



University of Basra-College of
Engineering
Petroleum Engineering Department



Subject: Numerical Methods & Reservoir Simulation **Class:** 4th Year

Lecturer: Dr. Amani J. Majeed Al-Husseini

Syllabus

- Interpolation, (Linear, Lagrange),
- Matrices, Review of matrix properties,
- Determinates, inverse of matrix,
- solution of system of linear equations (Gaussian elimination, Gauss Jordan method, Jacobi method, Gauss Seidel method),
- least Square method (linear equations, polynomial equations)
- Reservoir simulation (Introduction, types of simulators) flow through porous media (derivation of single phase, one-dimensional flow equation,
- two and three-dimensional flow equation),
- Finite difference method (Taylor series, forward difference, backward difference, central difference, concepts of explicit and implicit methods)

Numerical method

In numerical analysis, a **numerical method** is a mathematical tool designed to solve numerical problems. The implementation of a numerical method with an appropriate convergence check in a programming language is called a numerical algorithm.

Analytical vs Numerical Solutions

In mathematics, the problems can be solved analytically and numerically.

-An analytical solution involves framing the problem in a well-understood form and calculating the exact solution.

-A numerical solution means making guesses at the solution and testing whether the problem is solved well enough to stop.

Sometimes, the analytical solution is unknown and all we have to work with is the numerical approach.



Interpolation, (Linear, Lagrange),

Interpolation and Extrapolation

- Interpolation; is an estimation of a value within two known values in a sequence of values.
- Extrapolation is an estimation of a value based on extending a known sequence of values or facts beyond the area that is certainly known. In a general sense, to extrapolate is to infer something that is not explicitly stated from existing information.

* **Linear interpolation**

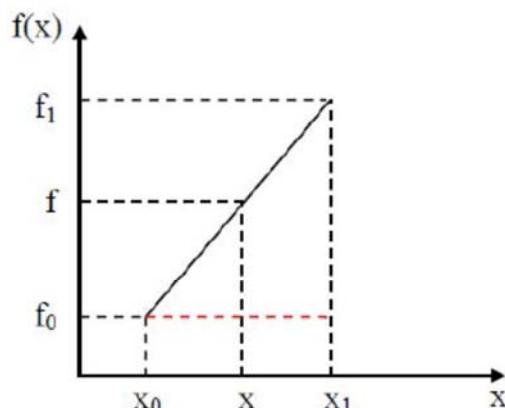
$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$

$$(f - f_0)h = (x - x_0)\Delta f_0$$

$$f = f_0 + \frac{x - x_0}{h}\Delta f_0$$

$$r = \frac{x - x_0}{h}$$

$$f = f_0 + r\Delta f_0 \quad \text{"linear"}$$



$$\begin{aligned} h &= x_1 - x_0 \\ &= x_{n+1} - x_n \quad \text{"step"} \end{aligned}$$

*Quadratic Interpolation derivative:

- If you have three data points, you can introduce some curvature for a better fitting.

- A second-order polynomial (quadratic polynomial) of the form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

To determine the values of the coefficients;

$$x = x_0 \Rightarrow b_0 = f(x_0)$$

$$x = x_1 \Rightarrow b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

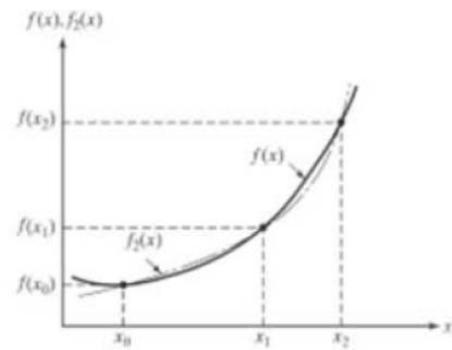
$$x = x_2 \Rightarrow$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)}$$

If you expand the terms, this is nothing different a general polynomial

Linear interpolation formula

Quadratic interpolation formula



finite divided difference of second derivative

*Newton interpolation derivative

Let f is a function of x then the procedure to find $f(x)$ for (n) order is as follow

| i | x_i | $f(x_i)$ | $f'(x_i)$ | $f''(x_i)$ | |
|---|-------|----------|-----------|------------|-------|
| 0 | x_0 | | | | |
| 1 | x_1 | | | | |
| 2 | x_2 | | | | |
| 3 | x_3 | | | | |
| 4 | x_4 | | | | |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| . | . | . | . | . | . |

| i | x_i | $f(x_i)$ | $f(x_l)$ | $f'(x_l)$ | $f''(x_l)$ | $f'''(x_l)$ |
|---|-------|----------|---------------------------------|---|---|--------------|
| 0 | x_0 | $f(x_0)$ | $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ | $\frac{f(x_2)-f(x_1)}{x_2-x_1} \quad \frac{f(x_1)-f(x_0)}{x_1-x_0}$ | $\frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \frac{f(x_2)-f(x_1)}{x_2-x_1} \quad \frac{f(x_2)-f(x_1)}{x_2-x_1} \quad \frac{f(x_1)-f(x_0)}{x_1-x_0}$ | \checkmark |
| 1 | x_1 | $f(x_1)$ | $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ | $\frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \frac{f(x_2)-f(x_1)}{x_2-x_1}$ | $\frac{f(x_4)-f(x_3)}{x_4-x_3} \quad \frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \frac{f(x_3)-f(x_2)}{x_3-x_2} \quad \frac{f(x_2)-f(x_1)}{x_2-x_1}$ | \square |
| 2 | x_2 | $f(x_2)$ | $\frac{f(x_3)-f(x_2)}{x_3-x_2}$ | $\frac{f(x_4)-f(x_3)}{x_4-x_3} \quad \frac{f(x_3)-f(x_2)}{x_3-x_2}$ | \square | \square |
| 3 | x_3 | $f(x_3)$ | $\frac{f(x_4)-f(x_3)}{x_4-x_3}$ | \square | \square | \square |
| 4 | x_4 | $f(x_4)$ | \square | \square | \square | \square |

$$f(x_n) = f(x_0) + f(x_0)(x - x_0) + f(x_0)(x - x_0)(x - x_1) + f(x_0)(x - x_0)(x - x_1)(x - x_2) \dots \dots \dots \dots$$

Or,

$$f(x) = f(x_n, x_{n-1}, \dots, x_0)(x - x_{n-1})$$

** Quadratic interpolation

$$f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0$$

****Newton Forward interpolation formula

$$f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 f + \dots$$

****Newton Backward interpolation formula

$$f = f_0 + r \nabla f_0 + \frac{r(r+1)}{2!} \nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_0 + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 f + \dots$$

Ex: If the values of $x = 8, 12$ and the $f(x) = \text{Log}(x)$, find $f(10)$ by using linear interpolation , What is the error?

Sol.

$$x_o = 8 \Rightarrow f(x_o) = \log(8)$$

$$x_1 = 12 \Rightarrow f(x_1) = \log(12)$$

$$\therefore f(x) = f(x_o) + \frac{x - x_o}{x_1 - x_o} (f(x_1) - f(x_o))$$

$$\therefore f(10) = 0.9911356$$

$$\text{Error} = \frac{\text{Exact} - \text{App.}}{\text{Exact}} * 100\%$$

$$\therefore \text{Error} = 0.88644$$



Ex. Find $f(-0.5)$, $f(1.25)$, $f(2.75)$ using linear and quadratic and 4-term Newton formula.

| | | | | | |
|--------|-----|-----|-----|-----|-----|
| x | 0 | 0.5 | 1 | 1.5 | 2.0 |
| $f(x)$ | 1.2 | 1.8 | 3.1 | 4.6 | 7.6 |

Sol.

$f(-0.5) = ?$ extrapolation

$$\text{linear : } f = f_0 + r \Delta f_0 \quad r = \frac{x - x_0}{h}$$

$$x_0 = 0, h = 0.5 \text{ "step"} \quad h = x_{n+1} - x_n \quad r = \frac{-0.5}{0.5} = -1$$

$$f(-0.5) = 1.2 + (-1)(0.6) = 0.6$$

$$\text{Quadratic : } f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0$$

$$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2} * 0.7 = 1.3$$

4-term Newton Forward formula

$$f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0$$

$$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2}(0.7) + \frac{(-1)(-2)(-3)}{6}(-0.5) = 1.8$$

$f(1.25) = ?$ Let $x_0 = 1.0$ **Note:** Assume the x_0 nearest to the desired value

$$\text{linear : } r = \frac{x - x_0}{h} = \frac{1.25 - 1}{0.5} = 0.5$$

$$f(1.25) = f_0 + r \Delta f_0 = 3.1 + 0.5 * 1.5 = 3.85$$

Quadratic :

$$f(1.25) = f_0 + r \Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0 = 3.1 + 0.5 * 1.5 + \frac{0.5 * (-0.5)}{2} * 1.5 = 3.66$$

4-term Newton formula Let $x_0 = 0.5$ *Because $x_0=1$ not found $\Delta^3 f_0$

$$r = \frac{x - x_0}{h} = \frac{1.25 - 0.5}{0.5} = 1.5$$

$$f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0$$

$$f(1.25) = 1.8 + 1.5 * 1.3 + \frac{1.5 * 0.5}{2} * 0.2 + \frac{1.5 * 0.5 * (-0.5)}{6} * 1.3 = 3.74$$

f(2.7) = ? Let $x_0 = 2.0$ **Note:** Using Backward Table because containing all terms

$$r = \frac{x - x_0}{h} = \frac{2.75 - 2}{0.5} = 1.5$$

linear : $f(2.75) = f_0 + r \nabla f_0 = 7.6 + 1.5 * 3.0 = 12.1$

Quadratic :

$$f(2.75) = f_0 + r \nabla f_0 + \frac{r(r+1)}{2} \nabla^2 f_0 = 7.6 + 1.5 * 3.0 + \frac{1.5 * 2.5}{2} * 1.5 = 14.91$$

4-term Newton formula

$$f = f_0 + r \nabla f_0 + \frac{r(r+1)}{2!} \nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_0$$

$$f(2.75) = 7.6 + 1.5 * 3.0 + \frac{1.5 * 2.5}{2} * 1.5 + \frac{1.5 * 2.5 * 3.5}{6} * 1.3 = 17.75$$

Note: Newton interpolation formula can be used only when the point are equally spaced ($h = \text{constant}$).

Explanation

For f (1.25):

Solution:

| i | x | F | F' | F'' | F''' | F'''' |
|---|-----|-----|-----|-----|------|-------|
| 1 | 0 | 1.2 | 0.6 | 0.7 | -0.5 | 0.8 |
| 2 | 0.5 | 1.8 | 1.3 | 0.2 | 1.3 | ■ |
| 3 | 1 | 3.1 | 1.5 | 1.5 | ■ | ■ |
| 4 | 1.5 | 4.6 | 3 | ■ | ■ | ■ |
| 5 | 2 | 7.6 | ■ | ■ | ■ | ■ |

$f(1.25) = ?$ Let $x_0 = 1.0$ **Note:** Assume the x_0 nearest to the desired value

linear : $r = \frac{x - x_0}{h} = \frac{1.25 - 1}{0.5} = 0.5$

$$f(1.25) = f_0 + r \Delta f_0 = 3.1 + 0.5 * 1.5 = 3.85$$

Quadratic :

$$f(1.25) = f_0 + r \Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0 = 3.1 + 0.5 * 1.5 + \frac{0.5 * (-0.5)}{2} * 1.5 = 3.66$$

4-term Newton formula Let $x_0 = 0.5$ *Because $x_0=1$ not found $\Delta^3 f_0$

$$r = \frac{x - x_0}{h} = \frac{1.25 - 0.5}{0.5} = 1.5$$

$$f = f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0$$

$$f(1.25) = 1.8 + 1.5 * 1.3 + \frac{1.5 * 0.5}{2} * 0.2 + \frac{1.5 * 0.5 * (-0.5)}{6} * 1.3 = 3.74$$

HW: If $f(x) = \ln(x)$ and $x_0=1$, $x_1=4$, $x_2=6$, find the $f(x)$ formula by using Quadratic interpolation

HW2

| | | | | | | | |
|------------------------|-------|-------|-------|-------|-------|-------|-------|
| $x = \text{height:}$ | 100 | 150 | 200 | 250 | 300 | 350 | 400 |
| $y = \text{distance:}$ | 10.63 | 13.03 | 15.04 | 16.81 | 18.42 | 19.90 | 21.27 |

Find the values of y when

- (i) $x = 160 \text{ ft.}$ (ii) $x = 410.$

