)-y(x)}{h}$$

$$y'(x) \approx \frac{y(x+h)-y(x)}{h} \qquad (\text{Difference} \sim h)$$

$$y'(x) \approx \frac{y(x)-y(x-h)}{h} \qquad (\text{Difference} \sim h)$$

$$y'(x) \approx \frac{y(x+h)-y(x-h)}{2h} \qquad \left(\text{Difference} \sim h^2\right)$$

$$\begin{aligned} y''(x) &\approx \frac{y'(x+h)-y'(x)}{h} \\ &\approx \frac{\frac{y(x+h)-y(x)}{h}-\frac{y(x)-y(x-h)}{h}}{h} \\ &= \frac{y(x+h)-2y(x)+y(x-h)}{h^2} \qquad \left(\text{Difference} \sim h^2\right) \end{aligned}$$

Discretize the Problem

1. Determine a grid x_n, $n = 0, \ldots, N$.
2. In the differential equation,
   replace $y'(x)$ with $\dfrac{y(x+h)-y(x-h)}{2h}$,
   replace $y''(x)$ with $\dfrac{y(x+h)-2y(x)+y(x-h)}{h^2}$.
3. In the resulting equation, replace x with x_n.
4. In the resulting equation,
   replace $y(x_n)$ with y_n,
   replace $y(x_n-h) = y(x_{n-1})$ with y_{n-1},
   replace $y(x_n+h) = y(x_{n+1})$ with y_{n+1}

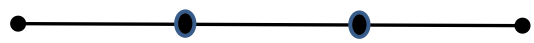
Ex. Solve the following equation by use finite difference.

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

Where u(2)=0.008, and u(6.5)=0.003. find u as function of r. use 4nodes to do the problem

Sol.

$h=\frac{r_n-r_0}{n}=\frac{6.5-2}{3}=1.5$



$$\left.\frac{d^2u}{dr^2}\right|_i \approx \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}$$

$$\left.\frac{du}{dr}\right|_i \approx \frac{u_{i+1}-u_i}{h}$$

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

$$\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}+\frac{1}{r_i}\frac{u_{i+1}-u_i}{h}-\frac{u_i}{r_i^2}=0$$

Node 1 $u_0=u(2)=0.008 \ldots..1$

Node 2, i=1

$$\frac{u_2-2u_1+u_0}{1.5^2}+\frac{1}{3.5}\frac{u_2-u_1}{1.5}-\frac{u_1}{3.5^2}=0$$

$$0.44444u_0-1.161u_1+0.63492u_2=0 \quad \ldots..2$$

Node 3, i=2

$$\frac{u_3-2u_2+u_1}{1.5^2}+\frac{1}{5}\frac{u_3-u_2}{1.5}-\frac{u_2}{5^2}=0$$

$$0.44444u_1-1.0622u_2+0.57778u_3=0 \quad \ldots..3$$

Node 4, i=3

$$u_3 = u(6.5) = 0.003 \quad \ldots\ldots 4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.44444 & -1.161 & 0.63492 & 0 \\ 0 & 0.44444 & -1.0622 & 0.57778 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.008 \\ 0 \\ 0 \\ 0.003 \end{bmatrix}$$

$$\therefore u_0 = 0.008, \qquad u_1 = 0.005128, \quad u_2 = 0.003778, \qquad u_3 = 0.003$$

Exact sol.

$$u = c_1 r + c_2 / r$$

$$u = 9.15 \times 10^{-5} r + \frac{0.015634}{r}$$