

Lagrange Interpolation formula

This method has less accuracy than Newton formula, but it has advantage of it can be used for equally or non-equally spaced data.

$$f(x) = \sum_{k=0}^n N_k(x) f_k$$

$$N_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x - x_j)}{(x_k - x_j)} = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x - x_j)}{(x_k - x_j)}$$

where, n: number of terms - 1

Derivation of the Lagrange formula

Let $y = f(x)$ is a function which takes $(n+1)$ values

$y_0, y_1, y_2, \dots, \dots, \dots, \dots, y_n$ corresponding to

$x_0, x_1, x_2, \dots, \dots, \dots, \dots, x_n$.

Let us consider the polynomial of the form;

$$\begin{aligned}
y = f(x) = & a_0(x - x_1)(x - x_2) \dots \dots \dots \dots \dots \dots \dots (x - x_n) \\
& + a_1(x - x_0)(x - x_2) \dots \dots \dots \dots \dots \dots \dots (x - x_n) \\
& + a_2(x - x_0)(x - x_1) \dots \dots \dots \dots \dots \dots \dots (x - x_n) \\
& + a_n(x - x_0)(x - x_1) \dots \dots \dots \dots \dots \dots \dots (x - x_{n-1})
\end{aligned}$$

----- Eq.(1)

To find the values of (a_0, a_1, \dots, a_n) we can use the following conditions:

1- $y = y_o$ at $x = x_o$, Sub. in Eq.(1)

$$\begin{aligned}
y_o = & a_0(x_o - x_1)(x_o - x_2) \dots \dots \dots \dots \dots \dots \dots (x_o - x_n) \\
& + a_1(x_o - x_o)(x_o - x_2) \dots \dots \dots \dots \dots \dots \dots (x_o - x_n) \\
& + a_2(x_o - x_o)(x_o - x_1) \dots \dots \dots \dots \dots \dots \dots (x_o - x_n) \\
& + a_n(x_o - x_o)(x_o - x_1) \dots \dots \dots \dots \dots \dots \dots (x_o - x_{n-1})
\end{aligned}$$

$$\gg a_o = \frac{y_o}{(x_o - x_1)(x_o - x_2) \dots \dots \dots \dots \dots \dots \dots (x_o - x_n)}$$

2- $y = y_1$ at $x = x_1$, Sub. in Eq.(1)

$$\begin{aligned}
y_1 = & a_0(x_1 - x_1)(x_1 - x_2) \dots \dots \dots \dots \dots \dots \dots (x_1 - x_n) \\
& + a_1(x_1 - x_o)(x_1 - x_2) \dots \dots \dots \dots \dots \dots \dots (x_1 - x_n) \\
& + a_2(x_1 - x_o)(x_1 - x_1) \dots \dots \dots \dots \dots \dots \dots (x_1 - x_n) \\
& + a_n(x_1 - x_o)(x_1 - x_1) \dots \dots \dots \dots \dots \dots \dots (x_1 - x_{n-1})
\end{aligned}$$

$$\gg a_1 = \frac{y_1}{(x_1 - x_o)(x_1 - x_2) \dots \dots \dots \dots (x_1 - x_n)}$$

Similarly for $a_2 \dots \dots \dots \dots \dots \dots a_n$

Where

$$\gg a_n = \frac{y_n}{(x_n - x_o)(x_n - x_1) \dots \dots \dots \dots (x_n - x_{n-1})}$$

If we sub. all the values in Eq. (1), we will get the Lagrange Interpolation formula;

$$\begin{aligned}
 y = f(x) = & \frac{(x - x_1)(x - x_2) \dots \dots \dots (x - x_n)}{(x_o - x_1)(x_o - x_2) \dots \dots \dots (x_o - x_n)} y_o \\
 & + \frac{(x - x_o)(x - x_2) \dots \dots \dots (x - x_n)}{(x_1 - x_o)(x_1 - x_2) \dots \dots \dots (x_1 - x_n)} y_1 \\
 & + \frac{(x - x_o)(x - x_1) \dots \dots \dots (x - x_n)}{(x_2 - x_1)(x_2 - x_2) \dots \dots \dots (x_2 - x_n)} y_2 \\
 & + \dots \dots \dots \dots \dots \dots \\
 & + \frac{(x - x_o)(x - x_1) \dots \dots \dots (x - x_{n-1})}{(x_n - x_o)(x_n - x_1) \dots \dots \dots (x_n - x_{n-1})} y_n
 \end{aligned}$$

Ex. if take $n = 2$

$$f(x) = \sum_{k=0}^2 N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$N_0 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)}$$

$$N_1 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)}$$

$$N_2 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)}$$

$$\therefore f(x) = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)} f_0 + \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)} f_1 + \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)} f_2$$

Ex. For the last example, find $f(1.25)$ using 3-term Lagrange interpolation formula.

		x_0	x_1	x_2		
$x = 1.25$	x	0	0.5	1	1.5	2.0
	$f(x)$	1.2	1.8	3.1	4.6	7.6

$f_0 \quad f_1 \quad f_2$

$$n = \text{No. of terms} - 1 = 3 - 1 = 2$$

$$f(x) = \sum_{k=0}^2 N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$N_{k=0}^0 = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)} = \frac{(1.25 - 1)}{(0.5 - 1)} \cdot \frac{(1.25 - 1.5)}{(0.5 - 1.5)} = -0.125$$

$$N_{k=1}^1 = \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)} = \frac{(1.25 - 0.5)}{(1 - 0.5)} \cdot \frac{(1.25 - 1.5)}{(1 - 1.5)} = 0.75$$

$$N_{k=2}^2 = \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)} = \frac{(1.25 - 0.5)}{(1.5 - 0.5)} \cdot \frac{(1.25 - 1)}{(1.5 - 1)} = 0.375$$

$$\therefore f(1.25) = -0.125 * 1.8 + 0.75 * 3.1 + 0.375 * 4.6 = \textcolor{red}{3.825}$$

Examples

Example (1):

For the following data (x_n, f_n) find $f(0.25)$, $f(1.3)$ and $f(-0.35)$

x	0.0	0.2	0.4	0.6	0.8	1.0
F	1.2	3.43	6.15	12.23	25.7	55.6

Using linear, quadratic interpolation and 4-trem Newton interpolation formula

Sol:-

n	X	f_n	ΔF_n	$\Delta^2 F_n$	$\Delta^3 F_n$
0	0.0	1.2	2.23	0.49	2.87
1	0.2	3.43	2.72	3.36	4.03
2	0.4	6.15	6.08	7.39	9.04
3	0.6	12.23	13.47	16.43	⊗
4	0.8	25.7	29.9	⊗	⊗
5	1.0	55.6	⊗	⊗	⊗

$$1 - F(0.25)$$

$$h = x_1 - x_0 = 0.2$$

$$\text{let } x_0 = 0.2, \quad F_0 = 3.43 \text{ and } \Delta F_0 = 2.72$$

$$\text{linear} \quad F = F_0 + r\Delta F_0$$

$$f(0.25) = 3.43 + \frac{0.25-0.2}{0.2} \times 2.72 = 4.11$$

$$\text{quadratic} \quad F = F_0 + r\Delta F_0 + \frac{r(r-1)}{2} \Delta^2 F_0$$

$$f(0.25) = 3.43 + 0.25 \times 2.72 + \frac{0.25(0.25-1)}{2} \times 3.36 = 3.795$$

4-trem Newton interpolation formula

$$r = 0.25$$

$$F = F_0 + r\Delta F_0 + \frac{r(r-1)}{2!} \Delta^2 F_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 F_0$$

$$F(0.25) = 3.43 + 0.25 \times 2.72 + \frac{0.25(0.25-1)}{2} \times 3.36 + \frac{0.25(0.25-1)(0.25-2)}{3 \times 2 \times 1} \times 4.03 = 4.015$$

2. F(1.3) HW

3. f(-0.35) HW

Example 2;

For the same data in the last example find $f(0.25)$ using 3-term Lagrange interpolation formula

$$f(0.25) = \frac{(0.25 - 0.2)(0.25 - 0.4)}{(0 - 0.2)(0 - 0.4)} \times 1.2 + \frac{(0.25 - 0)(0.25 - 0.4)}{(0.2 - 0)(0.2 - 0.4)} \times 3.43 + \frac{(0.25 - 0)(0.25 - 0.2)}{(0.4 - 0)(0.4 - 0.2)} \times 6.15 =$$

Hw. Complete the example and find $f(1.3)$ and $f(-0.35)$

Note:- Lagrange interpolation formula can be used for equally or non-equally spaced points.