Flow Through Saturated Soil

Lecture 3

9.0 EQUATION OF MOTION

9.1 CONFIND GROUNDWATER FLOW

If the velocities of flow in the cartesian coordinate directions x, y, z of the aquifer element, ΔV, are u, v and w respectively, the equation of continuity for the fluid flow is

$$\frac{\partial(\Delta M)}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] \qquad \text{------------(2.22)}$$

From Eq. (2.18) considering the differentials with respect to time and taking the limit as ΔV approaches zero

$$\frac{\partial(\Delta M)}{\partial t} = S_s \rho \frac{dh}{dt} \qquad \text{--------- (2.18 a)}$$

Further the aquifer is assumed to be isotropic with permeability coefficient K, so that the Darcy's equation for x, y and z directions can be written as

$$u = -K\frac{\partial h}{\partial x},\ v = -K\frac{\partial h}{\partial y} \text{ and } w = -K\frac{\partial h}{\partial z} \qquad \text{------- (2.23)}$$

Using Eqs. (2.23) and (2.13) and noting that $h = \dfrac{p}{\gamma} + z$, the various terms of the right-hand side of Eq. (2.22) are written as

$$\frac{\partial(\rho u)}{\partial x} = \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = -K\rho\frac{\partial^2 h}{\partial x^2} - K\rho^2\beta g\left(\frac{\partial h}{\partial x}\right)^2$$

$$\frac{\partial(\rho v)}{\partial y} = \rho\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} = -K\rho\frac{\partial^2 h}{\partial y^2} - K\rho^2\beta g\left(\frac{\partial h}{\partial y}\right)^2$$

$$\frac{\partial(\rho w)}{\partial z} = \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = -K\frac{\partial^2 h}{\partial z^2} - K\rho^2\beta g\left[\left(\frac{\partial h}{\partial z}\right)^2 - \frac{\partial h}{\partial z}\right]$$

Assembling these, Eq (2.22) can be written as

$$K\rho\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right] + K\rho^2\beta g\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \left(\frac{\partial h}{\partial z}\right)^2\right] - \frac{\partial h}{\partial z} = \rho S_s\frac{\partial h}{\partial t} \qquad \text{----------- (2.24)}$$

The second term on the left-hand side is neglected as very small, especially for $\partial h/\partial x \ll 1$, and Eq. (2.24) is rearranged to yield

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K}\frac{\partial h}{\partial t} \qquad \text{---------- (2.25)}$$

Defining $S_s B = S$, $K B = T$, and $\nabla^2 h = \left(\dfrac{\partial^2 h}{\partial x^2} + \dfrac{\partial^2 h}{\partial y^2} + \dfrac{\partial^2 h}{\partial z^2}\right)$, Eq. (2.25) reads as

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} \qquad \text{------------ (2.26)}$$

This is the basic differential equation governing unsteady groundwater flow in a homogeneous isotropic *confined aquifer*. This form of the equation is known as *diffusion equation*.

If the flow is steady, the $\partial h/\partial t$ term does not exist, leading to

$$\nabla^2 h = 0 \qquad \text{----------- (2.27)}$$

This equation is known as *Laplace equation* and is the fundamental equation of all potential flow problems. Being linear, the method of superposition is applicable in its solutions.

As an application of the Laplace equation (Eq. 2.27) a simple situation of ***steady state one-dimensional confined porous media flow*** is given below.

9.2 CONFIND GROUNDWATER FLOW BETWEEN TWO WATER BODIES

Figure 2.8 shows a very wide confined aquifer of depth B connecting to water bodies. A section of the aquifer of unit width is considered. The piezometric head at the upstream end is h_0 and at a distance x from the upstream end the head is h.
As the flow is in x direction only, Eq. (2.27) becomes

$$\frac{\partial^2 h}{\partial x^2} = 0 \qquad \text{--------- (2.28)}$$

On integrating twice $h = C_1 x + C_2$
On substitution of the boundary condition $h = h_0$ at $x = 0$

$$h = C_1 x + h_0$$

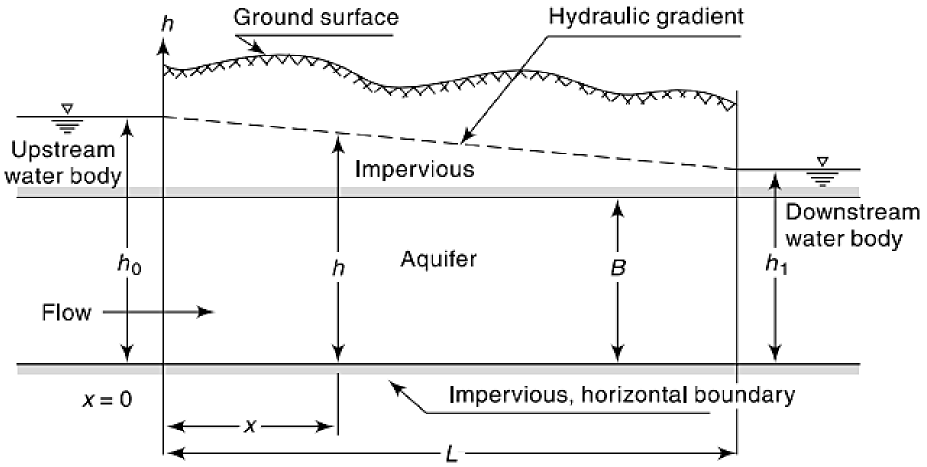

Fig. 2.8 : Confined Groundwater Flow between Two Water Bodies

Also at $x = L$, $h = h_1$ and hence $C_1 = -\left(\frac{h_0 - h_1}{L}\right)$

Thus
$$h = h_0 - \left(\frac{h_0 - h_1}{L}\right) x \qquad \text{----------- (2.29)}$$

This is the equation of the hydraulic grade line, which is shown to vary linearly from h_0 to h_1.
By Darcy law, the discharge per unit width of the aquifer is

$$q = -KB\frac{dh}{dx} = -KB\left(-(h_0 - h_1)/L\right)$$

$$q = \frac{(h_0 - h_1)}{L} K B \qquad \text{---------- (2.30)}$$