Flow Through Saturated Soil

Lecture 4

10.0 UNCONFINED FLOW BY DUPITs ASSUMPTION

While Eq.(2.26) is specifically for confined aquifers, Eq. (2.27)which is the Laplace equation in h is applicable to steady flow of both confined and unconfined aquifers. However, in unconfined aquifers the free surface of the water table, known as *phreatic surface,* has the boundary condition of constant pressure equal to atmospheric pressure. Also, in a section the line representing the water table, is also a streamline. These boundary conditions cause considerable difficulties in analytical solutions of steady unconfined flow problems by using the Laplace equation in h.

A simplified approach based on the assumptions suggested by Dupit (1863) which gives reasonably good results is described below. The basic assumptions of Dupit are:

- The curvature of the free surface is very small so that the streamlines can be assumed to be horizontal at all sections.
- The hydraulic grade line is equal to the free surface slope and does not vary with depth.

Consider an elementary prism of fluid bounded by the water table shown in Fig. 2.9 (a) .

Let V_x = gross velocity of groundwater entering the element in x direction

V_y = gross velocity of groundwater entering the element in y direction

Assume a horizontal impervious base and no vertical inflow from top due to recharge. By Dupit's assumptions, $\partial V_x/\partial z = 0$ and $\partial V_y/\partial z = 0$. Considering the X direction:

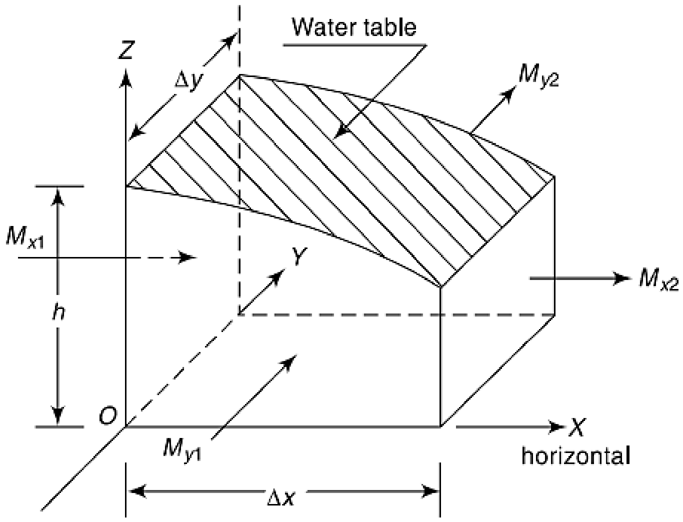

Fig. 2.9 (a) : Definition Sketch—Unconfined Groundwater Flow without Recharge

The mass flux entering the element $M_{x1} = \rho\, V_x\, h\, \Delta\, y$

The mass flux leaving the element $M_{x2} = \rho V_x h\, \Delta y + \dfrac{\partial}{\partial x}(\rho V_x\, h\, \Delta y)\, \Delta x$

The net mass efflux from the element in x direction, by considering the flow entering the element as positive and outflow as negative, is

$$M_{x1} - M_{x2} = \Delta\, M_x = -\frac{\partial}{\partial x}(\rho V_x h\, \Delta y)\, \Delta x$$

Similarly the net mass efflux in y direction

$$M_{y1} - M_{y2} = \Delta\, M_y = -\frac{\partial}{\partial y}(\rho V_y h\, \Delta x)\, \Delta y$$

Further, there is neither inflow or outflow in the Z direction. Thus for steady, incompressible flow, by continuity

$$\Delta M_x + \Delta M_y = 0 \qquad \text{------------ (2.31)}$$

Substituting for ΔM_x and ΔM_y and simplifying

$$\frac{\partial}{\partial x}(V_x h) + \frac{\partial}{\partial y}(V_y h) = 0 \qquad \text{------------ (2.32)}$$

By Darcy law $V_x = -K\frac{\partial h}{\partial x}$ and $V_y = -K\frac{\partial h}{\partial y}$

Hence Eq. (9.32) becomes

$$\frac{\partial}{\partial x}\left(-K\,h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(-K\,h\frac{\partial h}{\partial y}\right) = 0$$

or

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0$$

i.e.

$$\nabla^2 h^2 = 0 \qquad \text{--------- (2.33)}$$

Thus the steady unconfined groundwater flow with Dupit's assumptions is governed by Laplace equation in h^2.

UNCONFINED FLOW WITH RECHARGE If there is a recharge, i.e. infiltration of water from the top ground surface into the aquifer, at a rate of R (m^3/s per m^2 of horizontal area) as in Fig. 2.9(b), the continuity equation Eq. (2.31) is to be modified to take into account the recharge. Consider the element of an unconfined aquifer as in Fig. 2.9(b) situated on a horizontal impervious bed. Here, in addition to ΔM_x and ΔM_y there will be a net inflow into the element in the Z direction given by

$$\Delta M_z = \rho\, R\, \Delta x\, \Delta y$$

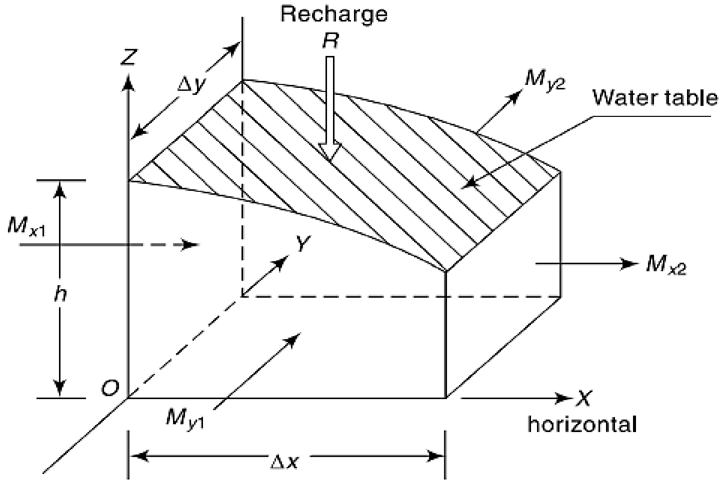

Fig. 2.9 (b) :Definition Sketch – Unconfined flow with Recharge

For steady, incompressible flow the continuity relationship for the element is

$$\Delta M_x + \Delta M_y + \Delta M_z = 0$$

i.e.

$$-\frac{\partial}{\partial x}(\rho\, V_x\, h\, \Delta x\, \Delta y) - \frac{\partial}{\partial y}(\rho\, V_y\, h\, \Delta x\, \Delta y) + \rho\, R\, \Delta x\, \Delta y = 0$$

Substituting $V_x = -K\frac{\partial h}{\partial x}$ and $V_y = -K\frac{\partial h}{\partial y}$ and simplifying

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = -\frac{2R}{K} \qquad \text{-------- (2.34)}$$

Equation (2.34) is the basic differential equation under Dupit's assumption for unconfined groundwater flow with recharge. Note that Eq.(2.33) is a special case of Eq.(2.34) with $R = 0$.

Use of Eq. (2.34) finds considerable practical application in finding the water table profiles in unconfined aquifers. A few examples are: (i) an unconfined aquifer separating two water bodies such as a canal and a river, (ii) various recharge situations, (iii) drainage problems, and (iv) infiltration galleries. To illustrate the use of Eq. (2.34) a situation of steady flow in an unconfined aquifer bounded by two water bodies and subjected to recharge from top is given below.

One Dimensional Dupit's Flow with Recharge

(1) The general case Consider an unconfined aquifer on a horizontal impervious base situated between two water bodies with a difference in surface elevation, as shown in Fig. 2.10. Further, there is a recharge at a constant rate of R m^3/s per unit horizontal area due to infiltration from the top of the aquifer. The aquifer is of infinite length and hence one dimensional method of analysis is adopted. A unit width of aquifer is considered for analysis.

From Eq. (2.34)
$$\frac{\partial^2 h^2}{\partial x^2} = -\frac{2R}{K} \qquad \text{------------- (2.35)}$$

On integration with respect to x twice,

$$h^2 = -\frac{R}{K}x^2 + C_1 x + C_2 \qquad \text{---------- (2.36)}$$

where C_1 and C_2 are constants of integration

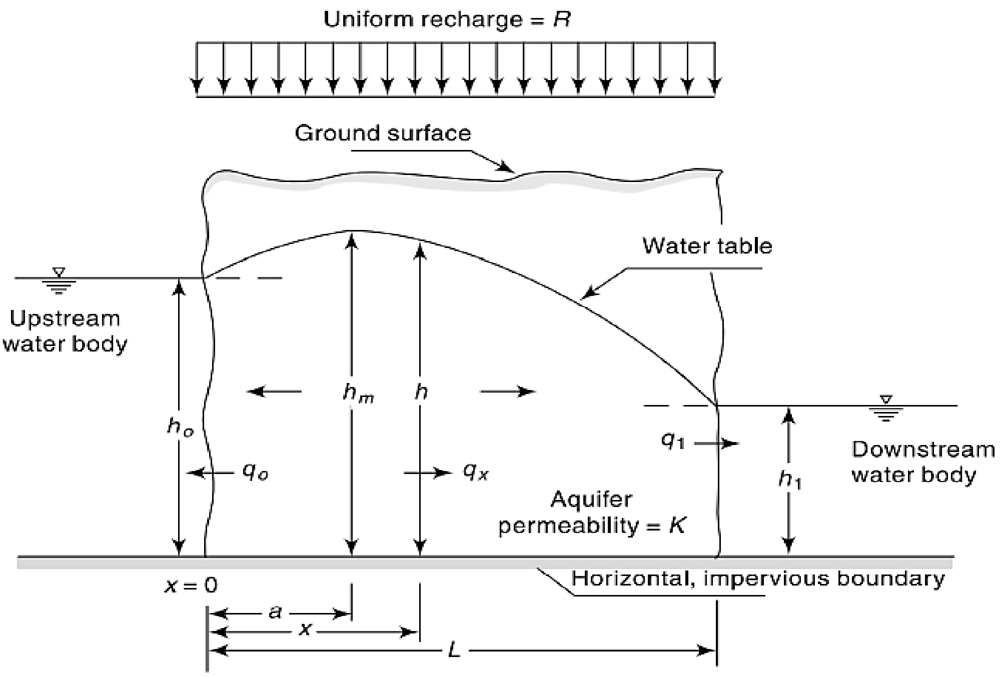

Fig. 2.10: One Dimensional Dupit Flow with Recharge

The boundary conditions are:

(i) at $x = 0, h = h_0$ hence, $C_2 = h_0^2$

(ii) at $x = L, h = h_1$ hence, $h_1^2 - h_0^2 = -\frac{R}{K}L^2 + C_1L$

or $$C_1 = -\frac{\left(h_0^2 - h_1^2 - \frac{RL^2}{K}\right)}{L}$$

Thus Eq. (2.36) becomes

$$h^2 = -\frac{Rx^2}{K} - \frac{\left(h_0^2 - h_1^2 - \frac{RL^2}{K}\right)}{L}x + h_0^2 \qquad \text{---------- (2.37)}$$

The water table is thus an ellipse represented by Eq. (2.37).The value of h will in general rise above h_0, reaches a maximum at $x = a$ and falls back to h_1 at $x = L$ as

shown in Fig. 2.10 .The value of a is obtained by equating $\frac{dh}{dx} = 0$, and is given by

$$a = \frac{L}{2} - \frac{K}{R}\left(\frac{h_0^2 - h_1^2}{2L}\right) \qquad \text{------------ (2.38)}$$

The location $x = a$ is called the *water divide.* Figure 2.10 shows the flow to the left of the divide will be to the upstream water body and the flow to the right of the divide will be to the downstream water body.

The discharge per unit width of aquifer at any location x is

$$q_x = -K\,h\frac{dh}{dx} = -K\left[-\frac{Rx}{K} - \frac{\left(h_0^2 - h_1^2 - \frac{RL^2}{K}\right)}{2L}\right]$$

$$q_x = R\left(x - \frac{L}{2}\right) + \frac{K}{2L}(h_0^2 - h_1^2) \qquad \text{----------- (2.39)}$$

It is obvious the discharge q_x varies with x. At the upstream water body, $x = 0$ and

Discharge $$q_0 = q_{x=0} = -\frac{RL}{2} + \frac{K}{2L}(h_0^2 - h_1^2) \qquad \text{--------- (2.40)}$$

At the downstream water body, $x = L$ and

$$q_1 = q_{x=L} = \frac{RL}{2} + \frac{K}{2L}(h_0^2 - h_1^2) = RL + q_0 \qquad \text{------- (2.40 a)}$$

(2) Flow without recharge When there is no recharge, $R = 0$ and the flow simplifies to that of steady one-dimensional flow in an unconfined aquifer as in Fig. 2.11.

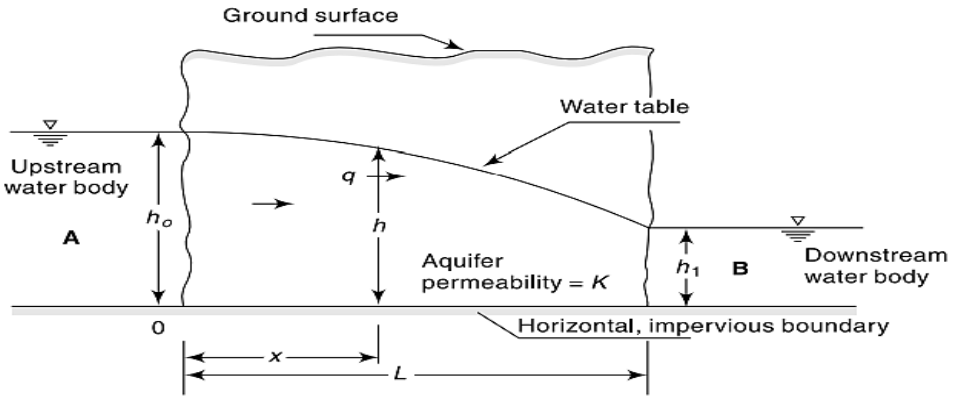

Fig. 2.11: One Dimensional Unconfined Flow without Recharge

By putting $R = 0$ in Eq. (2.37), the equation of the water table is given by

$$(h^2 - h_0^2) = \left(\frac{h_1^2 - h_0^2}{L}\right)x \qquad \text{----------- (2.41)}$$

This represents a parabola (known as *Dupit's parabola*) joining h_0 and h_1 on either side of the aquifer.

Differentiating Eq. (2.41) with respect to x

$$2\,h\frac{dh}{dx} = \frac{(h_1^2 - h_0^2)}{L}$$

The discharge q per unit width of the aquifer is

$$q = -K\,h\frac{dh}{dx} = \frac{(h_0^2 - h_1^2)}{2L}K \qquad \text{---------- (2.42)}$$

(3) Tile drain problem The provision of drains system is one of the most widely used method of draining waterlogged areas, the object being to reduce the level of the water table. Figure 2.12 shows a set of porous tile drains maintaining a constant recharge rate of R at the top ground surface.

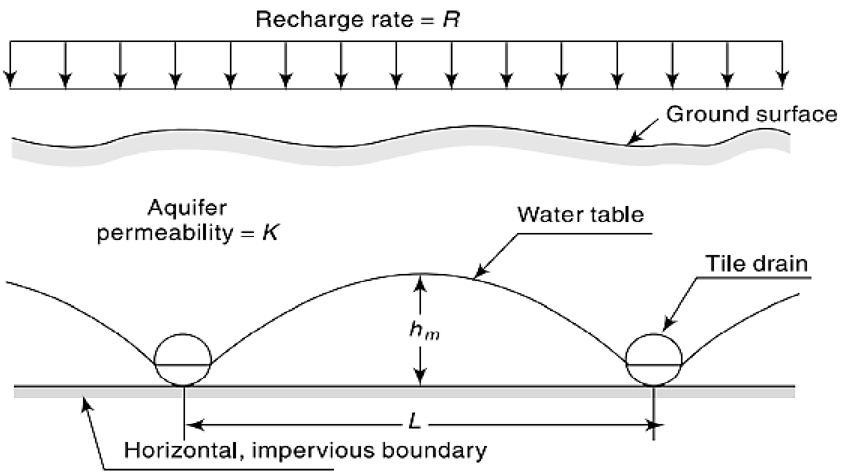

Fig. 2.12: Tile Drains under a Constant Recharge Rate

An approximate expression to the water table profile can be obtained by Eq. (2.37) by neglecting the depth of water in the drains, i.e. $h_0 = h_1 = 0$. The water table profile will then be

$$h^2 = \frac{R}{K}(L-x)\,x \qquad \text{--------- (2.43)}$$

The maximum height of the water table occurs at $a = L/2$ and is of magnitude

$$h_m = \frac{L}{2}\sqrt{R/K} \qquad \text{------------ (2.44)}$$

Considering a set of drains, since the flow is steady, the discharge entering a drain per unit length of the drain is

$$q = 2\left(R\frac{L}{2}\right) = RL \qquad \text{------------- (2.45)}$$

Example : *Two parallel rivers A and B are separated by a land mass as shown in* Fig. 2.13. *Estimate the seepage discharge from River A to River B per unit length of the rivers.*

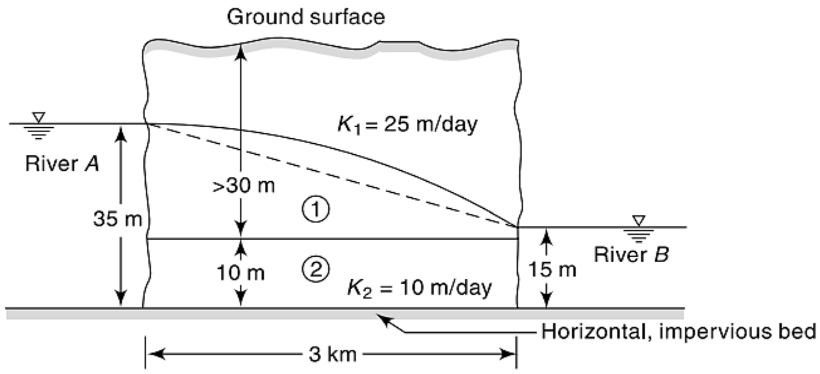

Fig. 2.13 : Schematic Layout of Example

SOLUTION: The aquifer system is considered as a composite of aquifers 1 and 2 with a horizontal impervious boundary at the interface. This leads to the assumptions:

(a) aquifer 2 is a confined aquifer with $K_2 = 10$ m/day
(b) aquifer 1 is an unconfined aquifer with $K_1 = 25$ m/day

Consider a unit width of the aquifers.
For the confined aquifer 2:

From Eq. (2.30) $q_2 = \dfrac{(h_0 - h_1)}{L} K\,B$

Here $h_0 = 35.0$ m, $h_1 = 15$ m,
$L - 3000$ m, $K_2 - 10$ m/day and $B - 10$ m

$$q_2 = \frac{(35-15)}{3000} \times 10 \times 10 = 0.667 \text{ m}^3\text{/day per metre width}$$

For the unconfined aquifer 1:

From Eq. (2.42) , $q_1 = \dfrac{(h_0^2 - h_1^2)}{2L} K$

Here $h_0 = (35 - 10) = 25$ m, $h_1 = (15 - 10) = 5$ m
$L = 3000$ m, $K_1 = 25$ m/day

$$q_2 = \frac{(25)^2 - (5)^2}{2 \times 3000} \times 25 = 2.5 \text{ m}^3\text{/day per metre width}$$

Total discharge from river A to river $B = q = q_1 + q_2$
$= 0.667 + 2.500 = 3.167$ m^3/day per unit length of the rivers