

# Analysis of Seepage from Polygon Channels

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**Abstract:** An exact analytical solution for the quantity of seepage from a trapezoidal channel underlain by a drainage layer at a shallow depth has been obtained using an inverse hodograph and a Schwarz-Christoffel transformation. The symmetry about the vertical axis has been utilized in obtaining the solution for half of the seepage domain only. The solution also includes relations for variation in seepage velocity along the channel perimeter and a set of parametric equations for the location of phreatic line. From this generalized case, particular solutions have also been deduced for rectangular and triangular channels with a drainage layer at finite depth and trapezoidal, rectangular, and triangular channels with a drainage layer and water table at infinite depth. Moreover, the analysis includes solutions for a slit, which is also a special case of polygon channels, for both cases of the drainage layer. These solutions are useful in quantifying seepage loss and/or artificial recharge of groundwater through polygon channels.

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## Introduction

Study of seepage from polygon (straight line boundary) channels is important due to its applications in areas of irrigation engineering, groundwater hydrology, and reservoir management. The loss of water due to seepage from irrigation canals constitutes a substantial part of the usable water (Rohwer and Stout 1948; Worstell 1976). The seepage loss results not only in depleted fresh water resources but also causes water logging, salinization, groundwater contamination, and health hazards. Canal lining is adopted to check seepage but cracks develop in the lining for various reasons and seepage from a canal with cracked lining is likely to approach the quantity of seepage from an unlined canal (Wachyan and Rushton 1987). Therefore, optimization of geometrical elements of polygon channels to minimize seepage loss is gaining importance (Kacimov 1992; Swamee et al. 2002a,b). On the other hand, an increased seepage in the form of artificial recharge of groundwater is practiced to mitigate the problems of groundwater depletion and its deleterious consequences. The seepage and recharge from a channel is governed by the same principle of flow through a porous medium and controlled by hydraulic conductivity of the subsoils, channel geometry, hydraulic gradient between the channel and the aquifer underneath, and the initial and boundary conditions (International Commission on Irrigation and Drainage 1967).

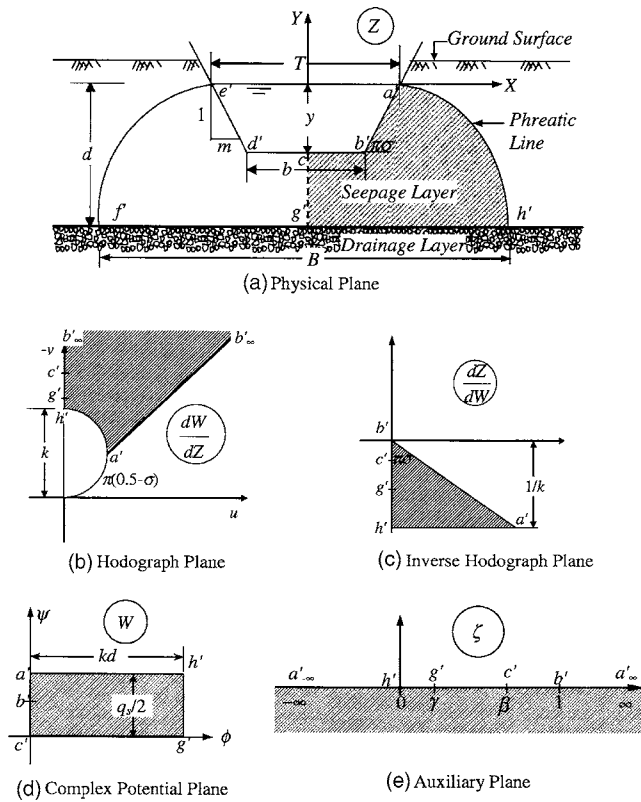
Vedernikov (Harr 1962) gave an exact mathematical solution to unconfined, steady-state seepage from a triangular and a trapezoidal channel in a homogeneous, isotropic, porous medium of large depth using inversion of hodograph and conformal mapping

techniques. The solution of a rectangular channel was given by Morel-Seytoux (1964) using conformal mapping and Green-Neumann functions. Chahar (2001) analyzed seepage from slit and strip channels as special cases of a polygon channel and also presented results for trapezoidal, triangular, and rectangular channels in graphical form. Muskat (1982) suggested an approximate solution using Zhukovsky functions and conformal mapping techniques for a trapezoidal channel in a porous medium of finite depth underlain by a drainage layer. Chahar (2000) and Swamee et al. (2001) obtained an analytical solution for seepage from a rectangular channel in a soil layer of finite depth overlying a drainage layer using inversion of hodograph and conformal mapping techniques. Bruch and Street (1967a,b) used the same method in computing seepage from a triangular channel underlain by a drainage layer at shallow depth. Seepage from polygon channels has also been estimated by several investigators for different boundary conditions using analytical methods (El Nimr 1963; Garg and Chawla 1970; Sharma and Chawla 1974, 1979), electrical analogue methods (Bouwer 1965) and numerical methods (Jeppson 1968; Remson et al. 1971; Pinder and Gray 1977; Liggett and Liu 1983). Approximate solutions by numerical methods have gained importance due to easy availability of high speed digital computers along with specialized software. However, generalized solutions in the functional form are not possible through numerical methods; instead they result only in a numerical value as a problem specific particular solution.

This review reveals that exact analytical solutions for computing seepage from a trapezoidal channel in porous medium of finite depth underlain by a drainage layer are not available using hodographs. Further, available analytical solutions for triangular, rectangular, and trapezoidal channels were obtained by different investigators using different methods or different point of openings in the mapping planes, so these solutions differ from expressions obtained as limiting or particular cases of the solution for the most general problem. In the present study, an exact analytical solution for the quantity of seepage from a trapezoidal channel underlain by a drainage layer at a shallow depth has been obtained using an inverse hodograph and Schwarz-Christoffel transformations for one half of the seepage domain. Moreover its

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**Fig. 1.** Seepage from a trapezoidal channel underlain by a drainage layer

particular cases corresponding to a drainage layer at finite and infinite depths have been dealt with.

## Analytical Solution

Consider a trapezoidal channel of bed width  $b$  (m), depth of water  $y$  (m), and side slope  $m$  (1 Vertical:  $m$  Horizontal) passing through a homogeneous isotropic porous medium of hydraulic conductivity  $k$  (m/s) underlain by a horizontal drainage layer at a depth  $d$  (m) below the water surface as shown in Fig. 1(a). The steady seepage discharge per unit length of channel  $q_s$  ( $\text{m}^2/\text{s}$ ) complying with Darcy's law can be expressed in the following simplest form (Chahar 2000; Swamee et al. 2000)

$$q_s = kyF_s \quad (1)$$

where  $F_s$  (dimensionless seepage function) = function of channel geometry and boundary conditions.

The pattern of seepage from the channel is shown in Fig. 1(a). The effects of capillarity, infiltration, and evaporation are ignored. In view of the significant length of the channel, the seepage flow can be considered two dimensional in the vertical plane. It is assumed that the water table is below the top of the drainage layer and hence atmospheric pressure prevails at the bottom of the seepage layer. The seepage domain has symmetry about vertical axis  $Y$  so half of the domain ( $a'b'c'g'h'a'$ ) has been used in the analysis. Defining complex potential  $W = \phi + i\psi$  [Fig. 1(d)] where  $\phi$  = velocity potential ( $\text{m}^2/\text{s}$ ) which is equal to  $k$  times the head  $h$  (m) and  $\psi$  = stream function ( $\text{m}^2/\text{s}$ ) which is constant along streamlines. If the physical plane is defined as  $Z = X + iY$  then Darcy's law yields  $u = \partial\phi/\partial X = -k \partial h/\partial X$  and  $v = \partial\phi/\partial Y = -k \partial h/\partial Y$ ;

where  $u$  and  $v$  = velocity or specific discharge vectors (m/s) in  $X$  and  $Y$  directions, respectively. The hodograph  $dW/dZ = u - iv$  [Fig. 1(b)] and the inverse hodograph  $dZ/dW$  [Fig. 1(c)] for half of the seepage flow domain ( $a'b'c'g'h'a'$ ) have been drawn following the standard steps (see Harr 1962; Polubarinova-Kochina 1962; or Strack 1989). The  $dZ/dW$  plane and  $W$  plane have been mapped on the lower half ( $\zeta \leq 0$ ) of an auxiliary ( $\zeta$ ) plane [Fig. 1(e)] using the Schwarz-Christoffel conformal transformation.

Mapping of the inverse hodograph plane on the auxiliary plane (see Appendix I) is

$$\frac{dZ}{dW} = \frac{-ie^{i\pi\sigma}}{kB(1/2, \sigma)} \int_1^\zeta \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} \quad (2)$$

where  $\pi\sigma = \cot^{-1} m$ ;  $t$  = dummy variable; and  $B(1/2, \sigma)$  = complete beta function. The  $W$  plane mapping on the auxiliary plane is

$$W = i \frac{k d \sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \int_\beta^\zeta \frac{dt}{\sqrt{t(\beta-\gamma)(t-\beta)}} \quad (3)$$

where  $\beta$  and  $\gamma$  = transformation variables; and  $K(\sqrt{(\beta-\gamma)/\beta})$  = complete elliptical integral of the first kind with a modulus ( $\sqrt{(\beta-\gamma)/\beta}$ ) (Byrd and Friedman 1971). Using Eq. (2) and the derivative of Eq. (3) and then integrating (see Appendix I) result in the mapping of physical plane on the auxiliary plane as

$$Z = \frac{B}{2} - id + \frac{e^{i\pi\sigma}}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \times \int_0^\zeta \left( \int_1^{\tau'} \frac{d\tau}{(\tau-1)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{\sqrt{t(\beta-t)(\gamma-t)}} \quad (4)$$

where  $B$  = width of saturated porous medium at the level of drainage layer (m); and  $\tau$  = another dummy variable. Eq. (4) defines the physical domain of the seepage flow  $a'b'c'g'h'a'$  [Fig. 1(a)]. For example, at  $g'$  ( $\zeta = \gamma$ ;  $Z = -id$ )

$$B = \frac{d\sqrt{\beta}}{K(\sqrt{(\beta-\gamma)/\beta})B(1/2, \sigma)} \int_0^\gamma \frac{F_1(t, \sigma) dt}{\sqrt{t(\beta-t)(\gamma-t)}} \quad (5)$$

where  $F_1(t, \sigma)$  is defined by Eq. (44). At the center of the channel  $c'$  ( $\zeta = \beta$ ;  $Z = -iy$ ) Eq. (4) reduces to

$$y = \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})B(1/2, \sigma)} \int_\gamma^\beta \frac{B_i(1/2, \sigma) dt}{\sqrt{t(\beta-t)(t-\gamma)}} \quad (6)$$

and at the corner of channel  $b'$  ( $\zeta = 1$ ;  $Z = b/2 - iy$ ) it yields

$$b = \frac{d\sqrt{\beta}}{K(\sqrt{(\beta-\gamma)/\beta})B(1/2, \sigma)} \int_\beta^1 \frac{F_1(t, \sigma) dt}{\sqrt{t(\beta-t)(t-\gamma)}} \quad (7)$$

where  $B_i(1/2, \sigma)$  = incomplete beta function (Abramowitz and Stegun 1972) given by Eq. (45).

The phreatic line  $a'h'$  ( $-\infty < \zeta < 0$ ) can be located by manipulating Eq. (4) and then separating the real and imaginary parts

$$X = \left( \frac{B}{2} - d\sqrt{\beta} \int_0^\zeta \frac{F_3(t, \sigma) dt}{\sqrt{(-t)(\beta-t)(\gamma-t)}} \right) / \left( 2K(\sqrt{(\beta-\gamma)/\beta})B(1/2, \sigma) \right) \quad (8a)$$

$$Y = \frac{d}{K(\sqrt{(\beta-\gamma)/\beta})} (K(\sqrt{(\beta-\gamma)/\beta}) - F(\sin^{-1}(\sqrt{\zeta/(\zeta-\gamma)}), \sqrt{(\beta-\gamma)/\beta})) \quad (8b)$$

where  $F_3(t, \sigma)$  is defined by Eq. (54);  $F(\sin^{-1}(\sqrt{\zeta/(\zeta-\gamma)}), \sqrt{(\beta-\gamma)/\beta})$  = incomplete elliptical integral of the first kind with modulus  $\sqrt{(\beta-\gamma)/\beta}$ , and amplitude  $\sin^{-1}\sqrt{\zeta/(\zeta-\gamma)}$  (Byrd and Friedman 1971).

### Computation of Seepage Quantity

Using the values at the point  $h'(\zeta=0; W=kd+iq_s/2)$  in Eq. (3)

$$kd + \frac{q_s}{2}i = \frac{ikd\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \int_{\beta}^0 \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} \quad (9)$$

which leads to

$$q_s = 2kdK(\sqrt{\gamma/\beta})/K(\sqrt{(\beta-\gamma)/\beta}) \quad (10)$$

Equating Eq. (1) with Eq. (10) gives

$$F_s = 2\frac{d}{y}K(\sqrt{\gamma/\beta})/K(\sqrt{(\beta-\gamma)/\beta}) \quad (11)$$

This involves two transformation parameters  $\beta$  and  $\gamma$ . Using Eqs. (6) and (7)

$$\frac{d}{y} = 2K(\sqrt{(\beta-\gamma)/\beta})B(1/2, \sigma) \left/ \sqrt{\beta} \int_{\gamma}^{\beta} \frac{B_t(1/2, \sigma)dt}{\sqrt{t(\beta-t)(t-\gamma)}} \right. \quad (12a)$$

$$\frac{b}{y} = 2 \int_{\beta}^1 \frac{F_1(t, \sigma)dt}{\sqrt{t(t-\beta)(t-\gamma)}} \left/ \int_{\gamma}^{\beta} \frac{B_t(1/2, \sigma)dt}{\sqrt{t(\beta-t)(t-\gamma)}} \right. \quad (12b)$$

Substituting  $d/y$  from Eqs. (12a) in Eq. (11)

$$F_s = 4K(\sqrt{\gamma/\beta})B(1/2, \sigma) \left/ \sqrt{\beta} \int_{\gamma}^{\beta} \frac{B_t(1/2, \sigma)dt}{\sqrt{t(\beta-t)(t-\gamma)}} \right. \quad (13)$$

Simultaneous solution of Eqs. (12a) and (12b) for the given channel dimensions ( $b, y$ , and  $\sigma$ ) and depth of the drainage layer ( $d$ ) results in parameters  $\beta$  and  $\gamma$ . Using these values in Eq. (13), the seepage function and then the quantity of seepage can be determined. Further, these values can be used in Eq. (5) to find the width of seepage domain at the drainage layer. Finally the phreatic line can be plotted using  $\beta$  and  $\gamma$  in Eqs. (8a) and (8b). However these equations involve complicated integrals with implicit transformation variables. These integrals (complete and incomplete beta functions, complete and incomplete elliptical integrals, and remaining improper integrals) can be evaluated using numerical integration (Press et al. 1992) after converting the improper integrals into proper integrals (Chahar 2005).

As the depth of the drainage layer becomes very large the transformation variable  $\gamma$  approaches a value equal to zero and the seepage function changes to (refer to Appendix I)

$$F_s = \frac{2\pi B(1/2, \sigma)}{\sqrt{\beta} \sin \pi \sigma} \left/ \int_1^{\infty} \frac{F_2(t, \sigma)dt}{t\sqrt{t-\beta}} \right. \\ = \frac{2\pi B(1/2, \sigma)}{\sqrt{\beta}} \left/ \int_0^{\beta} \frac{B_t(t, \sigma)dt}{t\sqrt{\beta-t}} \right. \quad (14)$$

where the transformation variable  $\beta$  for this case is given by

$$\frac{b}{y} = 2 \int_{\beta}^1 \frac{F_1(t, \sigma)dt}{t\sqrt{t-\beta}} \left/ \int_0^{\beta} \frac{B_t(t, \sigma)dt}{t\sqrt{\beta-t}} \right. \quad (15)$$

### Variation in Seepage Velocity

From Eq. (2)

$$\frac{dZ}{dW} = \frac{1}{u-iv} = \frac{u+iv}{u^2+v^2} = \frac{-ie^{i\pi\sigma}}{kB(1/2, \sigma)} \int_1^{\zeta} \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} \quad (16)$$

Equating the real and imaginary parts and then squaring and adding them along the bed of the channel  $c'b'(\beta \leq \zeta \leq 1)$  results

$$\frac{V}{k} = B(1/2, \sigma) \left/ \int_{\zeta}^1 \frac{dt}{(1-t)^{1-\sigma}\sqrt{t}} \right. \quad (17)$$

where  $V = \sqrt{u^2+v^2}$  is the resultant velocity of seeping water. The denominator is zero when  $\zeta = 1$ ; hence at the corner of the channel

$$\frac{V}{k} = \infty \quad (18)$$

The minimum seepage velocity along the bed of the channel occurs at the center  $c'(\zeta = \beta)$  and is equal to

$$\frac{V}{k} = B(1/2, \sigma) \left/ \int_{\beta}^1 \frac{dt}{(1-t)^{1-\sigma}\sqrt{t}} \right. \quad (19)$$

The relationship for the seepage velocity along the side slope of the channel  $b'a'(1 \leq \zeta \leq \infty)$  is

$$\frac{V}{k} = B(1/2, \sigma) \left/ \int_1^{\zeta} \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} \right. \quad (20)$$

The minimum seepage velocity along the side slope of the channel at the water surface  $a'(\zeta = \infty)$  is

$$\frac{V}{k} = B(1/2, \sigma) \left/ \int_1^{\infty} \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} \right. \\ = B(1/2, \sigma)/B(1/2 - \sigma, \sigma) \\ = \cos \pi \sigma \quad (21)$$

This shows that the seepage velocity at the water surface is zero for a rectangular channel shape (i.e.,  $\sigma=0.5$ ) and is equal to the hydraulic conductivity of the porous medium for a strip channel (i.e.,  $\sigma=0$ ) for which ponded water depth is small and water seeps vertical downwards with unit hydraulic gradient in saturated porous medium (Chahar 2001).

The expressions for the variation in the seepage velocity when the drainage layer is at infinite depth are identical to the case with the drainage layer at a shallow depth because the inverse hodograph mapping is identical in both cases except for the location of  $g'$ , which does not take part in Schwarz-Christoffel transformation due to vertex angle being equal to  $\pi$  in the drainage layer for the shallow depth case.

### Rectangular Channel

A rectangular section is a special case of a trapezoidal section with vertical side slopes. See Appendix I for details. Eq. (8a) for the  $X$ -coordinate of the phreatic line changes to

$$X = \frac{d\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \left( \int_0^\gamma \frac{\pi/2 - \tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(\gamma-t)}} dt - \int_\zeta^0 \frac{\sinh^{-1}\sqrt{-t}}{\sqrt{(-t)(\beta-t)(\gamma-t)}} dt \right) \quad (22)$$

while Eq. (8b) for the  $Y$  coordinate remains unaltered.

Since the  $W$  plane mapping for a rectangular channel is identical to a trapezoidal channel, the expressions for  $q_s$  [Eq. (10)] and  $F_s$  [Eq. (11)] remain unaltered, however,  $\beta$  and  $\gamma$  have values different than a trapezoidal channel. These transformation parameters can be obtained from the simultaneous solution of

$$\frac{d}{y} = \pi K(\sqrt{(\beta-\gamma)/\beta}) / \sqrt{\beta} \int_\gamma^\beta \frac{\tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(t-\gamma)}} dt \quad (23a)$$

$$\frac{b}{y} = \int_\beta^1 \frac{\pi/2 - \tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(t-\beta)(t-\gamma)}} dt / \int_\gamma^\beta \frac{\tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(t-\gamma)}} dt \quad (23b)$$

Hence Eq. (13) changes to

$$F_s = 2\pi K(\sqrt{\gamma/\beta}) / \sqrt{\beta} \int_\gamma^\beta \frac{\tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(t-\gamma)}} dt \quad (24)$$

For the drainage layer at large depth Eqs. (23b) and (24) transform to

$$\frac{b}{y} = \int_\beta^1 \frac{\pi/2 - \tan^{-1}\sqrt{t/(1-t)}}{t\sqrt{t-\beta}} dt / \int_0^\beta \frac{\tan^{-1}\sqrt{t/(1-t)}}{t\sqrt{\beta-t}} dt \quad (25)$$

$$F_s = \pi^2 / \sqrt{\beta} \int_0^\beta \frac{\tan^{-1}\sqrt{t/(1-t)}}{t\sqrt{\beta-t}} dt \quad (26)$$

The seepage velocity distribution for the drainage layer at shallow as well as large depth cases along the bed of the rectangular channel ( $\beta \leq \zeta \leq 1$ ) is

$$\frac{V}{k} = \frac{\pi}{\pi - 2 \tan^{-1}\sqrt{\zeta/(1-\zeta)}} \quad (27)$$

and along the sides of the channel  $b'a'(1 \leq \zeta \leq \infty)$  is

$$\frac{V}{k} = \frac{\pi}{2 \tanh^{-1}\sqrt{(\zeta-1)/\zeta}} \quad (28)$$

At the water surface  $\zeta = \infty$ , so the velocity of the seeping water is zero.

### Triangular Channel

A trapezoidal channel with zero bed width ( $b=0$ ) is a triangular channel. In the different mapping planes [Figs. 1(b–e)] the point  $b'$  coincides with the point  $c'$  and hence the transformation variable  $\beta$  vanishes after attaining a value equal to unity. Through  $\beta=1$ , various relations for a trapezoidal channel can be deduced for a triangular channel, for example, Eq. (13) for the seepage function reduces to

$$F_s = 4K(\sqrt{\gamma})B(1/2, \sigma) / \int_\gamma^1 \frac{B_t(1/2, \sigma) dt}{\sqrt{t(1-t)(t-\gamma)}} \quad (29)$$

and Eq. (12a) for the transformation parameter  $\gamma$  reduces to

$$\frac{d}{y} = 2K(\sqrt{1-\gamma})B(1/2, \sigma) / \int_\gamma^1 \frac{B_t(1/2, \sigma) dt}{\sqrt{t(1-t)(t-\gamma)}} \quad (30)$$

For a large depth of drainage layer Eq. (29) becomes

$$F_s = 2\pi B(1/2, \sigma) / \int_0^1 \frac{B_t(1/2, \sigma) dt}{t\sqrt{1-t}} \quad (31)$$

The seepage velocity distribution is identical to the variation in the seepage velocity along the sides in the trapezoidal channel.

### Slit

A very narrow and deep polygon channel can be assumed as a slit. For a slit, the width at water surface ( $T$ ) approaches zero, i.e.,  $T/y \rightarrow 0$ . This means  $m \rightarrow 0$  (or  $\sigma \rightarrow 0.5$ ) for a triangular section;  $b/y \rightarrow 0$  (or  $\beta \rightarrow 1$ ) for a rectangular channel; and both  $m \rightarrow 0$  and  $b/y \rightarrow 0$  for a trapezoidal channel. Thus Eq. (24) with  $\beta=1$  or Eq. (29) with  $\sigma=0.5$  results in the seepage function for a slit as

$$F_s = 2\pi K(\sqrt{\gamma}) / \int_\gamma^1 \frac{\tan^{-1}\sqrt{t/(1-t)}}{\sqrt{t(1-t)(t-\gamma)}} dt \quad (32)$$

For infinite depth of the drainage layer, Eq. (32) with  $\gamma=0$ , Eq. (26) with  $\beta=1$ , or Eq. (31) with  $\sigma=0.5$  yields

$$F_s = \pi^2 / \int_0^1 \frac{\tan^{-1}\sqrt{t/(1-t)}}{t\sqrt{1-t}} dt = \frac{\pi^2}{4G} \approx \pi(4-\pi) \quad (33)$$

where  $G=0.915965594\dots$ =Catalan's constant. The outcome is identical to Chahar's (2000, 2001) solution.

### Example

As an example, we can compute the quantity of seepage/recharge from a trapezoidal channel having a bed width=3.0 m, a depth of flow=2 m, and side slopes=1 vertical: 1.5 horizontal, passing through a porous medium having hydraulic conductivity= $3 \times 10^{-6}$  m/s and underlain by a highly pervious drainage layer at a depth of 4.0 m.

For the given data  $b/y=1.5$  and  $d/y=2.0$ . Eqs. (12a) and (12b) should be solved simultaneously to get  $\beta$  and  $\gamma$ . However, since these equations are highly nonlinear and contain improper integrals, an indirect method has been used to find  $\beta$  and  $\gamma$ . The method consists of minimization of an objective function by Powell's conjugate search method (Press et al. 1992). The objective function is defined as

$$f(\beta, \gamma) = \left( \frac{d}{y} - f_1(\sigma, \beta, \gamma) \right)^2 + \left( \frac{b}{y} - f_2(\sigma, \beta, \gamma) \right)^2 \quad (34)$$

where  $f_1(\sigma, \beta, \gamma)$  and  $f_2(\sigma, \beta, \gamma)$  are right-hand sides of Eqs. (12a) and (12b), respectively. The minimum of this function is zero, which can be attained only when both of the parts of the function reach zero values and hence satisfy Eqs. (12a) and (12b). After removing singularities (Chahar 2005) and using Gaussian quadratures (96 points for weights and abscissa for both inner and outer integrals) for numerical integration (Abramowitz and Ste-



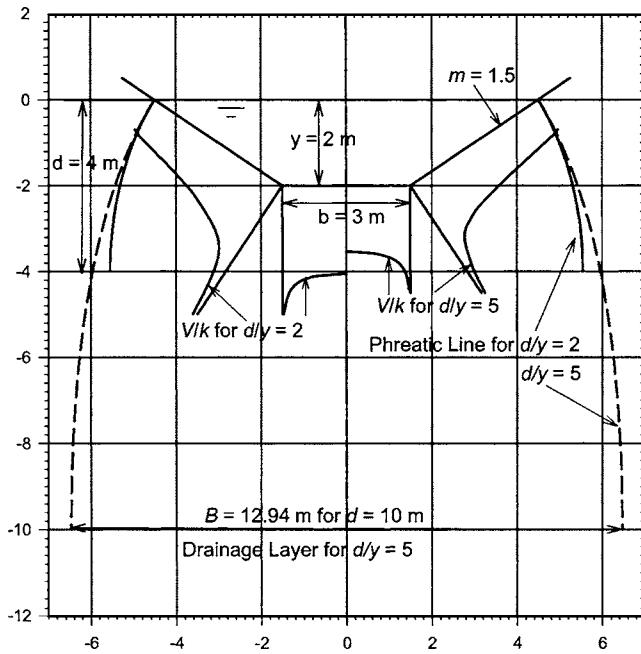


Fig. 2. Seepage velocity distribution and phreatic lines

gun 1972), the function was minimized for  $\beta$  and  $\gamma$  for given set of  $\sigma$ ,  $b/y$ , and  $d/y$  to get

$$\beta = 0.9353, \quad \gamma = 0.9145$$

Making use of these values in Eqs. (5) and (13)

$$B = 11.088 \text{ m}, \quad F_s = 8.3610$$

and finally from Eq. (1)

$$q_s = 5.0166 \times 10^{-5} \text{ m}^3/\text{s per meter run of the channel}$$

Adopting a similar procedure for Muskat's approximate solution [Eqs. (61a)–(61c)], the corresponding values are  $\beta=0.9854$ ;  $\gamma=0.96867$ ;  $F_s=8.0185$ , and  $q_s=4.8111 \times 10^{-5} \text{ m}^3/\text{s/m}$ , therefore it underestimates the seepage loss by 4.10%.

If the depth of the drainage layer is increased to 10.0 m (i.e.,  $d/y=5$ ),  $\beta=0.7217$  and  $\gamma=0.1146$  and hence,  $B=12.940 \text{ m}$ ,  $F_s=6.9384$ , and  $q_s=4.16304 \times 10^{-5} \text{ m}^3/\text{s/m}$ . It can be seen that the seepage loss is reduced by 17.01%. The comparisons between the two cases for phreatic surfaces and seepage velocity distributions along the channel perimeter are shown in Fig. 2.

## Discussions

Following the procedure as outlined in the above-presented example the seepage function was calculated for various values of  $d/y$  and  $b/y$  for a fixed value of  $\sigma$  (i.e.,  $m=1.5$ ). The resultant values were plotted in Fig. 3. Similar graphs can be prepared for other values of  $m$ . Fig. 3 shows that the quantity of seepage is very sensitive to the presence of a drainage layer at a shallow depth (i.e.,  $d/y$  close to one) while the quantity of seepage varies a little with change in position of the drainage layer at large depth (i.e.,  $d/y > 5$ ) for given  $b/y$  and  $m$ .

Eqs. (17) and (20) include  $\sigma$  and  $\zeta$  only, as if the variations in the seepage velocities along the bed and the side slopes were independent of  $b$ ,  $y$ , and  $d$ . Therefore, they result in identical values of the seepage velocity for a same set of  $\sigma$  and  $\zeta$  regardless of

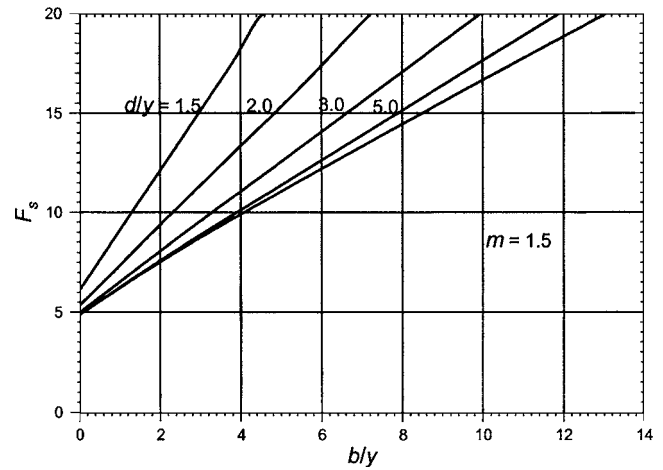


Fig. 3. Variation in seepage function with  $d/y$  and  $b/y$

$b$ ,  $y$ , and  $d$ . However, the transformation parameters ( $\beta$  and  $\gamma$ ) are controlled by  $\sigma$ ,  $b$ ,  $y$ , and  $d$ , so  $\zeta$  maps at different locations along the channel perimeter based on  $\sigma$ ,  $b$ ,  $y$ , and  $d$ , resulting in change in the seepage velocity with change in the channel dimensions or depth of the drainage layer. To show this, plots similar to Bouwer (1965) have been added in Fig. 2 for seepage velocity distributions for  $d/y=2$  and  $d/y=5$  for the worked example. The seepage velocities are higher for lower values of  $d/y$ . However, the seepage velocity at the originating point of the phreatic line is independent of  $d/y$  and  $b/y$  and a unique function of  $\sigma$  given by Eq. (21).

The solution given by Eqs. (10)–(13) can be compared with the existing Muskat's (1982) approximate solution [see Eq. (61)]. Similarly the present solutions for the rectangular channel [Eqs. (23)–(26)] and the triangular channel [Eqs. (29)–(31)] can also be compared with the existing solutions given in Appendix II. It can be seen that the present solution is consistent and convenient in obtaining limiting or particular solutions for channels of rectangular, triangular, and slit shapes both for drainage layer at shallow and large depths. On the other hand, the existing solutions by different investigators using different methods or different point of openings in the mapping planes differ from each other as well as cannot be deduced from each other. Further, the existing solutions lack in expressions for the velocity distribution along the channel perimeter.

## Conclusions

An exact analytical solution for the quantity of seepage from a trapezoidal channel underlain by a drainage layer at a shallow depth can be obtained using an inverse hodograph and Schwarz-Christoffel transformation for one half of the seepage domain. From this general solution, other special cases like a trapezoidal channel without a drainage layer, a rectangular channel underlain by a drainage layer at a shallow depth, a triangular channel underlain by a drainage layer at a shallow depth, a rectangular channel without a drainage layer, and a triangular channel without a drainage layer can be deduced. The analysis can also include solutions for the variation in the seepage velocity along the channel perimeter and the quantity of seepage from slit-like channels. Therefore the solution is exact, complete, consistent, and general. However, the solutions for the quantity of seepage, location of the

phreatic line, and width of seepage at the drainage layer contain improper integrals which can only be evaluated by numerical integration.

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## Appendix I. Mapping Details

### Drainage Layer at Shallow Depth

#### Mapping of Inverse Hodograph Plane

Mapping of  $dZ/dW$  plane on the  $\zeta$  plane results in

$$\frac{dZ}{dW} = C_1 \int_0^\zeta \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} + C_2 \quad (35)$$

where  $C_1$  and  $C_2$  = constants. These constants can be found by using values of  $dZ/dW$  and  $\zeta$  at two points in  $dZ/dW$  plane and  $\zeta$  plane. Using the values at point  $b'$  ( $\zeta=1; dZ/dW=0$ ) and at the point  $h'$  ( $\zeta=0; dZ/dW=-i/k$ ) in Eq. (35) and solving simultaneously

$$C_1 = \frac{i}{k} \int_0^1 \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} = \frac{-ie^{i\pi\sigma}}{kB(1/2, \sigma)}, \quad C_2 = -\frac{i}{k} \quad (36)$$

where

$$\int_0^1 \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} = \frac{1}{(-1)^{1-\sigma}} \int_0^1 (1-t)^{\sigma-1} (t)^{(1/2)-1} dt = \frac{B(1/2, \sigma)}{(e^{-i\pi})^{1-\sigma}} \quad (37)$$

Substitution of  $C_1$  and  $C_2$  in Eq. (35) gives Eq. (2).

#### Mapping of Complex Potential Plane

The  $W$  plane mapping on the  $\zeta$  plane is

$$W = C_3 \int_0^\zeta \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} + C_4 \quad (38)$$

The constants  $C_3$  and  $C_4$  have been determined using the values at points  $c'$  ( $\zeta=\beta; W=0$ ) and  $g'$  ( $\zeta=\gamma; W=kd$ ). After substituting  $C_3$  and  $C_4$ , Eq. (38) becomes

$$W = kd \int_\beta^\zeta \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} \Big/ \int_\beta^\gamma \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} \quad (39)$$

We know that

$$\int_\beta^\gamma \frac{dt}{\sqrt{t(t-\gamma)(t-\beta)}} = -i \int_\gamma^\beta \frac{dt}{\sqrt{t(t-\gamma)(\beta-t)}} = -i \frac{2}{\sqrt{\beta}} K(\sqrt{(\beta-\gamma)/\beta}) \quad (40)$$

Combining Eqs. (39) and (40) results in Eq. (3). Differentiating Eq. (3) with respect to  $\zeta$  gives

$$\frac{dW}{d\zeta} = i \frac{kd\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \frac{1}{\sqrt{\zeta(\zeta-\gamma)(\zeta-\beta)}} \quad (41)$$

#### Mapping of Physical Plane

Since  $dZ/d\zeta = (dZ/dW)(dW/d\zeta)$ , substitution of  $dZ/dW$  from Eq. (2) and  $dW/d\zeta$  from Eq. (41) results in

$$\frac{dZ}{d\zeta} = \frac{e^{i\pi\sigma}}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \times \left( \int_1^\zeta \frac{dt}{(t-1)^{1-\sigma}\sqrt{t}} \right) \frac{1}{\sqrt{\zeta(\zeta-\beta)(\zeta-\gamma)}} \quad (42)$$

Integrating Eq. (42) and applying the condition at  $h'$  ( $\zeta=0; Z=B/2-id$ ) gives Eq. (4). Along the drainage layer  $h'g'$  [Fig. 1(a)],  $0 \leq \zeta \leq \gamma$  and hence Eq. (4) at  $g'$  ( $\zeta=\gamma; Z=-id$ ) becomes

$$\frac{B}{2} = \frac{1}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta-\gamma)/\beta})} \times \int_0^\gamma \left( \int_t^1 \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{\sqrt{t(\beta-t)(\gamma-t)}} \quad (43)$$

Let

$$F_1(t, \sigma) = \int_t^1 \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} = \int_0^1 \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} - \int_0^t \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} = B(1/2, \sigma) - B_1(1/2, \sigma) \quad (44)$$

where

$$B_1\left(\frac{1}{2}, \sigma\right) = \int_0^t \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} = 2\sqrt{t} {}_2F_1\left(\frac{1}{2}, 1-\sigma; \frac{3}{2}; t\right) \quad (45)$$

in which  ${}_2F_1$  = Gauss-Hypergeometric series (Abramowitz and Stegun 1972) given by

$${}_2F_1(a, b; c; t) = 1 + \frac{ab}{c}t + \frac{a(a+1)b(b+1)}{c(c+1)(1)(2)}t^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)(1)(2)(3)}t^3 + \dots \quad (46)$$

Therefore Eq. (43) can be rewritten as Eq. (5). Along the center line of channel  $g'c'$  ( $\gamma \leq \zeta \leq \beta$ ), Eq. (4) becomes

$$Z = -id + \frac{i}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta - \gamma)/\beta})} \times \int_{\gamma}^{\zeta} \left( \int_t^1 \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{\sqrt{t(\beta-t)(t-\gamma)}} \quad (47)$$

so that at the center of the channel  $g'$  ( $\zeta = \beta; Z = -iy$ ) this reduces to Eq. (6). Further, Eq. (4) along the bed of the channel  $c'b'$  ( $\beta \leq \zeta \leq 1$ ) becomes

$$Z = -iy + \frac{1}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta - \gamma)/\beta})} \times \int_{\beta}^{\zeta} \left( \int_t^1 \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{\sqrt{t(t-\beta)(t-\gamma)}} \quad (48)$$

at the corner of channel  $b'$  ( $\zeta = 1; Z = b/2 - iy$ ) Eq. (48) yields Eq. (7). Finally Eq. (4) along the side slope of the channel  $b'a'$  ( $1 \leq \zeta \leq \infty$ ) is

$$Z = \frac{b}{2} - iy + \frac{e^{i\pi\sigma}}{B(1/2, \sigma)} \frac{d\sqrt{\beta}}{2K(\sqrt{(\beta - \gamma)/\beta})} \times \int_1^{\zeta} \left( \int_1^t \frac{d\tau}{(\tau-1)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{\sqrt{t(t-\beta)(t-\gamma)}} \quad (49)$$

Therefore at water surface  $a'$  ( $\zeta = \infty; Z = b/2 + y \cot \pi\sigma$ ), it gives

$$y = \frac{d\sqrt{\beta} \sin \pi\sigma}{2K(\sqrt{(\beta - \gamma)/\beta})B(1/2, \sigma)} \int_1^{\infty} \frac{F_2(t, \sigma) dt}{\sqrt{t(t-\beta)(t-\gamma)}} \quad (50)$$

in which

$$F_2(t, \sigma) = \int_1^t \frac{d\tau}{(\tau-1)^{1-\sigma}\sqrt{\tau}} \quad (51)$$

Eqs. (50) and (6) must give identical results for  $y$ , therefore

$$\int_{\gamma}^{\beta} \frac{B_1(1/2, \sigma) dt}{\sqrt{t(\beta-t)(t-\gamma)}} = \sin \pi\sigma \int_1^{\infty} \frac{F_2(t, \sigma) dt}{\sqrt{t(t-\beta)(t-\gamma)}} \quad (52)$$

### Position of Phreatic Line

The phreatic line  $a'h'$  ( $-\infty < \zeta < 0$ ) can be located by manipulating Eq. (4) as

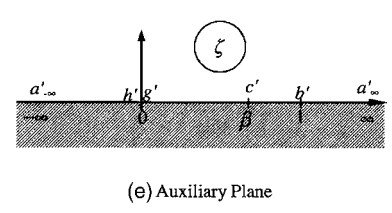
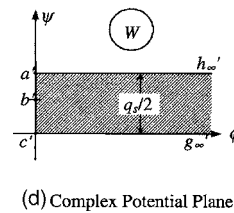
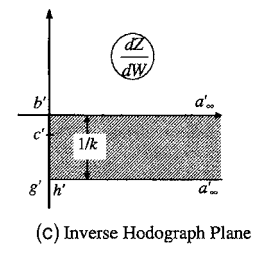
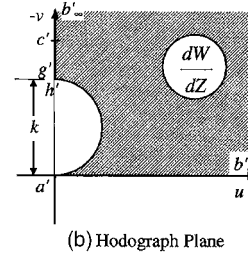
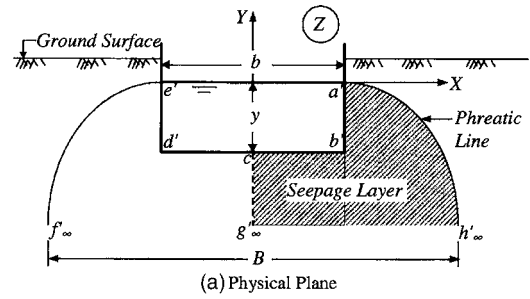
$$Z = \frac{B}{2} - id + \frac{id\sqrt{\beta}}{2K(\sqrt{(\beta - \gamma)/\beta})} \int_0^{\zeta} \left( 1 + \frac{i}{B(1/2, \sigma)} \times \int_0^t \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{-\tau}} \right) \frac{dt}{\sqrt{(-t)(\beta-t)(\gamma-t)}} \quad (53)$$

Letting

$$\int_0^t \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{-\tau}} = F_3(t, \sigma) \quad (54)$$

and separating the real and imaginary parts leads to Eqs. (21) and (22), where

$$\int_{\zeta}^0 \frac{dt}{\sqrt{(-t)(\beta-t)(\gamma-t)}} = \frac{2}{\sqrt{\beta}} F(\sin^{-1} \sqrt{\zeta/(\zeta - \gamma)}, \sqrt{(\beta - \gamma)/\beta}) \quad (55)$$



**Fig. 4.** Seepage from a rectangular channel with a drainage layer at infinite depth

### Rectangular Channel

For a rectangular channel  $m=0$ , so  $\sigma=0.5$  [See Fig. 4(a)] and hence the mappings in the hodograph and inverse hodograph planes are modified as shown in Figs. 4(b and c), respectively (except that the points  $g'$  and  $h'$  map to separate locations, similar to the trapezoidal case). However, the mapping in the complex potential plane remains unaltered, i.e., similar to Fig. 1(d). With  $\sigma=0.5$

$$B\left(\frac{1}{2}, \sigma\right) = \int_0^1 \frac{dt}{(1-t)^{1-\sigma}\sqrt{t}} = \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \pi \quad (56a)$$

$$F_1(\sigma, t) = \int_t^1 \frac{dt}{(1-\tau)^{1-\sigma}\sqrt{\tau}} = \int_t^1 \frac{dt}{\sqrt{\tau(1-\tau)}} = \pi - 2 \tan^{-1} \sqrt{t/(1-t)} \quad (56b)$$

$$F_2(\sigma, t) = \int_1^t \frac{d\tau}{(\tau-1)^{1-\sigma}\sqrt{\tau}} = \int_1^t \frac{d\tau}{\sqrt{\tau(\tau-1)}} = 2 \tanh^{-1} \sqrt{(t-1)/t} \quad (56c)$$

and

$$F_3(\sigma, t) = \int_0^t \frac{d\tau}{(1-\tau)^{1-\sigma}\sqrt{-\tau}} = \int_0^t \frac{d\tau}{\sqrt{-\tau(1-\tau)}} = -2 \sinh^{-1} \sqrt{-\tau} \quad (56d)$$

Therefore, the relevant equations transform to

$$\frac{dZ}{dW} = \frac{i}{k} \int_1^\zeta \frac{dt}{\sqrt{t(t-1)}} \bigg/ \int_0^1 \frac{dt}{\sqrt{t(t-1)}} = \frac{2 \tanh^{-1} \sqrt{(\zeta-1)/\zeta}}{\pi k} \quad (57a)$$

$$Z = \frac{B}{2} - id + \frac{id\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \int_0^\zeta \frac{\tanh^{-1} \sqrt{(t-1)/t}}{\sqrt{t(t-\beta)(t-\gamma)}} dt \quad (57b)$$

$$\frac{B}{2} = \frac{d\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \int_0^\gamma \frac{\pi/2 - \tan^{-1} \sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(\gamma-t)}} dt \quad (57c)$$

$$y = \frac{d\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \int_\gamma^\beta \frac{\tan^{-1} \sqrt{t/(1-t)}}{\sqrt{t(\beta-t)(t-\gamma)}} dt$$

$$= \frac{d\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \int_1^\zeta \frac{\tanh^{-1} \sqrt{(t-1)/t}}{\sqrt{t(t-\beta)(t-\gamma)}} dt \quad (57d)$$

and

$$b = \frac{d\sqrt{\beta}}{\pi K(\sqrt{(\beta-\gamma)/\beta})} \int_\beta^1 \frac{\pi/2 - \tan^{-1} \sqrt{t/(1-t)}}{\sqrt{t(t-\beta)(t-\gamma)}} dt \quad (57e)$$

### Drainage Layer and Water Table at Infinite Depth

#### Trapezoidal Channel

When the drainage layer and water table both lie at infinite depth in a homogeneous isotropic porous medium of infinite extent both the points  $g'$  and  $h'$  approach each other in the hodograph and inverse hodograph mapping planes [Figs. 1(b and c)]. At infinite depth the hydraulic gradient becomes unity and the seepage velocity acquires a uniform magnitude equal to the hydraulic conductivity over a horizontal plane. However, at infinite depth the velocity potential becomes infinite, so the mapping in the  $W$  plane converts to a semi-infinite strip as shown in Fig. 4(d). With these adjustments, the transformation variable  $\gamma$  in Fig. 1(e) vanishes from the transformation after attaining a value equal to zero and the changed relations turn into

$$W = \frac{iq_s\sqrt{\beta}}{2\pi} \int_\beta^\zeta \frac{dt}{t\sqrt{t-\beta}} \quad (58a)$$

$$Z = -iy + \frac{q_s\sqrt{\beta}}{2\pi} \frac{e^{i\pi\sigma}}{kB(1/2,\sigma)} \int_\beta^\zeta \left( \int_1^t \frac{d\tau}{(\tau-1)^{1-\sigma}\sqrt{\tau}} \right) \frac{dt}{t\sqrt{t-\beta}} \quad (58b)$$

$$b = \frac{q_s\sqrt{\beta}}{\pi kB(1/2,\sigma)} \int_\beta^1 \frac{F_1(t,\sigma) dt}{t\sqrt{t-\beta}} \quad (58c)$$

$$y = \frac{q_s\sqrt{\beta}}{2\pi} \frac{\sin \pi\sigma}{kB(1/2,\sigma)} \int_1^\infty \frac{F_2(t,\sigma) dt}{t\sqrt{t-\beta}} = \frac{q_s\sqrt{\beta}}{2\pi kB(1/2,\sigma)} \int_0^\beta \frac{B_t(t,\sigma) dt}{t\sqrt{\beta-t}} \quad (58d)$$

$$X = \frac{b}{2} + y \cot \pi\sigma$$

$$+ y \int_{-\infty}^\zeta \frac{(F_3(t,\sigma) - B(1/2,\sigma)\cot \pi\sigma) dt}{-t\sqrt{\beta-t}} \bigg/ \int_1^\infty \frac{F_2(t,\sigma) dt}{t\sqrt{t-\beta}} \quad (58e)$$

$$Y = y \int_{-\infty}^\zeta \frac{(\cot \pi\sigma F_3(t,\sigma) - B(1/2,\sigma)) dt}{-t\sqrt{\beta-t}} \bigg/ \int_1^\infty \frac{F_2(t,\sigma) dt}{t\sqrt{t-\beta}} \quad (58f)$$

and width of seepage flow at infinity

$$B = b + 2y \cot \pi\sigma$$

$$+ 2y \int_{-\infty}^0 \frac{(F_3(t,\sigma) - B(1/2,\sigma)\cot \pi\sigma) dt}{-t\sqrt{\beta-t}} \bigg/ \int_1^\infty \frac{F_2(t,\sigma) dt}{t\sqrt{t-\beta}} \quad (58g)$$

Comparing Eq. (1) with Eq. (58d) gives Eq. (14)

#### Rectangular Channel

Fig. 4 shows mappings in different planes for a rectangular channel passing through a homogeneous, isotropic porous medium of infinite extent, i.e., both the drainage layer and water table lie at an infinite depth. The inverse hodograph mapping relations are similar to a rectangular channel having drainage layer at finite depth. On the other hand, the complex potential plane mapping becomes similar to a trapezoidal channel with the drainage layer and water table lying at an infinite depth given by Eq. (58a). With  $\gamma=0$  and  $\sigma=0.5$ , the other relations alter as follows:

$$Z = -iy + \frac{iq_s\sqrt{\beta}}{\pi^2 k} \int_\beta^\zeta \frac{\tanh^{-1} \sqrt{(t-1)/t}}{t\sqrt{t-\beta}} dt \quad (59a)$$

$$b = \frac{q_s\sqrt{\beta}}{\pi^2 k} \int_\beta^1 \frac{\pi/2 - \tan^{-1} \sqrt{t/(1-t)}}{t\sqrt{t-\beta}} dt \quad (59b)$$

$$y = \frac{q_s\sqrt{\beta}}{\pi^2 k} \int_1^\infty \frac{\tanh^{-1} \sqrt{(t-1)/t}}{t\sqrt{t-\beta}} dt = \frac{q_s\sqrt{\beta}}{\pi^2 k} \int_0^\beta \frac{\tan^{-1} \sqrt{t/(1-t)}}{t\sqrt{\beta-t}} dt \quad (59c)$$

$$X = \frac{b}{2} + \frac{q_s\sqrt{\beta}}{\pi^2 k} \int_{-\infty}^\zeta \frac{\sinh^{-1} \sqrt{-t}}{-t\sqrt{\beta-t}} dt \quad (59d)$$

$$Y = -\frac{q_s\sqrt{\beta}}{2\pi k} \int_{-\infty}^\zeta \frac{dt}{-t\sqrt{\beta-t}} = \frac{q_s}{2\pi k} \ln \left( \frac{\sqrt{\beta-\zeta} - \sqrt{\beta}}{\sqrt{\beta-\zeta} + \sqrt{\beta}} \right) \quad (59e)$$

and

$$\frac{B}{2} = \frac{b}{2} + \frac{q_s\sqrt{\beta}}{\pi^2 k} \int_{-\infty}^0 \frac{\sinh^{-1} \sqrt{-t}}{-t\sqrt{\beta-t}} dt \quad (59f)$$

For  $\beta=1$ , the integrals in Eq. (59c) have the following special value:

$$\int_1^\infty \frac{\tanh^{-1} \sqrt{(t-1)/t}}{t\sqrt{t-1}} dt = \int_0^1 \frac{\tan^{-1} \sqrt{t/(1-t)}}{t\sqrt{1-t}} dt = 4G \quad (60)$$



## Appendix II. Comparison with existing solutions

### Drainage Layer at Shallow Depth

Using the Zhukovsky function and conformal mapping, Muskat (1982) gave the following approximate solution for the seepage function for a trapezoidal channel

$$F_s = \frac{2mK(\beta)}{K(\beta) - F(\beta, \sin^{-1}\gamma)} \quad (61a)$$

where the transformation variables  $\beta$  and  $\gamma$  are given by the simultaneous solution of the following:

$$\frac{d}{y} = \frac{mK(\sqrt{1-\beta^2})}{K(\beta) - F(\beta, \sin^{-1}\gamma)} \quad (61b)$$

$$\frac{b}{y} = \frac{2mF(\beta, \sin^{-1}\gamma)}{K(\beta) - F(\beta, \sin^{-1}\gamma)} - \frac{2K(\gamma)}{K(\sqrt{1-\gamma^2})} \quad (61c)$$

Chahar (2000) and Swamee et al. (2001) obtained an exact analytical solution for a rectangular channel using an inverse hodograph and Schwarz Christoffel transformations but mapping the vertices of the polygons in the  $dZ/dW$  and  $W$  planes onto the auxiliary plane at different locations. The solution was

$$F_s = \frac{2\pi}{\sqrt{\beta}} K(\sqrt{(\beta-1)/\beta}) \left/ \int_0^1 \frac{\tan^{-1}\sqrt{(\beta-\tau)/(\gamma+\tau)}}{\sqrt{\tau(1-\tau)(\beta-\tau)}} d\tau \right. \quad (62a)$$

where  $\beta$  and  $\gamma$  were given by simultaneous solution of

$$\frac{d}{y} = \frac{\pi}{\sqrt{\beta}} K(1/\sqrt{\beta}) \left/ \int_0^1 \frac{\tan^{-1}\sqrt{(\beta-\tau)/(\gamma+\tau)}}{\sqrt{\tau(1-\tau)(\beta-\tau)}} d\tau \right. \quad (62b)$$

$$\frac{b}{y} = 2 \left( \frac{\pi}{\sqrt{\beta}} F\left(\sqrt{\frac{\beta-1}{\beta}}, \sin^{-1}\sqrt{\frac{\gamma}{1+\gamma}}\right) - \int_0^\gamma \frac{\tan^{-1}\sqrt{\frac{\beta+\tau}{\gamma-\tau}} d\tau}{\sqrt{\tau(1+\tau)(\beta+\tau)}} \right) \left/ \int_0^1 \frac{\tan^{-1}\sqrt{\frac{\beta-\tau}{\gamma+\tau}} d\tau}{\sqrt{\tau(1-\tau)(\beta-\tau)}} \right. \quad (62c)$$

In the present solution  $0 \leq \gamma \leq \beta \leq 1$ ; while in Chahar's (2000) solution  $0 \leq \gamma \leq \infty$  and  $1 \leq \beta \leq \infty$ .

Using an inverse hodograph and conformal mapping, Bruch and Street (1967a, b) gave the following expression for the seepage function for a triangular channel, but with a different auxiliary plane mapping

$$F_s = 2 \frac{d}{y} \frac{K(1/\sqrt{\gamma})}{K(\sqrt{(\gamma-1)/\gamma})} \quad (63a)$$

where the transformation variable  $\gamma$  was given by

$$1 - \frac{d}{y} = \text{Im} \left( (m+i) \frac{\int_0^{-\gamma} \left( \int_0^t \frac{d\tau}{\tau^{1-\sigma}(1+\tau)^{0.5+\sigma}\sqrt{\tau+\gamma}} \right) \frac{dt}{\sqrt{t(1+t)(t+\gamma)}}}{\int_0^{-1} \left( \int_0^t \frac{d\tau}{\tau^{1-\sigma}(1+\tau)^{0.5+\sigma}\sqrt{\tau+\gamma}} \right) \frac{dt}{\sqrt{t(1+t)(t+\gamma)}}} \right) \quad (63b)$$

where  $\text{Im}$ =imaginary part. In the present solution  $0 \leq \gamma \leq 1$ ; while in Bruch's solution  $1 \leq \gamma \leq \infty$ .

### Drainage Layer and Water Table at Infinite Depth

Vedernikov (Harr 1962) gave an exact solution to the present case using an inversion of the hodograph and conformal mapping technique as

$$F_s = \pi m \int_\beta^1 \frac{tdt}{(1-t^2)^{(0.5+\sigma)}(t^2-\beta^2)^{(1-\sigma)}} \left/ \int_\beta^1 \frac{t \cos^{-1}tdt}{(1-t^2)^{(0.5+\sigma)}(t^2-\beta^2)^{(1-\sigma)}} \right. \quad (64a)$$

where  $\beta$ =transformation variable was given by

$$\frac{b}{y} = 2 \int_0^\beta \frac{t \sin^{-1}tdt}{(1-t^2)^{(0.5+\sigma)}(\beta^2-t^2)^{(1-\sigma)}} \left/ \sin \pi\sigma \int_\beta^1 \frac{t \cos^{-1}tdt}{(1-t^2)^{(0.5+\sigma)}(t^2-\beta^2)^{(1-\sigma)}} \right. \quad (64b)$$

Vedernikov obtained his solution using a full seepage domain while the present solution utilized the advantage of the symmetry of the seepage flow in the vertical plane and hence only half of the seepage domain is used in the solution.

Using conformal mapping and Green-Neumann functions, Morel-Seytoux (1964) gave the solution for a rectangular channel as follows:

$$F_s = \pi^2 \left/ 2 \int_\beta^\infty \ln\left[\frac{\sqrt{1+t^2} + \sqrt{t^2-\beta^2}}{\sqrt{1+\beta^2}}\right] \frac{dt}{1+t^2} \right. \quad (65a)$$

where transformation variable  $\beta$  was given by

$$\frac{b}{y} = \int_0^\beta \cos^{-1}\left(\frac{2t^2+1-\beta^2}{1+\beta^2}\right) \frac{dt}{1+t^2} \left/ \int_\beta^\infty \ln\left[\frac{\sqrt{1+t^2} + \sqrt{t^2-\beta^2}}{\sqrt{1+\beta^2}}\right] \frac{dt}{1+t^2} \right. \quad (65b)$$

The existing Vedernikov's solution (Harr 1962) for a triangular channel is

$$F_s = \pi m \int_0^1 (1-t^2)^{-(0.5+\sigma)} t^{-(1-2\sigma)} dt \left/ \int_0^1 \cos^{-1} t(1-t^2)^{-(0.5+\sigma)} t^{-(1-2\sigma)} dt \right. \quad (66)$$

## Notation

The following symbols are used in this paper:

- $a', b', c', \dots$  = points on flow domain (dimensionless);
- $B$  = seepage width at drainage layer (m);
- $B(\dots)$  = complete beta function (dimensionless);
- $B_t(\dots)$  = incomplete beta function (dimensionless);
- $b$  = bed width of channel (m);
- $d$  = depth of drainage layer/aquifer (m);

$F_s$  = seepage function (dimensionless);  
 $F(.,.)$  = incomplete elliptical integral of the first kind (dimensionless);  
 $F_1(t, \sigma)$  = integral defined by Eq. (44) (dimensionless);  
 $F_2(t, \sigma)$  = integral defined by Eq. (51) (dimensionless);  
 $F_3(t, \sigma)$  = integral defined by Eq. (54) (dimensionless);  
 ${}_2F_1$  = Gauss-hypergeometric series (dimensionless);  
 $G$  = Catalan constant (dimensionless);  
 $i$  = imaginary number (dimensionless);  
 $K(.)$  = complete elliptical integral of the first kind (dimensionless);  
 $k$  = hydraulic conductivity (m/s);  
 $m$  = side slope of channel (1 vertical:  $m$  horizontal) (dimensionless);  
 $q_s$  = seepage discharge per unit length of channel ( $m^2/s$ );  
 $T$  = width of channel at water surface (m);  
 $u$  = velocity of seeping water along  $X$  axis (m/s);  
 $V$  = resultant velocity of seeping water (m/s);  
 $v$  = velocity of seeping water along  $Y$  axis (m/s);  
 $W$  =  $\phi + i\psi$  complex potential ( $m^2/s$ );  
 $X$  = real axis of the complex plane (m);  
 $Y$  = imaginary axis of the complex plane (m);  
 $y$  = water depth in channel (m);  
 $Z$  =  $X + iY$  complex plane variable (m);  
 $\beta, \gamma$  = transformation variables (dimensionless);  
 $\Gamma$  = gamma function (dimensionless);  
 $\zeta$  = complex variable in auxiliary plane (dimensionless);  
 $\sigma$  =  $(1/\pi)\cot^{-1}m$  = (dimensionless);  
 $\tau, t$  = dummy variables (dimensionless);  
 $\phi$  = velocity potential ( $m^2/s$ ); and  
 $\psi$  = stream function ( $m^2/s$ ).

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