Flow in Unsaturated Soil

Lecture 4

4.1.4 Steady-state horizontal flow

Consider a one-dimensional and steady flow through a homogeneous soil column as shown in the Fig. 13.

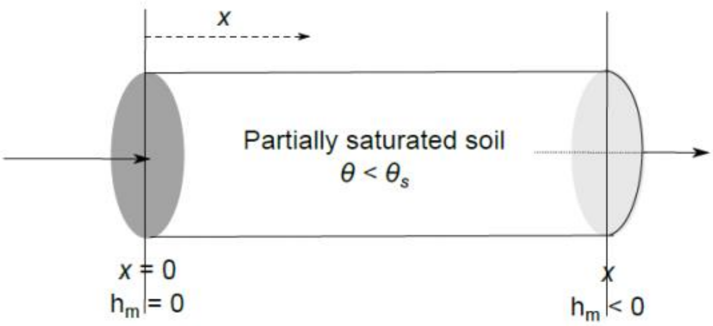

Fig. 13. Illustration of 1D steady-state flow through unsaturated soils

Flow through unsaturated soils is assumed to be described by modified Darcy's law which is expressed as:

$$q = -k(h)\frac{dh}{dx} \quad \text{--------(29)}$$

Where: k is the hydraulic conductivity expressed in functional form for unsaturated soils, h is the total energy potential (total head), and x is the distance measured in the direction of flow.

The total energy potential is sum of the matric suction head and the elevation head i.e., $h_m + x$ in the absence of solute.

If we assume that the Gardner's (1958) functional form for the hydraulic conductivity is valid The Darcy's law for steady flow in unsaturated soils may be written as:

$$q = -k_s \exp(\alpha h_m)\frac{dh_m}{dx} \quad \text{-------- (30)}$$

where α is the fitting parameter related to air entry head (1/cm). Integrating the above expression and imposing the boundary condition results

$$q\int_0^x dx = -k_s \int_0^{h_m} \exp(\alpha h_m) dh_m \quad \text{--------- (31)}$$

which gives to

$$h_m = \frac{1}{\alpha}\ln\left(1 - \frac{q\alpha x}{k_s}\right) \quad \text{-----------(32)}$$

which is a non-linear distribution when a non-linear functional relationship is assumed for hydraulic conductivity.

Similarly, one can also assume a simple linear relationship (Richards, 1931) for hydraulic conductivity and derive the expression for head distribution. The hydraulic conductivity function by Richards (1931) is given as:

$$k = a + bh_m \qquad \text{-------(33)}$$

The discharge flux can be expressed by

$$q = -\left(a + bh_m\right)\frac{dh_m}{dx} \qquad \text{--------(34)}$$

After simplification and integration,

$$q\int_0^x dx = -\int_0^{h_m}\left(a + bh_m\right)dh_m = -ah_m - \frac{bh_m^2}{2} \qquad \text{---------(35)}$$

which can be expressed in the quadratic form as

$$\frac{bh_m^2}{2} + ah_m + qx = 0 \qquad \text{------(36)}$$

which is has two solutions (roots) as shown below

$$h_m = \frac{1}{b}\left(-a \pm \sqrt{a^2 - 2bqx}\right) \qquad \text{-----------(37)}$$

Substituting the lower boundary ($x = 0$, $h = 0$) provides the correct root which is

$$h_m = -\frac{1}{b}\left(a - \sqrt{a^2 - 2bqx}\right) \qquad \text{---------(38)}$$

which is a non-linear distribution.

Therefore, the head distribution in unsaturated soils is non-linear irrespective of the functional form assumed for hydraulic conductivity.

4.2 Transient flow (Unsteady-state flow)

The flow of water in unsaturated soils may vary both spatially and temporally due to several factors. Time dependent changes in the boundary conditions (infiltrationevaporation) can significantly influence the flow mechanism. Such changes are accounted by the theoretical models by considering these changes in terms of boundary conditions for the soil domain. Other effects due to soil hydraulic characteristics are captured in the governing equation.

The governing one-dimensional, transient flow equation in soils can be expressed as:

$$-\rho_w \frac{\partial q}{\partial x} = \frac{\partial(\rho_w \theta)}{\partial t} \quad \text{--------(39)}$$

Where: ρ is the density of water (kg/m^3) and q is the water flux (m/s) in the x direction.

In case of flow through saturated soils, the volumetric water content is equal to the porosity, n, of the soil. After combining the Eq. (39) with Darcy's equation and writing n for θ gives:

$$D\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \quad \text{------(40)}$$

where *h* is the total head, *D* the hydraulic diffusivity (m^2/s), which is equal to S_s/k, and S_s the specific storage. The specific storage is defined as

$$S_s = \rho_w g(\alpha_s + n\beta_w) \quad \text{--------------(41)}$$

where α_s is the bulk compressibility of soil (m^2/N) and β_w the compressibility of pore water (m^2/N).

The flow through unsaturated soils can be described using the Darcy's law as

$$\frac{\partial}{\partial z}\left[k(h_m)\left(\frac{\partial h_m}{\partial z} + 1\right)\right] = \frac{\partial \theta}{\partial t} \quad \text{--------(42)}$$

where the additional term added to suction gradient is the gradient due to elevation. Using the chain rule,

$$\frac{\partial}{\partial z}\left[k(h_m)\left(\frac{\partial h_m}{\partial z} + 1\right)\right] = \frac{\partial \theta}{\partial h_m}\frac{\partial h_m}{\partial t} \quad \text{---------(43)}$$

where $\partial\theta/\partial h_m$ is the slope of the soil water characteristic curve, which is called specific moisture capacity, C. As SWCC is non-linear, C is expressed as a function of matric suction head. The resulting equation is called Richards' equation which is written as

$$\frac{\partial}{\partial z}\left[k(h_m)\left(\frac{\partial h_m}{\partial z}+1\right)\right]=C(\theta)\frac{\partial h_m}{\partial t} \qquad \text{--------(44)}$$

The Richards' equation may often be expressed in terms of volumetric water content as shown in the following equation

$$\frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}+k(\theta)\right]=\frac{\partial \theta}{\partial t} \qquad \text{---------(45)}$$

The numerical solution of the Richards equation is one of the most challenging problems in earth science. Richards' equation has been criticized for being computationally expensive and unpredictable because there is no guarantee that a solver will converge for a particular set of soil constitutive relations. This prevents use of the method in general applications where the risk of non-convergence is high. The method has also been criticized for over-emphasizing the role of capillarity, and for being in some ways 'overly simplistic' .In one dimensional simulations of rainfall infiltration into dry soils, fine spatial discretization less than one cm is required near the land surface. which is due to the small size of the representative elementary volume for multiphase flow in porous media. In three-dimensional applications the numerical solution of Richards' equation is subject to aspect ratio constraints where the ratio of horizontal to vertical resolution in the solution domain should be less than about 7.

i.e :

The **specific storage** is the amount of water that a portion of an aquifer releases from storage, per unit mass or volume of aquifer, per unit change in hydraulic head, while remaining fully saturated.

Mass specific storage is the mass of water that an aquifer releases from storage, per mass of aquifer, per unit decline in hydraulic head:

$$(S_s)_m = \frac{1}{m_a}\frac{dm_w}{dh}$$

where

$(S_s)_m$ is the mass specific storage ([L^{-1}]);

m_a is the mass of that portion of the aquifer from which the water is released ([M]);

dm_w is the mass of water released from storage ([M]); and

dh is the decline in hydraulic head ([L]).

Volumetric specific storage (or **volume specific storage**) is the volume of water that an aquifer releases from storage, per volume of aquifer, per unit decline in hydraulic head (Freeze and Cherry, 1979):

$$S_s = \frac{1}{V_a}\frac{dV_w}{dh} = \frac{1}{V_a}\frac{dV_w}{dp}\frac{dp}{dh} = \frac{1}{V_a}\frac{dV_w}{dp}\gamma_w$$

where

S_s is the volumetric specific storage ($[L^{-1}]$);
V_a is the bulk volume of that portion of the aquifer from which the water is released ($[L^3]$);
dV_w is the volume of water released from storage ($[L^3]$);
dp is the decline in pressure($N{\cdot}m^{-2}$ or $[ML^{-1}T^{-2}]$) ;
dh is the decline in hydraulic head ([L]) and
γ_w is the specific weight of water ($N{\cdot}m^{-3}$ or $[ML^{-2}T^{-2}]$).

In hydrogeology, **volumetric specific storage** is much more commonly encountered than **mass specific storage**. Consequently, the term **specific storage** generally refers to **volumetric specific storage**.

In terms of measurable physical properties, specific storage can be expressed as

$$S_s = \gamma_w(\beta_p + n \cdot \beta_w)$$

where

γ_w is the specific weight of water ($N{\cdot}m^{-3}$ or $[ML^{-2}T^{-2}]$)
n is the porosity of the material (dimensionless ratio between 0 and 1)
β_p is the compressibility of the bulk aquifer material (m^2N^{-1} or $[LM^{-1}T^2]$), and
β_w is the compressibility of water (m^2N^{-1} or $[LM^{-1}T^2]$)

The compressibility terms relate a given change in stress to a change in volume (a strain). These two terms can be defined as:

$$\beta_p = -\frac{dV_t}{d\sigma_e}\frac{1}{V_t}$$

$$\beta_w = -\frac{dV_w}{dp}\frac{1}{V_w}$$

where

σ_e is the effective stress (N/m^2 or $[MLT^{-2}/L^2]$)

These equations relate a change in total or water volume (V_t or V_w) per change in applied stress (effective stress — σ_e or pore pressure — p) per unit volume. The compressibilities (and therefore also S_s) can be estimated from laboratory consolidation tests (in an apparatus called a consolidometer), using the consolidation theory of soil mechanics (developed by Karl Terzaghi).