Finite Element Method

M.Sc. Course in Structural Engineering

University of Basrah – 2022

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Finite Element Method:

- The finite element method is a numerical method for solving problems of engineering and mathematical physics.
- Usually the problem addressed is too complicated to be solved satisfactorily by classical analytical methods.
- The FEM originated as a method of stress analysis, but today it is used to analyze problems of

❑Heat transfer

❑Fluid flow

❑Lubrication

❑Mass transport

❑Electric and magnetic fields

- The Finite Element procedure produces ***many simultaneous algebraic equations***, which are generated and solved on digital computer.
- The results are usually ***not exact***, however, ***errors can decreased*** to an acceptable values with reasonable cost.

The need for using non-analytical solution:

- For problems involving:

✓ Complicated geometries,

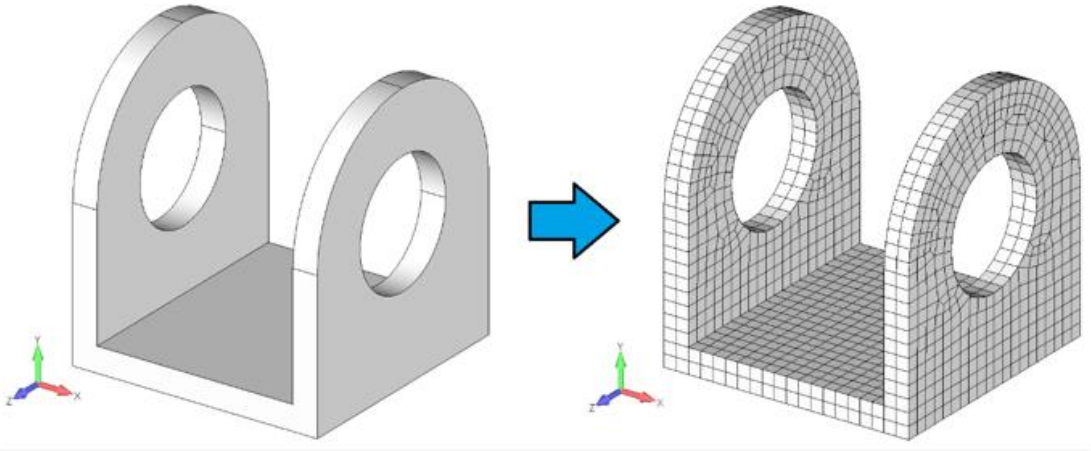
✓Complicated loadings, and

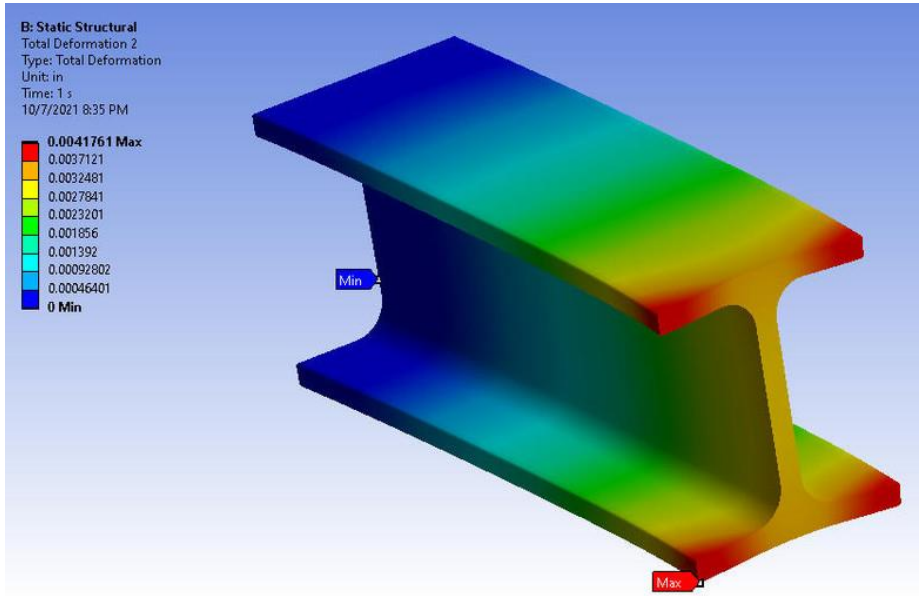
✓Complicated material properties,

it is generally not possible to obtain ***analytical mathematical solutions***.

- Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities ***at any location*** in a body (here total structure or physical system of interest) and are thus ***valid for an infinite number of locations*** in the body.
- These analytical solutions generally require the solution of ordinary or partial differential equations, which because of the complicated geometries, loadings, and material properties, are not usually obtainable.

- Hence we need to rely on numerical methods, such as the **finite element method**, for acceptable solutions.
- The finite element formulation of the problem results in a ***system of simultaneous algebraic equations*** for solution, rather than requiring the solution of differential equations.
- These numerical methods ***yield approximate values*** of the unknowns at ***discrete*** numbers of points in the continuum.
- Hence this process of modeling a body by dividing it into an equivalent system of smaller bodies or units (***finite elements***) interconnected at points common to two or more elements (***nodal points or nodes***) and/or boundary lines and/or surfaces is called ***Discretization***.




B: Static Structural
Total Deformation 2
Type: Total Deformation
Unit: in
Time: 1 s
10/7/2021 8:35 PM
0.0041761 Max
0.0037121
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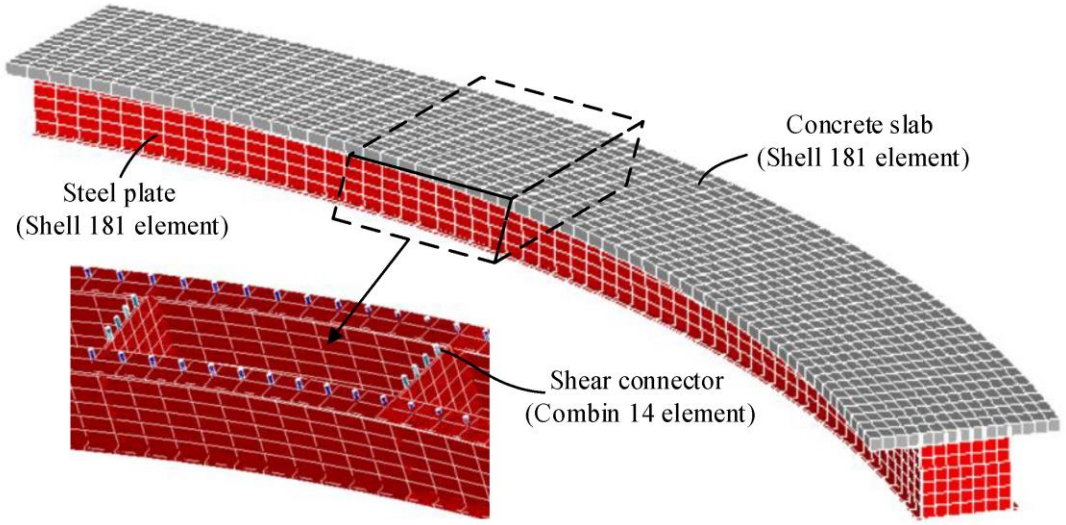
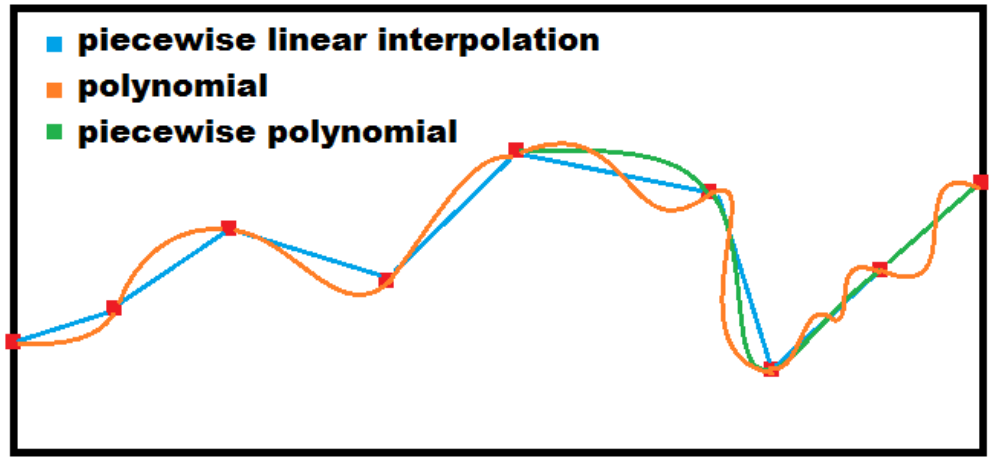

Concrete slab
(Shell 181 element)
Steel plate
(Shell 181 element)
Shear connector
(Combin 14 element)

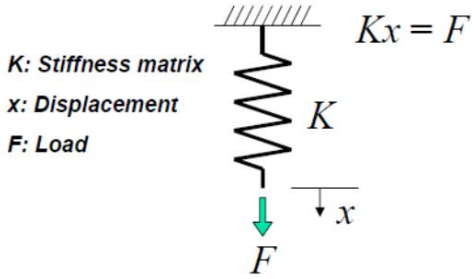
- In the finite element method, instead of solving the problem for the entire body in one operation, we **formulate the equations for each finite element** and combine them to obtain the solution of the whole body.
- **Briefly**, the solution for structural problems typically refers to determining the ***displacements*** at each node and the **stresses** within each element making up the structure that is subjected to applied loads.
- In nonstructural problems, the nodal unknowns may, for instance, be ***temperatures*** or ***fluid pressures*** due to thermal or fluid fluxes.

- FEM uses the concept of ***piecewise polynomial interpolation***.
- By connecting elements together, the field quantity becomes interpolated over the entire structure in piecewise fashion.
- A set of simultaneous algebraic equations at nodes.



$$[\mathbf{K}]\{\mathbf{u}\} = \{\mathbf{F}\}$$

Property — Behavior — Action



- In this method of analysis, ***a complex region*** defining a continuum is ***discretized into simple geometric shapes*** called ***finite elements***.
- The **material properties** and the **governing relationships** are considered over these elements and expressed in terms of ***unknown values*** at element corners.
- An ***assembly*** process, duly considering the ***loading and constraints***, results in a ***set of equations***.
- Solution of these equations gives us the approximate behavior of the continuum.

Problem Classification, modeling and discretization:

1. Problem classification:

- The first step in solving problems is to identify it.
- What are the more important physical phenomena involved?
- Is the problem is time dependent or time independent, i.e. static or dynamic?
- Is nonlinearity involved, so that iterative solution is necessary?
- What results are required from analysis?
- What accuracy is required?

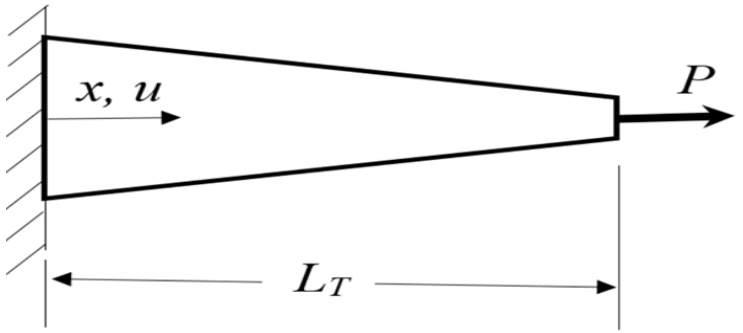
- A complicated problems may not lie in one category.
- An example is a ***fluid – structure interaction*** problem, such as earthquake excitation of a storage tank that contains liquid.
- Motion of the liquid makes a thin walled tank deflect, and deflection of the thin walls modifies the liquid motion.
- Therefore, structural displacement and fluid motion fields cannot be considered separately.
- So that, calculations must take their ***interaction*** into account.
- This is known as ***direct or mutual coupling***, in which each field influence the other.
- Other problems may be called as ***indirect or sequential coupling***, in which only one field influences the other.
- An example, is the thermal stresses, where temperature influences stresses but stresses have no influence on temperature.

2. **Modelling:**

- The equations is applied to a model rather than to an actual physical problem.
- A geometric model becomes a mathematical model when its behavior is described, or approximated by selected differential equations and boundary conditions.
- The mathematical model is an idealization, in which geometry, material properties, loads and boundary conditions are simplified.
- The simplification depends on what features are important or unimportant in obtaining the required results.
- As an example in stress analysis, materials may be regarded as homogeneous, isotropic, and linearly elastic.
- A load distributed over a small area may be regarded as concentrated at a point (which is not physically possible).
- A support may be represented as fixed (actually no support is completely fixed)
- A flat structure may be modeled as two dimensional if stress variation in in the thickness direction is neglected.

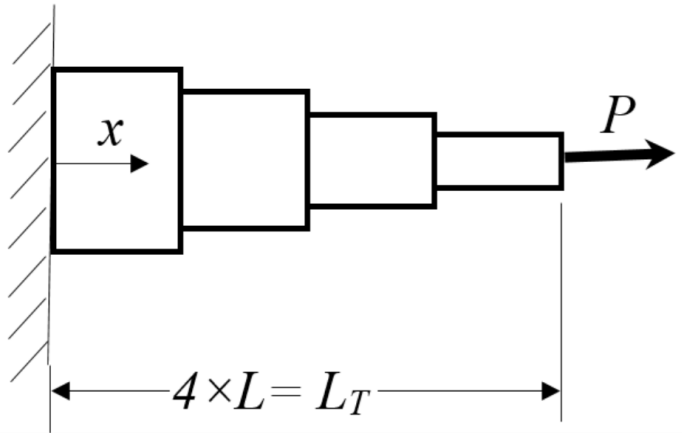
3. Discretization:

- A mathematical model is discretized by dividing it into a mesh of finite elements.
- Thus a fully continuous field is represented by a ***piecewise continuous*** field defined by a finite number of nodal quantities and simple interpolation within each element.

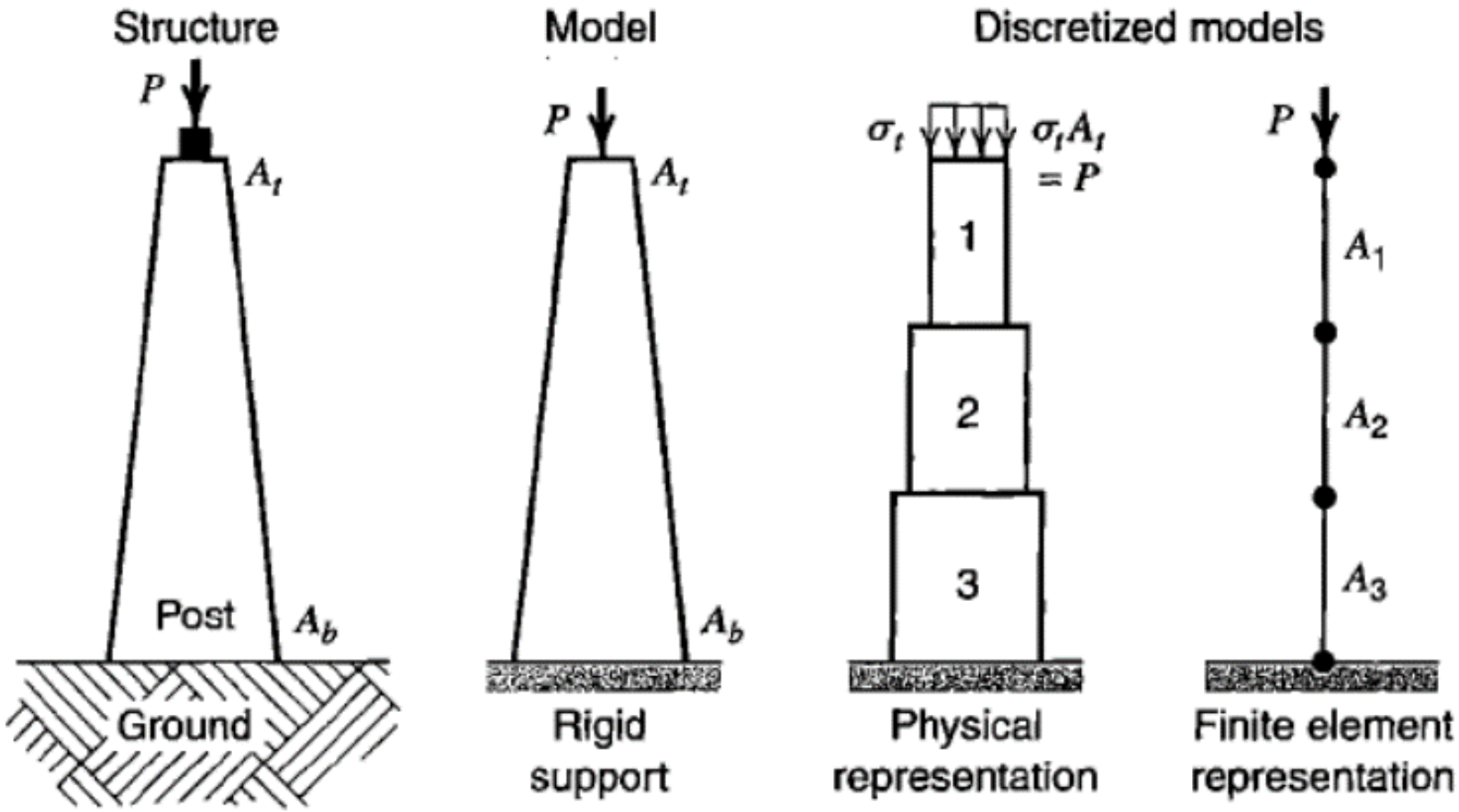


- Consider the tapered bar shown.
- The displacement of the right end is required.
- The classical approach is to write the differential equation of the tapered bar, then solve it for axial displacement ***u*** as a function of ***x***, and finally substitute $\boldsymbol{x = L_T}$ to find the required end displacement.

- The finite element approach to this problem does not begin with a differential equation.
- Instead, the bar is discretized by modeling it as a series of segments of certain dimensions (***finite elements***).
- Each element is of uniform but of different cross sectional area A.
- In each element, the displacement ***u*** varies linearly with local ***x*** of each element.
- Therefore, for $0 < x < L_T$, ***u*** is a piecewise smooth function of ***x***.
  - The elongation of each element can be determined from the elementary formula ***PL/AE***.
  - The end displacement, at $x = L_T$, is the sum of the elemental elongations.
  - The accuracy is improved as the number of elements are increased.



- In general, the finite element method models a structure as an assemblage of small parts (elements).
- Each element is of simple geometry and therefore is much easier to analyze than the actual structure.
- In essence, the complicated solution is approximated by a model that consists of piecewise-continuous simple solutions.
- Elements are called "***Finite***" to distinguish them from differential elements used in calculus (infinite elements).



Modeling and discretization

HISTORY OF FINITE ELEMENT METHOD:

- Basic ideas of the finite element method originated from advances in aircraft structural analysis.
- In 1941, Hrenikoff presented a solution of elasticity problems using the “frame work method.”
- Courant’s paper, which used piecewise polynomial interpolation over triangular subregions to model torsion problems, was published in 1943.
- Turner et al. derived stiffness matrices for truss, beam, and other elements and presented their findings in 1956.
- The term *finite element* was first coined and used by Clough in 1960.

- In the early 1960s, engineers used the method for approximate solution of problems in stress analysis, fluid flow, heat transfer, and other areas.
- A book by Argyris in 1955 on energy theorems and matrix methods laid a foundation for further developments in finite element studies.
- The first book on finite elements by Zienkiewicz and Cheung was published in 1967.
- In the late 1960s and early 1970s, finite element analysis was applied to nonlinear problems and large deformations.
- Oden’s book on nonlinear continua appeared in 1972.
- Mathematical foundations, new element development, convergence studies, and other related areas fall in this category were laid in the 1970s.

Advantages of Finite Element Method:

- Can readily handle very complex geometry:
- Can handle a wide variety of engineering problems

- Solid mechanics - Dynamics - Heat problems - Fluids - Electrostatic problems

- Can handle complex restraints

- Indeterminate structures can be solved.

- Can handle complex loading

- Nodal load (point loads)

- Element load (pressure, thermal, inertial forces)

- Time or frequency dependent loading

Disadvantages of Finite Element Method:

- A general closed-form solution, which would permit one to examine system response to changes in various parameters, is not produced.
- The FEM obtains only "approximate" solutions.
- The FEM has "inherent" errors.
- Mistakes by users can be fatal.

Procedure of FEA:

1. Physical Problem (Structural, heat, flow, ..)
2. Mathematical Model (Differential equations)
3. Material properties, Loading, Boundary conditions.
4. Finite element solution:

    4 a- choice of element type:

- 1D (truss element, Beam element)
- 2D (Shell element, Plate element)
- 3D (Solid Element)

    4 b- mesh density (number of elements)

    4 c- solution parameters

- 5. Representation of loading and boundary conditions
- 6. Type and method of solution (Linear, Nonlinear, Structural Static Analysis, Modal Analysis, Transient Dynamic Analysis, Buckling Analysis, Contact, Steady- state Thermal Analysis, Transient Thermal Analysis
- 7. Obtaining the results: Displacement, Stress, Strain, Natural frequency, Temperature, Time history, pressure)

Basic Steps and Properties of Finite Element Method

General Steps of the Finite Element Method:

- Typically, for the structural stress-analysis problem, the engineer seeks to determine displacements and stresses throughout the structure, which is in equilibrium and is subjected to applied loads.
- For many structures, it is difficult to determine the distribution of deformation using conventional methods, and thus the finite element method is necessarily used.
- There are three primary methods that can be used to derive the finite element equations of a physical system. These are

(1) the *direct method* or *direct equilibrium method for structural analysis problems*,
(2) the *variational methods consisting of among the subsets energy methods and the principle of virtual work*, and
(3) the *weighted residual methods*.

(1) Direct Approaches:

- The direct methods, being the simplest and yielding a clear physical insight into the finite element method, is recommended in the initial stages of learning the concepts of the finite element method.
- However, the direct method is limited in its application to deriving element stiffness matrices for one-dimensional elements involving springs, uniaxial bars, trusses, and beams.
- There are two general direct approaches traditionally associated with the finite element method as applied to structural mechanics problems.
- One approach, called the *force* (*or flexibility*) *method*, the second is called the *stiffness (or displacement) method*.

1.1 Flexibility (Force) Method:

- The force, or flexibility, method, uses internal forces as the unknowns of the problem.
- To obtain the governing equations, first the equilibrium equations are used.
- Then necessary additional equations are found by introducing compatibility equations.
- The result is a set of algebraic equations for determining the redundant or unknown forces.

1.2 Stiffness (Displacement) method:

- The displacement, or stiffness, method, assumes the displacements of the nodes as the unknowns of the problem.
- For instance, compatibility conditions requiring that elements connected at a common node, along a common edge, or on a common surface before loading remain connected at that node, edge, or surface after deformation takes place are initially satisfied.
- Then the governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

- These two direct approaches result in different unknowns (forces or displacements) in the analysis and different matrices associated with their formulations (flexibilities or stiffnesses).
- It has been shown that, for computational purposes, the displacement (or stiffness) method is more desirable because its formulation is simpler for most structural analysis problems.
- Furthermore, a vast majority of general-purpose finite element programs have incorporated the displacement formulation for solving structural problems.
- Consequently, only the displacement method will be used throughout this course.

(2) Variational Methods:

- A second general method that can be used to develop the governing equations for both structural and nonstructural problems is the variational method.
- The variational method is much easier to use for deriving the finite element equations for two- and three-dimensional elements than the direct method.
- However, it requires the existence of a "**functional**", that upon minimizing yields the stiffness matrix and related element equations.
- The variational method includes a number of principles, such as

1) The principle of minimum potential energy, and
2) The principle of virtual work.

2.1 Principle of minimum potential energy:

- For structural/stress analysis problems, we can use the "**principle of minimum potential energy**" as the functional.
- This principle is extensively used for it is relatively easy physical concept to understand and has likely been introduced in an undergraduate course in basic applied mechanics.
- The theorem of minimum potential energy applies to materials behaving in a linear-elastic manner.
- A functional analogous to that used in the theorem of minimum potential energy can be employed to develop the finite element equations for the nonstructural problem such as heat transfer.

2.2 Principle of Virtual Work:

- Another variational principle often used to derive the governing equations is the principle of virtual work.
- This principle applies more generally to materials that behave in a linear-elastic fashion, as well as those that behave in a nonlinear fashion.
- The principle of virtual work can be used for developing the general governing finite element equations that can be applied specifically to bars, beams, and two- and three-dimensional solids in either static or dynamic systems.

(3) Weighted Residual Methods:

- The weighted residual methods allow the finite element method to be applied directly to **any differential equation** without having the existence of a variational principle.
- The widely used weighted residual methods are:

a) Galerkin's method,

b) Collocation Method,

c) Subdomain Method, and

d) Least Squares Method

Step 1: Discretization and Selecting the Element Types:

- First step involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type to model most closely the actual physical behavior.
- The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment.
- The elements must be made small enough to give usable results and yet large enough to reduce computational effort.
- Small elements (and possibly higher order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur; large elements can be used where results are relatively constant.
- The discretized body or mesh is often created with mesh-generation programs or preprocessor programs available to the user.

Selecting the appropriate Element:

- The choice of elements used in a finite element analysis depends on the physical state of the body under actual loading conditions and on how close to the actual behavior the analyst wants the results to be.
- Judgment concerning the appropriateness of one-, two-, or three-dimensional idealizations is necessary.
- Moreover, the choice of the most appropriate element for a particular problem is one of the major tasks that must be carried out by the designer/analyst.

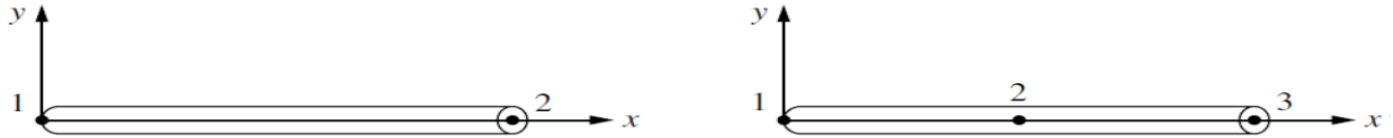
Types and Shapes of Element:

- Based on the problem considered the elements may be on of the following types:

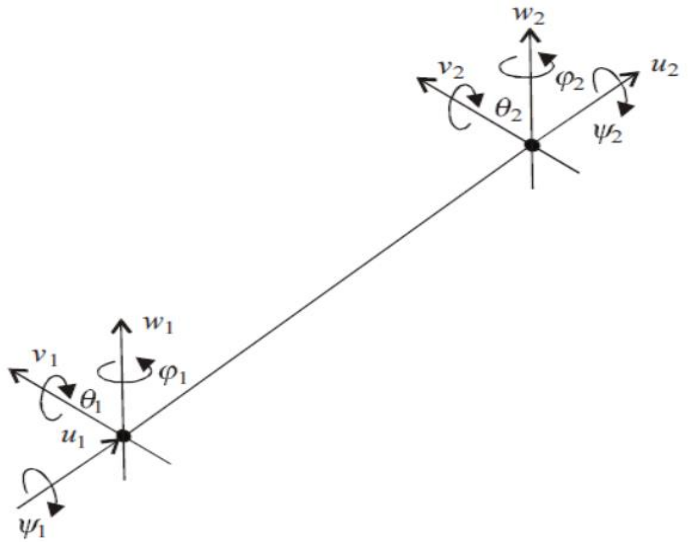
1) One dimensional elements
2) Two dimensional elements
3) Axi-symmetric elements and
4) Three dimensional elements.

1. One dimensional element:

- These elements are suitable for the analysis of one dimensional problem and may be called as line elements.



- Examples of the one dimensional elements are: Spring, Bar, Truss, and Beam (frame) elements.
- One dimensional elements usually have two nodes (one at each end).
- Each node may has one or more degree(s) of freedom.
- Frame Element have 6 DOF at each node.



- The line elements have a cross-sectional area but are usually represented by line segments.
- In general, the cross-sectional area within the element can vary, however constant is usually used.
- The simplest line element (called a linear element) has two nodes, one at each end, although higher-order elements having three nodes or more (called quadratic, cubic, etc. elements) also exist.
- The line elements are the simplest of elements to consider.

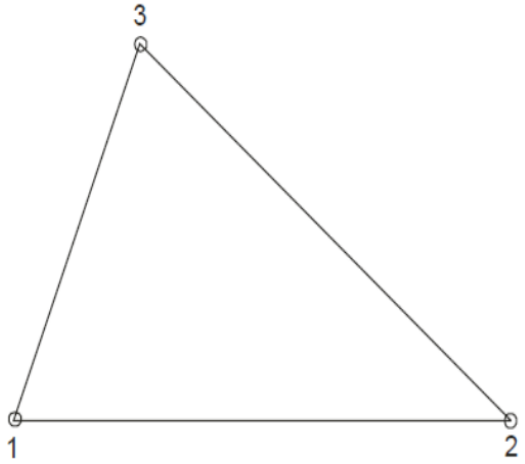
2. Two dimensional elements:

- Two dimensional elements are used to solve two dimensional problems.
- Common two dimensional problems in stress analysis are plane stress, plane strain and plate problems.
- Examples of two dimensional problems are: a plate subjected to in-plane stresses and bending of slab.

- Two dimensional elements often used is three noded triangular elements.
- It has the distinction of being the first and most used element.

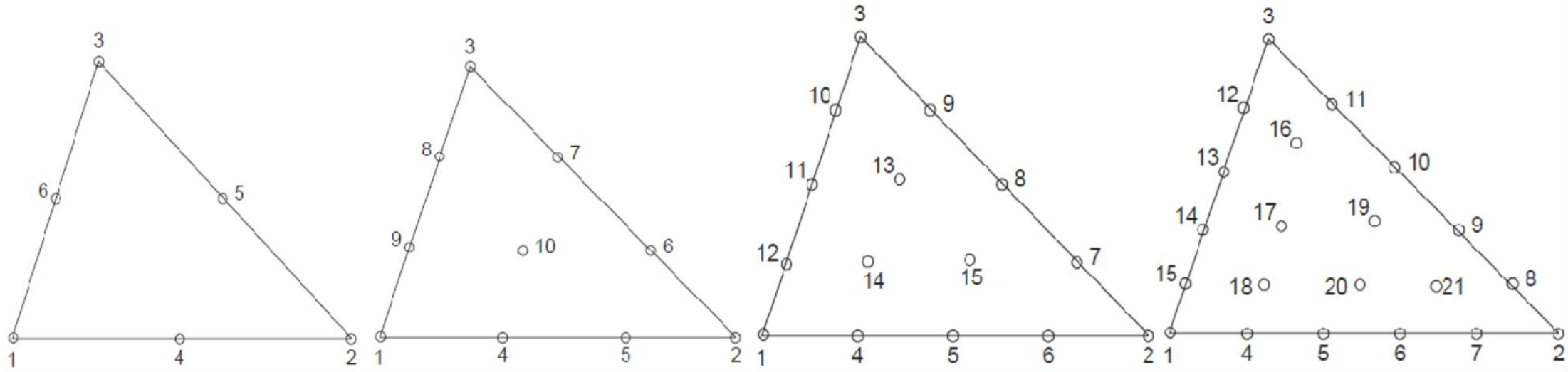
- These elements are known as **Constant Strain Triangles**

  (CST) or Linear Displacement Triangles.



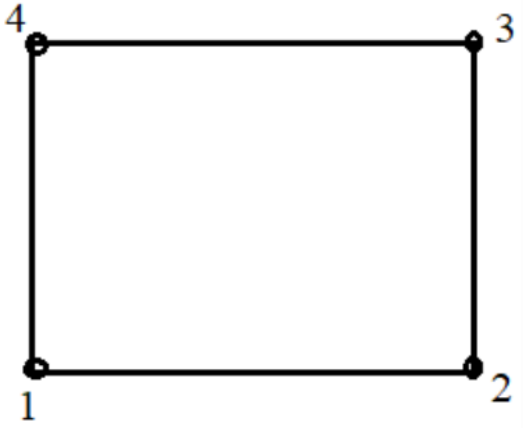
Constant Strain

- Six noded and ten noded triangular elements are also used by the analysts.
- Six noded triangular element is known as **Linear Strain Triangle (LST) or as Quadratic Displacement Triangle**.
- Ten noded triangular elements are known as **Quadratic Strain Triangles (QST)** or Cubic Displacement Triangles.
- 15 noded and 21 noded elements may also used.



Linear Strain Quadratic Strain Cubic Strain Quartic Strain

- A simple but less used two dimensional element is the four noded rectangular element whose sides are parallel to the global coordinate systems.



- This elements are easy to construct automatically but it is not well suited to approximate inclined boundaries.
- Rectangular elements of higher order also can be used.
- **Lagrange rectangle** in which nodes are in the form of grid points.
- **Serendipity rectangles** which are having nodes only along the external boundaries.

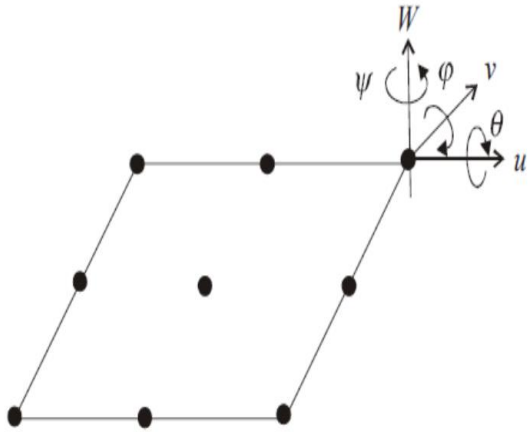
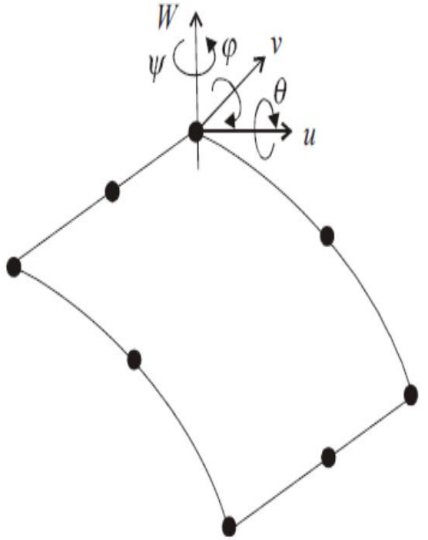
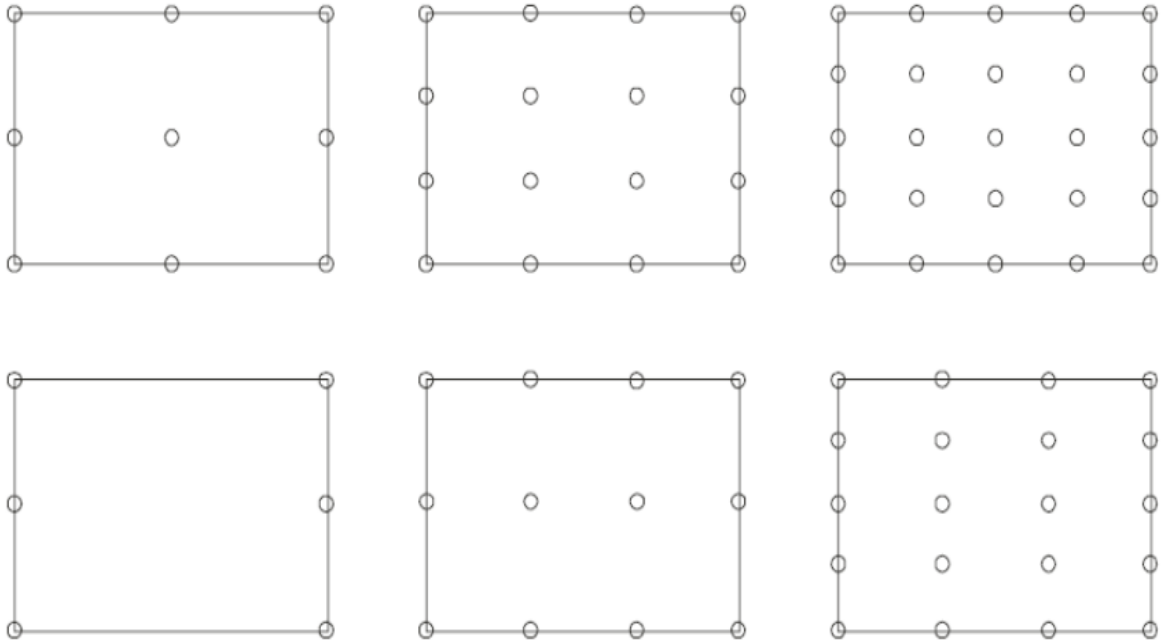


Plate Element
6 DOF per node

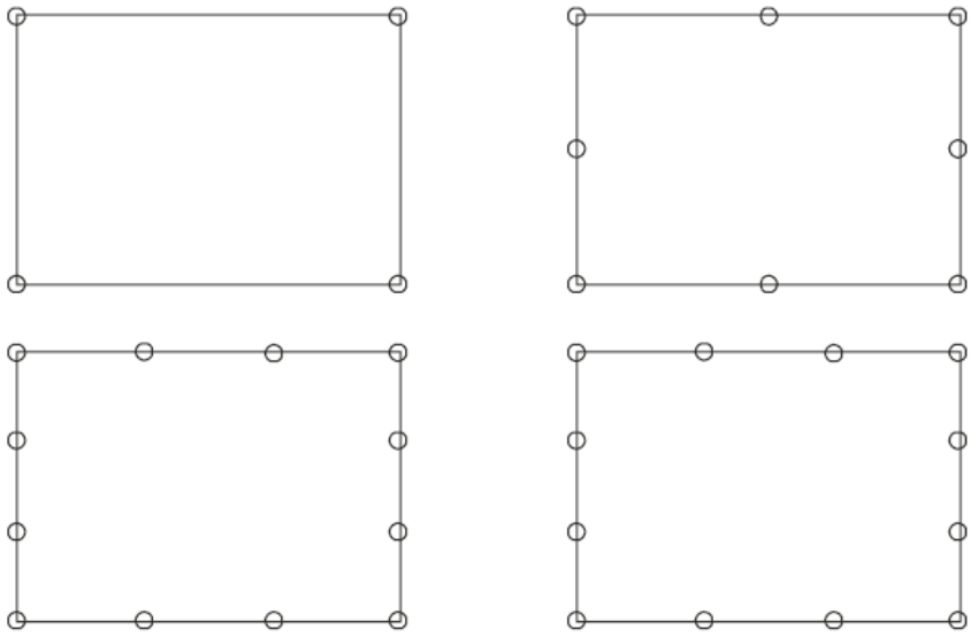


Shell Element
6-DOF per node



Lagrange family rectangular elements

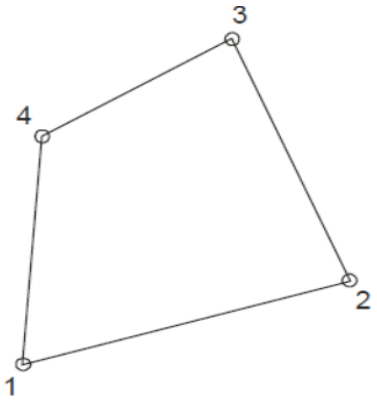
Lagrange Elements have nodes in the form of grid points.



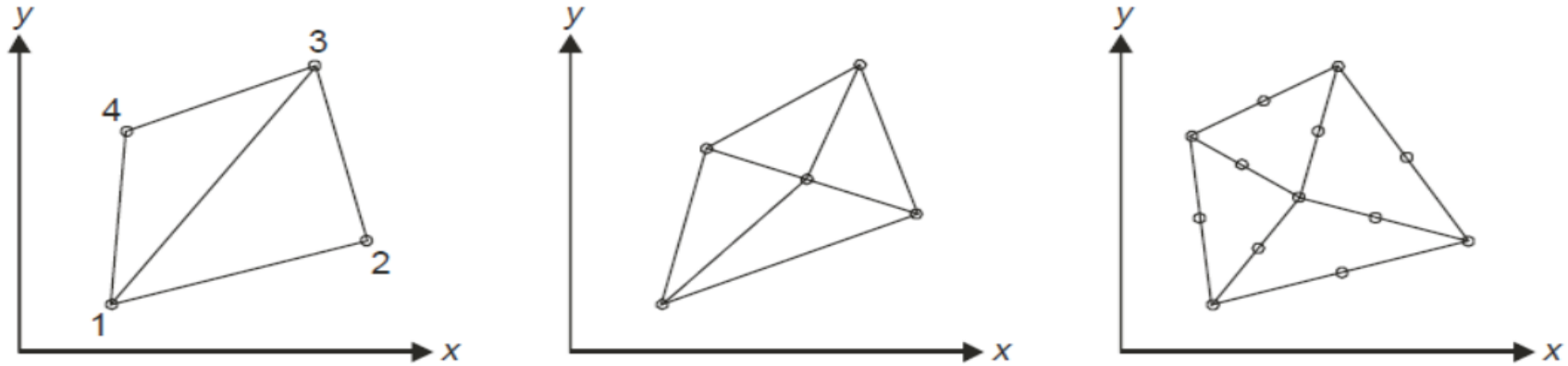
Serendipity family rectangular elements

Serendipity Elements have nodes only along the external boundaries.

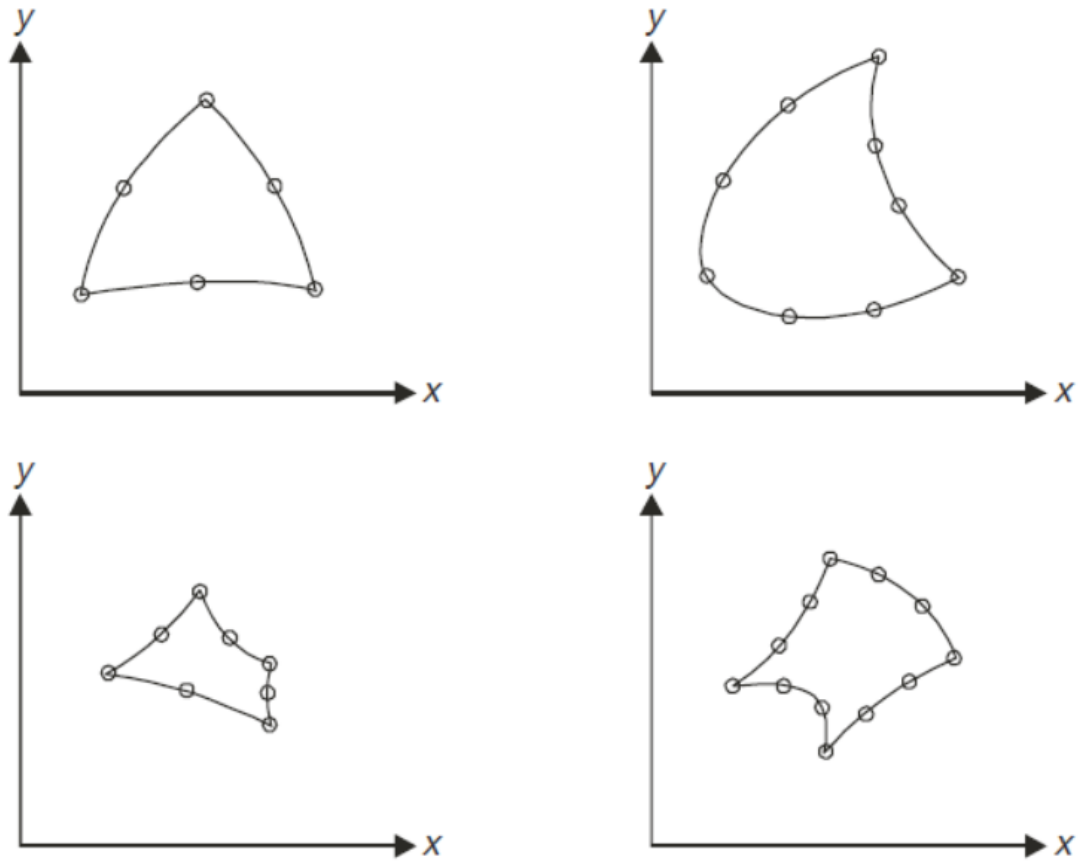
- **Quadrilateral Elements** are also used in finite element analysis
- Initially quadrilateral elements were developed by combining triangular elements.
- But it has taken back stage after isoparametric concept was developed.



- Isoparametric concept is based on using same functions for defining geometries and nodal unknowns.
- Even higher order triangular elements may be used to generate quadrilateral elements.

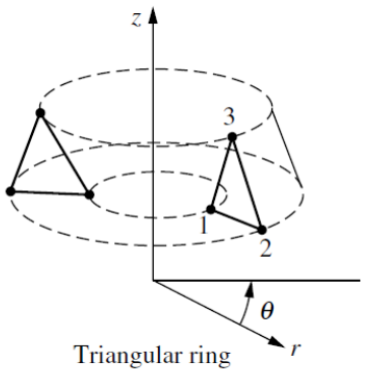


- Using **<u>isoparametric</u>** concept even curved elements are developed to take care of boundaries with curved shapes.

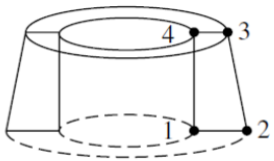


3. Axisymmetric Elements:

- These are also known as ring type elements.
- These elements are useful for the analysis of axi-symmetric problems such as analysis of cylindrical storage tanks, shafts, rocket nozzles.
- Axi-symmetric elements can be constructed from one or two dimensional elements.
- One dimensional axi-symmetric element is a conical surface.
- Two dimensional axi-symmetric element is a ring with a triangular or quadrilateral cross section.



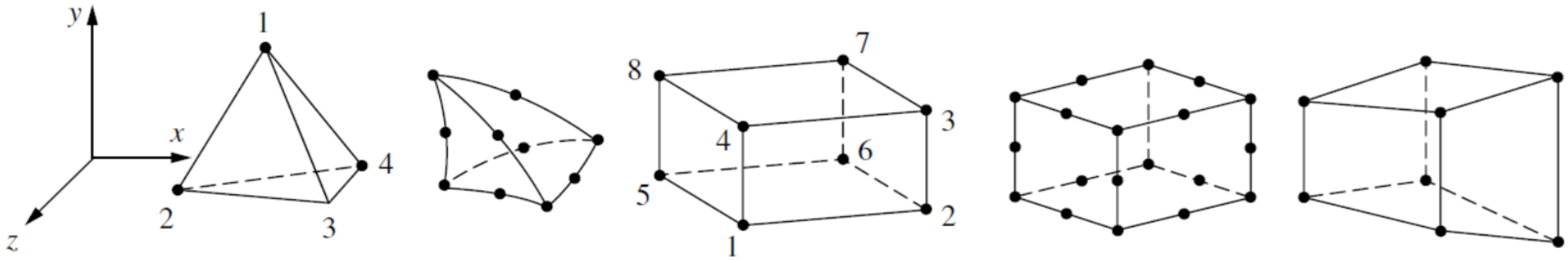
Triangular ring



Quadrilateral ring

4. Three dimensional elements:

- Similar to the triangle for two dimensional problems tetrahedron is the basic element for three dimensional problems.
- Tetrahedron having four nodes, one at each corner.
- Three dimensional elements with eight nodes are either in the form of a general hexahedron or a rectangular prism, which is a particular case of a hexahedron.
- The rectangular prism element is many times called as a **brick element** also.
- Higher order three dimensional elements are also used.



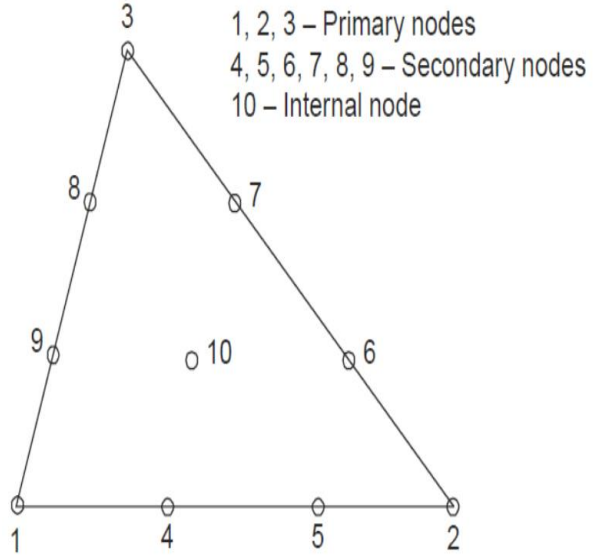
The Nodes:

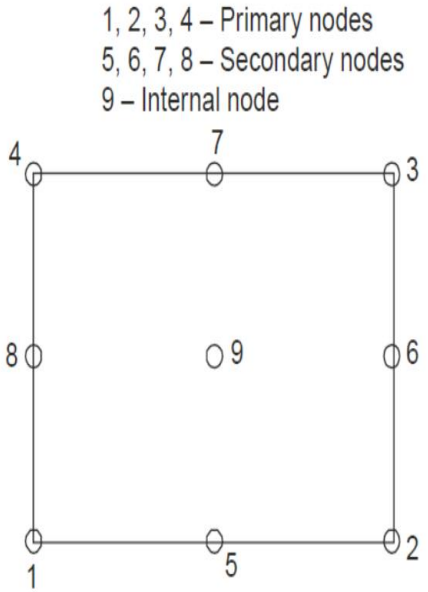
- Nodes are the selected finite points at which basic unknowns (displacements in elasticity problems, temperature in heat transfer problems, … etc.) are to be determined in the finite element analysis.
- The basic unknowns at any point inside the element are determined by using approximating/interpolation functions named as **Shape Functions** in terms of the nodal values of the element.
- There are two types of nodes; external nodes and internal nodes.
- External nodes are those which occur on the edges/surface of the elements and they may be common to two or more elements.
- These nodes may be further classified as (i) Primary nodes and (ii) Secondary nodes.

External and Internal Nodes:

- Primary nodes occur at the ends of one dimensional elements or at the corners in the two or three dimensional elements.
- Secondary nodes occur along the side of an element but not at corners.

- Internal nodes are the one which occur inside an element.
- They are specific to the element selected i.e. there will not be any other element connecting to this node.
- Such nodes are selected to satisfy the requirement of geometric isotropy while choosing interpolation functions.





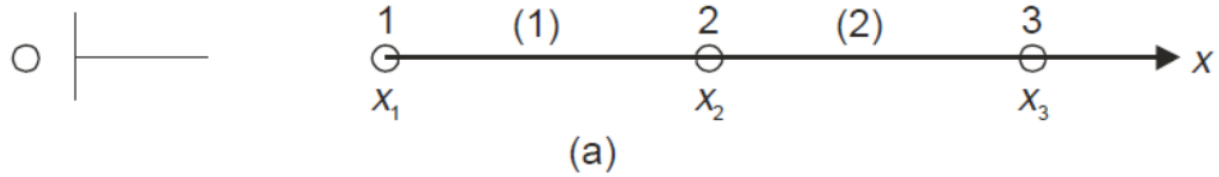
Coordinates System:

- The following terms are commonly referred in FEM

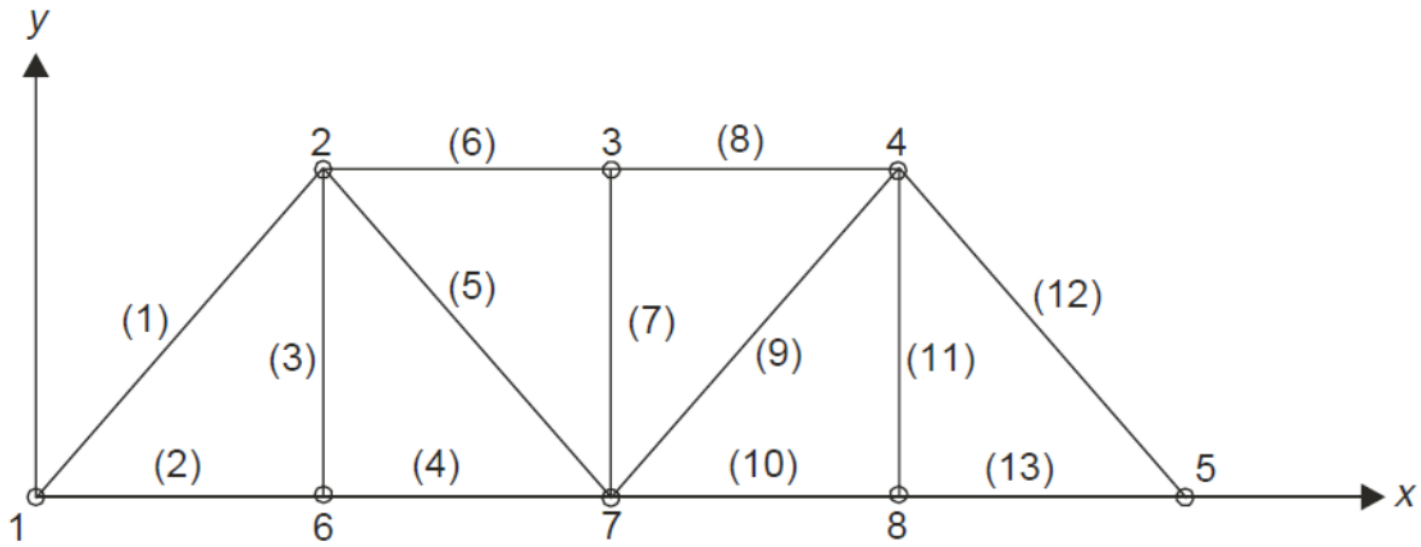
1. Global coordinates
2. Local coordinates, and
3. Natural coordinates.

Global Coordinates:

- The coordinate system used to define the points in the entire structure is called global coordinate system.

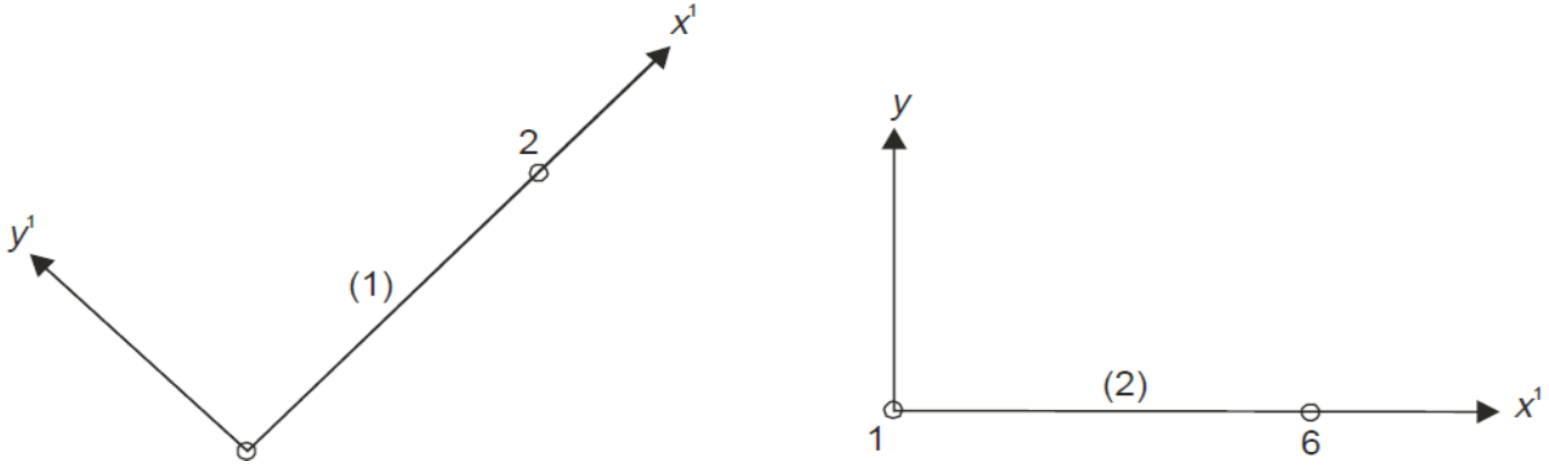


(a)



Local Coordinates:

- For the convenience of deriving element properties, in FEM many times for each element a separate coordinate system is used.
- The local coordinates are measured from the node point 1 of each element.
- However, the final equations are to be formed in the common coordinate system i.e. global coordinate system only.

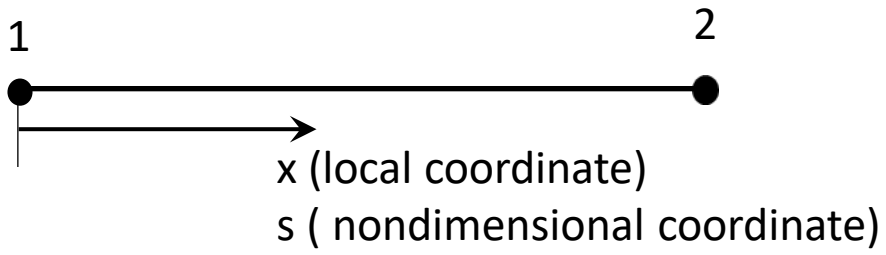


Non-dimensional local coordinates:

- Often it is more convenient to express the local coordinate as a nondimensional number; such a procedure can considerably facilitate the integrations and differentiations involved in the subsequent computation.
- The non-dimensional local coordinate for 1D element can be obtained by dividing the local coordinate by the element length, i.e.

$$s = x/l$$

Where l is the length of the element.



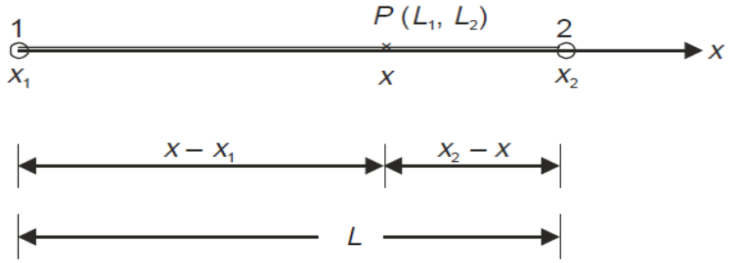
- The value of s varies from 0 at node 1 to 1.0 at node 2.

Natural Coordinates:

- It is obtained by assigning weightages to the nodal coordinates in defining the coordinate of any point inside the element.
- Hence such system has the property that *i*th coordinate has unit value at node *i* of the element and zero value at all other nodes.
- The use of natural coordinate system is advantageous in assembling element properties (stiffness matrices), since closed form integrations formulae are available when the expressions are in natural coordinate systems.
- For 1-D elements, the natural coordinates system gives each point two natural coordinates L_1 and L_2.
- The Cartesian coordinates of node 1 and node 2 are x_1 and x_2, respectively.
- Since natural coordinates are weightage to the nodal coordinates, total weightage at any point is unity i.e.,

$$L_1 + L_2 = 1 \qquad (1)$$

and also $$L_1 x_1 + L_2 x_2 = x \qquad (2)$$



In matrix form

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

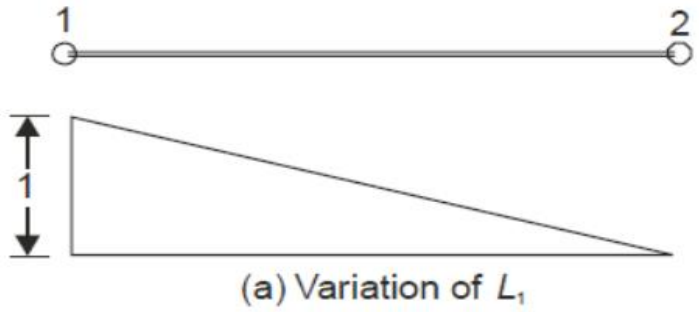
$$\therefore \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \times \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ -x_1 + x \end{Bmatrix}$$

Noting that $x_2 - x_1$ = length of the element (l), then,

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \frac{x_2 - x}{l} \\ \frac{x - x_1}{l} \end{Bmatrix}$$

- L_1 is 1 at node 1 and is zero at node 2.
- L_2 is zero at node 1 and is one at node 2.
- The variation is linear.



(a) Variation of L_1

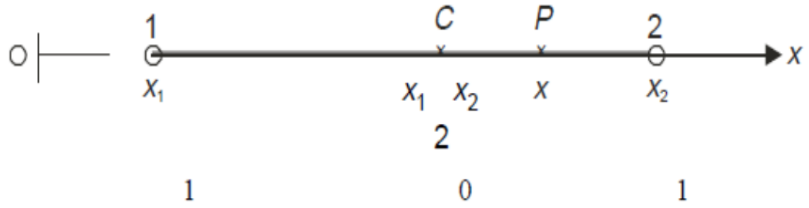
(b) Variation of L_2

Natural coordinates ξ:

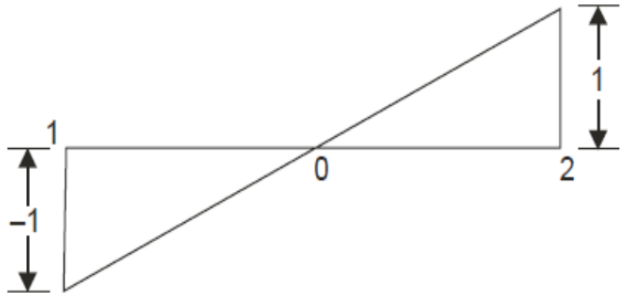
- In the second alternative, we can attach the origin of the local system at an intermediate point in the element, say, at the midpoint.
- Here the natural coordinate is written as

$$\xi = x / (l/2)$$

- ξ = -1 at node 1 and +1 at node 2.



Natural coordinate ξ



Variation of ξ

Step 2: Select a Displacement Function:

- This step involves choosing a displacement function within each element.
- The function is defined within the element using the nodal values of the element.
- Linear, quadratic, and cubic polynomials are frequently used functions because they are simple to work with in finite element formulation.
- However, trigonometric series can also be used.
- For a two-dimensional element, the displacement function is a function of the coordinates in its plane (say, the *x*-*y* plane).

- The functions are expressed in terms of the nodal unknowns (in the two-dimensional problem, in terms of an x and a y component).
- The same general displacement function can be used repeatedly for each element.
- Hence the finite element method is one in which a continuous quantity, such as the displacement throughout the body, is approximated by a discrete model composed of a set of piecewise-continuous functions defined within each finite domain or finite element.
- For the one-dimensional spring and bar elements, the displacement function is a function of a single coordinate (say x, along the axis of the spring or bar).

Interpolation functions:

- One of the main ideas in the finite element method is to describe the variation of the field variable (e.g. the displacement) throughout the element by a trial approximate functions.
- This implies that since it is difficult to find a closed form or exact solution, we guess a solution shape or distribution of displacement by using an appropriate mathematical function.
- In choosing this function, we must follow the laws, principles, and constraints or boundary conditions inherent in the problem.
- The most common functions used are polynomials.
- In the initial stages of the finite element method, the polynomials used were expressed in terms of generalized coordinates; however, now most finite element work is done by using interpolation functions, which can often be considered as transformed generalized coordinate functions.

- The simplest polynomial that we can use is the one that gives linear variation of displacements within the element.

$$u = a_1 + a_2 x$$

at node 1, $u_1 = a_1 + a_2 x_1$

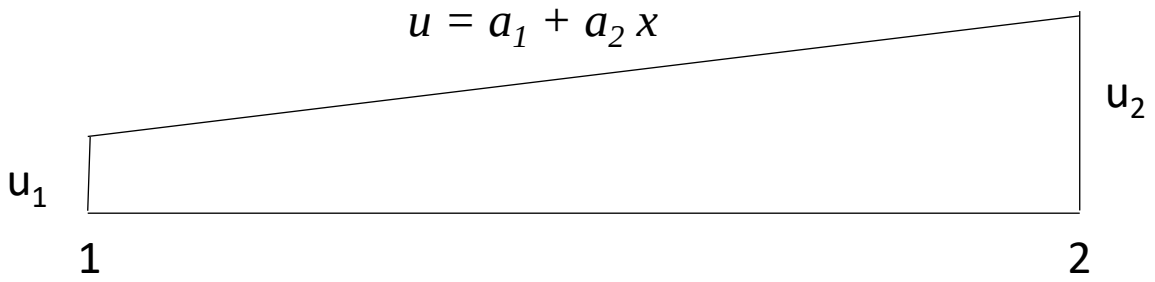
at node 2, $u_2 = a_1 + a_2 x_2$

In matrix form,

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

or

$$\{u\} = [x]\{a\}$$



Solving for [a], yields,

$\{a\} = [x]^{-1}\{u\}$. i.e.

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

From which, $a_1 = \frac{1}{l}(x_2\, u_1 - x_1\, u_2)$, and $a_2 = \frac{1}{l}(u_2 - u_1)$

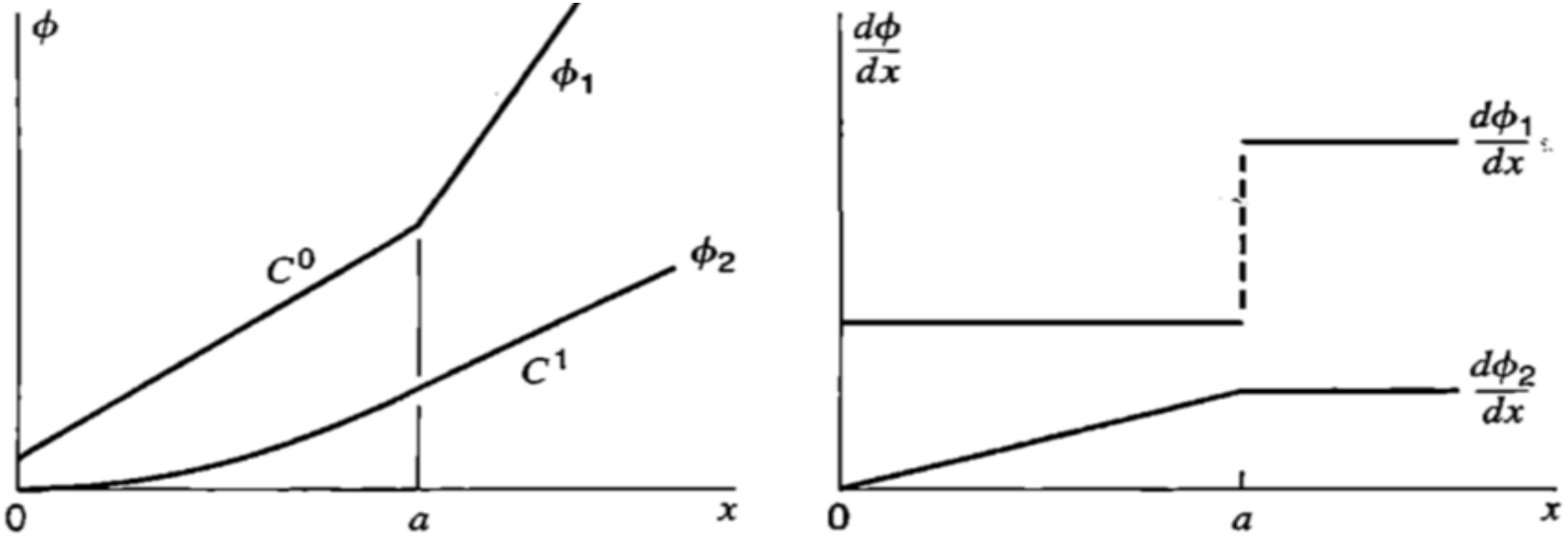
Thus, $u = [1 \quad x] \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = [\ 1 \quad \mathrm{x}] \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

$$u = \frac{1}{l}[x_2 - x \qquad -x_1 + x] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Nodal Unknowns:

- Basic unknowns may be displacements for stress analysis, temperatures for heat flow problems and the potentials for fluid flow or in the magnetic field problems.
- In the problems like truss analysis, plane stress and plane strain, it is enough if the continuity of only displacements are satisfied, since there is no change in the slopes at any nodal point.
- Such problems are classified as ‘zeroth’ continuity problems and are indicated as C^0-continuity problems.

- In case of beams and plates, not only the continuity of displacements, but the slope continuity also should be ensured.
- Since the slope is the first derivative of displacement, this type of problems are classified as '**First order continuity problems** and are denoted as **C^1–continuity problems**.



Step 3: Define the Strain/Displacement and Stress/Strain Relationships:

- Strain/displacement and stress/strain relationships are necessary for deriving the equations for each finite element.
- In the case of one-dimensional deformation, say, in the *x* direction, we have strain ε_x related to displacement *u* by

$$\varepsilon_x = \frac{du}{dx} \qquad \text{(for small strains)}$$

- In addition, the stresses must be related to the strains through the stress/ strain law (generally called the *constitutive law)*.
- The ability to define the material behavior accurately is most important in obtaining acceptable results.
- The simplest of stress/strain laws (Hooke's law) which is often used in stress analysis, is given by

$$\sigma_x = E\varepsilon_x$$

- where σ_x is the stress in the *x* direction and *E* is the modulus of elasticity.

Step 4: Derive the Element Stiffness Matrix and Equations:

- Initially, the development of element stiffness matrices and element equations was based on the concept of stiffness influence coefficients, which presupposes a background in structural analysis.
- Later, other methods such as the variational and weighted residual methos are also used to derive the element matrices and equations.

Step 5: Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions:

- In this step the individual element nodal equilibrium equations generated in step 4 are assembled into the global nodal equilibrium equations.
- Another more direct method of superposition (called the *direct stiffness method*), whose basis is nodal force equilibrium, can be used to obtain the global equations for the whole structure.
- Implicit in the direct stiffness method is the concept of continuity, or compatibility, which requires that the structure remain together and that no tears occur anywhere within the structure.

- The final assembled or global equation written in matrix form is

$$\{F\} = [K]\{d\} \qquad (1)$$

- where $\{F\}$ is the vector of global nodal forces, $[K]$ is the structure global or total stiffness matrix, (for most problems, the global stiffness matrix is square and symmetric) and $\{d\}$ is the vector of known and unknown structure nodal degrees of freedom or generalized displacements.
- It can be shown that at this stage, the global stiffness matrix $[K]$ is a singular matrix because its determinant is equal to zero.
- To remove this singularity problem, we must invoke certain boundary conditions (or constraints or supports) so that the structure remains in place instead of moving as a rigid body.
- At this time it is sufficient to note that invoking boundary or support conditions results in a modification of the global Eq. (1).
- We also emphasize that the applied known loads have been accounted for in the global force matrix $\{F\}$.

Step 6: Solve for the Unknown Degrees of Freedom (or Generalized Displacements):

- Equation (1), modified to account for the boundary conditions, is a set of simultaneous algebraic equations that can be written in expanded matrix form as

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{Bmatrix}$$

- where now *n* is the structure total number of unknown nodal degrees of freedom.
- These equations can be solved for the *d*s by using an elimination method (such as Gauss's method) or an iterative method (such as the Gauss-Seidel method).
- The *d*s are called the *primary unknowns*, because they are the first quantities determined using the stiffness (or displacement) finite element method.

Step 7: Solve for the Element Strains and Stresses:

- For the structural stress-analysis problems, important secondary quantities of strain and stress (or moment and shear force) can be obtained because they can be directly expressed in terms of the displacements determined in step 6.
- Typical relationships between strain and displacement and between stress and strain for one-dimensional problems are:

$$\varepsilon_x = \frac{du}{dx} \qquad \text{and} \qquad \sigma_x = E\varepsilon_x$$

Step 8: Interpret the Results:

- The final goal is to interpret and analyze the results for use in the design/analysis process.
- Determination of locations in the structure where large deformations and large stresses occur is generally important in making design/analysis decisions.
- Postprocessor computer programs help the user to interpret the results by displaying them in graphical form.

Direct Stiffness Method (Displacement Method)

Direct Stiffness Method:

- A main step of the finite element analytical process is the determination of the Stiffness Matrix.
- All the applications of FEM involves the derivation of this matrix.
- Before doing that it is helpful to layout some general ideas about the nature of this important matrix and the significance of the terms within it.
- The standard form of matrix displacement equation is,

$$[k]\ \{u\} = \{F\}$$

where $[k]$ is stiffness matrix

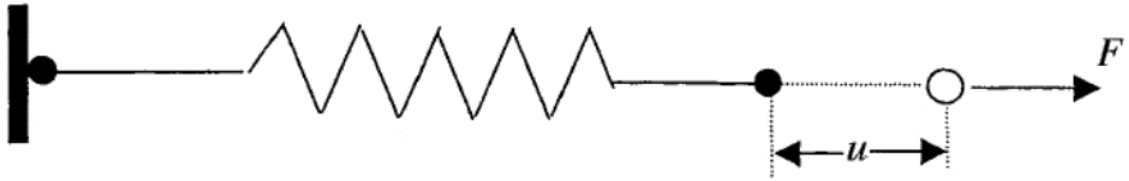
$\{u\}$ is displacement vector and

$\{F\}$ is force vector in the coordinate directions

- The element k_{ij} of stiffness matrix may be defined as the force at coordinate i due to unit displacement in coordinate direction j.
- The direct method of assembling stiffness matrix for few standard cases is briefly given in this article.

1. Stiffness matrix for a spring:

- Consider a coiled spring fixed at one end and with a force *F* applied at the other.



- The force required to produce unit extension of the spring is *k* (N/m).
- The right direction is positive.
- The relation between the force produced by a displacement *u* of the free end of the spring is given by the relation

$$F = k\,u \qquad \text{-------- (a)}$$

- Consider the case when end 2 is fixed and end 1 is given a positive displacement u_1.

- The force at end 1 due to displacement at 1 is

$$F_{11} = k\, u_1$$

The equilibrium requires that an equal and opposite force at end 2 is developed, i.e.

$$F_{21} = - F_{11}$$

This means that the force at 2 due to displacement at 1 is

$$F_{21} = - k\, u_1$$

- Similarly, when the end 1 is fixed while end 2 is given a displacement u_2
- The force at end 2 due to displacement at 2 is

$$F_{22} = k\, u_2$$

And the force at 2 due to displacement at 1 is

$$F_{12} = -\, k\, u_2$$

- If the two ends can displaced, the resultant force at each end is the algebraic sum of the forces developed from the two cases, i.e.

$$F_1 = F_{11} + F_{12} \qquad \text{and} \qquad F_2 = F_{21} + F_{22}$$

Or,

$$F_1 = k\, u_1 - k\, u_2$$

And $$F_2 = -\, k\, u_1 + k\, u_2$$

- In matrix form

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

The rectangular matrix

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

is known as the stiffness matrix of a spring element

And is usually given the symbol [k], i.e.

$$[\mathrm{k}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

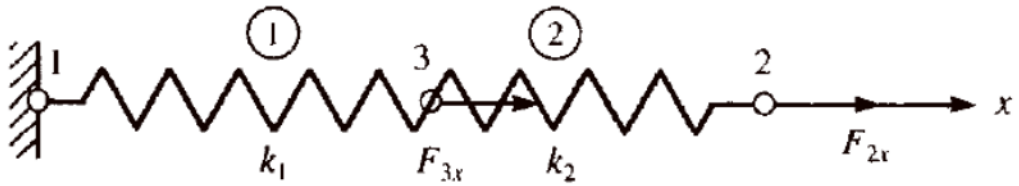
or $$[k] = \mathrm{k} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

[*k*] is called the *local stiffness matrix* for the element.

Properties of stiffness matrix:

1. It is symmetric (that is, $k_{ij} = k_{ji}$ for $i \neq j$). This is proven by the reciprocal theorem of Rayleigh and Betti.
2. It is square (the number of rows equals the number of columns in $[k]$) as it relates the same number of nodal forces to nodal displacements.
3. It is singular, that is, the determinant of $[k]$ is equal to zero, so $[k]$ cannot be inverted.

Example: Determine the nodal displacements of the system of two springs shown.



Solution:

$$[\mathrm{k}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$[k^{(1)}] = \begin{array}{c} \\ u_1 \\ u_3 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_3 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array}$$

$$[k^{(2)}] = \begin{array}{c} \\ u_3 \\ u_2 \end{array} \begin{array}{c} \begin{array}{cc} u_3 & u_2 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array}$$

$$k_1 \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \quad \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix}$$

$$k_2 \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \quad \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix}$$

$$\begin{Bmatrix} f_{1x}^{(1)} \\ 0 \\ f_{3x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

$$k_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} + k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

- For this simple example, it is easy to expand the element stiffness matrices and then superimpose them to arrive at the total stiffness matrix.
- However, for problems involving a large number of degrees of freedom, it will become tedious to expand each element stiffness matrix to the order of the total stiffness matrix.
- To avoid this expansion of each element stiffness matrix, we suggest a direct, or shortcut, form of the direct stiffness method to obtain the total stiffness matrix.
- For the spring assemblage example, the rows and columns of each element stiffness matrix are labeled according to the degrees of freedom associated with them as follows:

$$[k^{(1)}] = \begin{array}{c} \\ u_1 \\ u_3 \end{array} \begin{array}{c} \begin{array}{cc} u_1 & u_3 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \qquad [k^{(2)}] = \begin{array}{c} \\ u_3 \\ u_2 \end{array} \begin{array}{c} \begin{array}{cc} u_3 & u_2 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array}$$

- [K] is then constructed simply by directly adding terms associated with degrees of freedom in [$k^{(1)}$] and [$k^{(2)}$] into their corresponding identical degree-of-freedom locations in [K] as follows:
- The u_1 row, u_1 column term of [K] is contributed only by element 1, as only element 1 has degree of freedom u_1, that is, $k_{11}=k_1$.
- The u_3 row, u_3 column of [K] has contributions from both elements 1 and 2, as the u_3 degree of freedom is associated with both elements.
- Therefore, $k_{33}= k_1+ k_2$.
- Similar reasoning results in [K] as

$$[K] = \begin{matrix} \\ u_1 \\ u_2 \\ u_3 \end{matrix} \begin{matrix} \begin{matrix} u_1 & u_2 & u_3 \end{matrix} \\ \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \end{matrix}$$

- Here elements in [K] are located on the basis that degrees of freedom are ordered in increasing node numerical order for the total structure.

Boundary Conditions:

- We must specify boundary (or support) conditions for structure models such as the spring assemblage, or [K] will be singular; that is, the determinant of [K] will be zero, and its inverse will not exist.
- This means the structural system is unstable.
- Without our specifying adequate kinematic constraints or support conditions, the structure will be free to move as a rigid body and not resist any applied loads.
- In general, the number of boundary conditions necessary to make [K] nonsingular is equal to the number of possible rigid body modes.
- Boundary conditions relevant for spring assemblages are associated with nodal displacements.
- These conditions are of two types:

A. Homogeneous boundary conditions—the more common— occur at locations that are completely prevented from movement, and

B. Nonhomogeneous boundary conditions occur where finite nonzero values of displacement are specified, such as the settlement of a support.

- In the mathematical sense in regard to solving boundary value problems, we encounter two general classifications of boundary conditions when imposed on an ordinary or partial differential equation or derived upon taking the first variation of a functional.
- The first type—primary, essential, or Dirichlet—boundary condition [named after Johann Dirichlet (1805–1859)], specifies the values a solution, such as the displacement, must satisfy on the boundary of the domain.
- The second type—natural or Neumann—boundary condition [named after Carl Neumann (1832–1925)], specifies the values that the derivatives of a solution must satisfy on the boundary of the domain.

Homogeneous Boundary Conditions:

- We first consider the case of homogeneous boundary conditions. Hence all boundary conditions are such that the displacements are zero at certain nodes.
- In the example of spiring system we have $u_1 = 0$ because node 1 is fixed.
- Therefore, the global equation can be written as

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

- written in expanded form, becomes

$$\begin{aligned} k_1(0) + (0)u_2 - k_1u_3 &= F_{1x} \\ 0(0) + k_2u_2 - k_2u_3 &= F_{2x} \\ -k_1(0) - k_2u_2 + (k_1 + k_2)u_3 &= F_{3x} \end{aligned}$$

- where F_{1x} is the unknown reaction and F_{2x} and F_{3x} are known applied loads.

- Writing the second and third in matrix form, we have

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix}$$

- We have now effectively partitioned off the first column and row of $[K]$ and the first row of $\{d\}$ and $\{F\}$ to arrive the reduced system of equations.

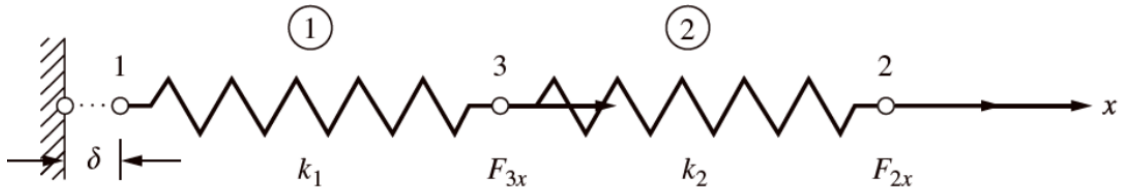
Rule:

- For homogeneous boundary conditions, the reduced system resulting from application of B.Cs. could have been obtained directly by deleting the rows and columns corresponding to the zero-displacement degrees of freedom.
- In the spring system, row 1 and column 1 are deleted because column 1 of $[K]$ is really multiplied by $u_1 = 0$.
- However, F_{1x} is not necessarily zero and can be determined once u_2 and u_3 are solved for.

Nonhomogeneous Boundary Conditions:

- In the case of nonhomogeneous boundary conditions one or more of the specified displacements are nonzero.
- For simplicity's sake, let $u_1 = \delta$,

where δ is a known displacement,



- we now have

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

- written in expanded form becomes

$$\begin{aligned} k_1\delta + 0u_2 - k_1u_3 &= F_{1x} \\ 0\delta + k_2u_2 - k_2u_3 &= F_{2x} \\ -k_1\delta - k_2u_2 + (k_1 + k_2)u_3 &= F_{3x} \end{aligned}$$

- where F_{1x} is now a reaction from the support that has moved an amount δ.
- Considering the second and third of equations because they have known right-side nodal forces F_{2x} and F_{3x} , we obtain

$$0\delta + k_2u_2 - k_2u_3 = F_{2x}$$
$$-k_1\delta - k_2u_2 + (k_1 + k_2)u_3 = F_{3x}$$

- Transforming the known δ terms to the right side yields

$$k_2u_2 - k_2u_3 = F_{2x}$$
$$-k_2u_2 + (k_1 + k_2)u_3 = +k_1\delta + F_{3x}$$

- Rewriting in matrix form, we have

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ k_1\delta + F_{3x} \end{Bmatrix}$$

- Therefore, when dealing with nonhomogeneous boundary conditions, we cannot initially delete row 1 and column 1 corresponding to the nonhomogeneous boundary condition, because we are multiplying each element by a nonzero number.
- Had we done so, the $k_1\delta$ term would have been neglected, resulting in an error in the solution for the displacements.

<u>Rule:</u>

- For nonhomogeneous boundary conditions, we must, in general, transform the terms associated with the known displacements to the right-side force matrix before solving for the unknown nodal displacements.
- This was illustrated by transforming the $k_1\delta$ term to the right side of the second equation.
- We could now solve for the displacements in a manner similar to that used for homogeneous boundary conditions.

Properties of Global Stiffness Matrix:

- Following are some properties of the global stiffness matrix that are also applicable to the generalization of the finite element method:

1) $[K]$ is square, as it relates the same number of forces and displacements.
2) $[K]$ is symmetric, as is each of the element stiffness matrices. In structural mechanics, the symmetry property in not surprising. It can be proved by using the reciprocal laws.
3) $[K]$ is singular (its determinant is equal to zero), and thus, no inverse exists until sufficient boundary conditions are imposed to remove the singularity and prevent rigid body motion.
4) The main diagonal terms of $[K]$ are always positive. Otherwise, a positive nodal force F_i could produce a negative displacement δ_i —a behavior contrary to the physical behavior of any actual structure.
5) $[K]$ is positive semidefinite (that is $\{x\}^T[K]\{x\} > 0$ for all non-zero vector $\{x\}$ with real numbers).

Example: For the spring assemblage with arbitrarily numbered nodes shown in figure below, obtain

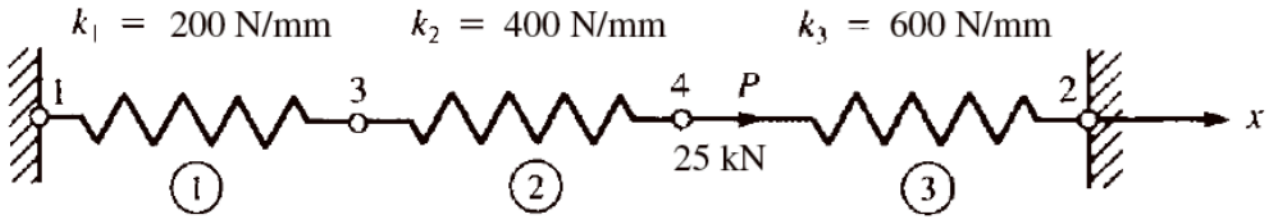
(a) the global stiffness matrix,

(b) the displacements of nodes 3 and 4,

(c) the reaction forces at nodes 1 and 2, and

(d) the forces in each spring. A force of 25 kN is applied at node 4 in the *x* direction.

The spring constants are given in the figure. Nodes 1 and 2 are fixed.



Example: For the spring assemblage shown in Figure 2–12, obtain

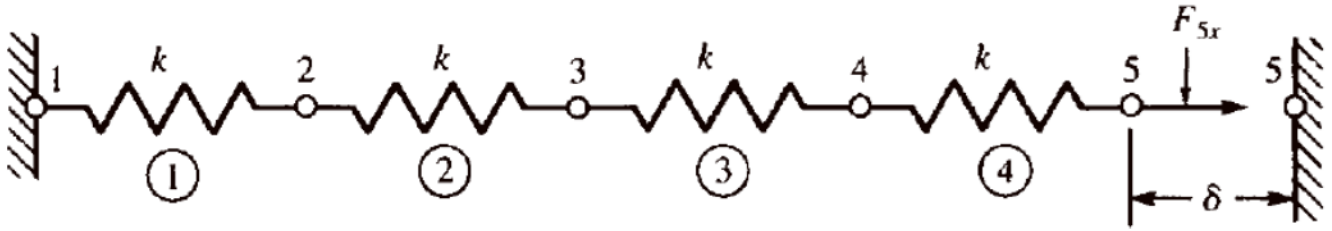
(a) the global stiffness matrix,

(b) the displacements of nodes 2–4,

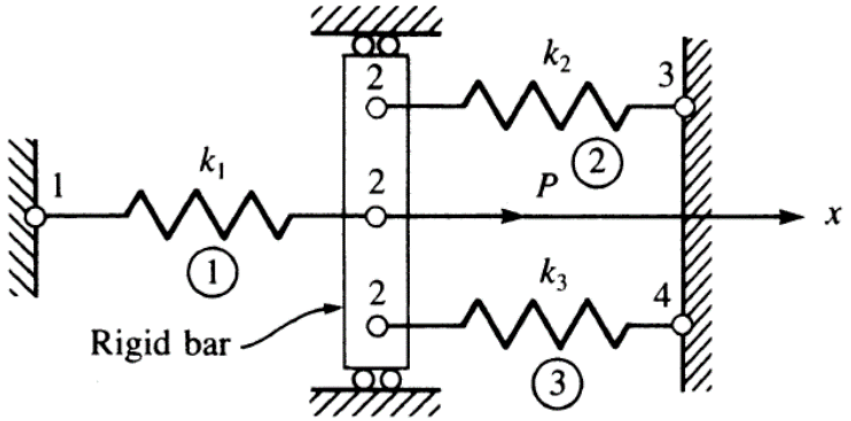
(c) the global nodal forces, and

(d) the local element forces.

Node 1 is fixed while node 5 is given a fixed, known displacement d 5 20.0 mm. The spring constants are all equal to k = 200 kN/m.

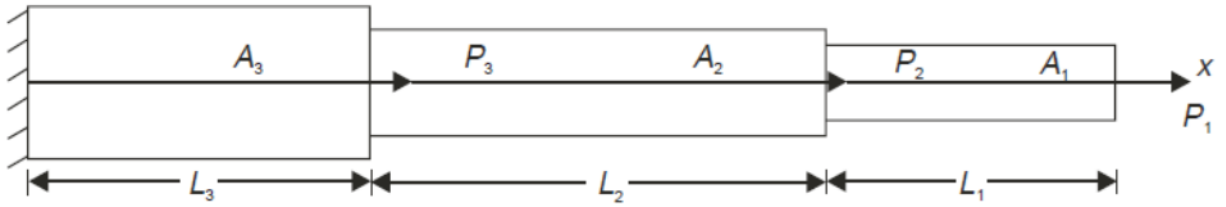


Example: Using the direct stiffness method, formulate the global stiffness matrix and equation, and specify the boundary and compatibility conditions.



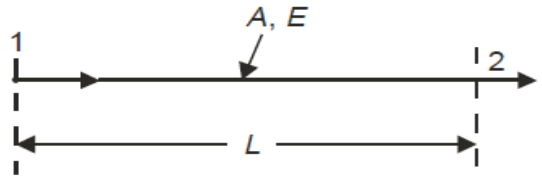
2. Bar element:

Consider the bar shown



- Consider one part of this bar, of uniform sectional area A and modulus of elasticity E and length L.
- For such bar, the extension/shortening is given by

$$\Delta = PL/EA$$



- From which

$$P = (EA/L)\,\Delta$$

- If end 1 is displaced by u_1, while end 2 is fixed, the force at end 1 will be

$$F_{11} = (EA/L)\,u_1$$

- The force at end 2 due to displacement at 1 is

$$F_{21} = -(EA/L)\,u_1$$

- If end 2 is displaced by u_2, while end 1 is fixed,
- the force at end 1 due to displacement at 2 will be

$$F_{12} = -(EA/L)\, u_2$$

- The force at end 2 due to displacement at 2 is

$$F_{22} = (EA/L)\, u_2$$

- The resultant forces at the two end developed from the two displacements will be:

$$F_1 = (EA/L)\, u_1 - (EA/L)\, u_2$$

And

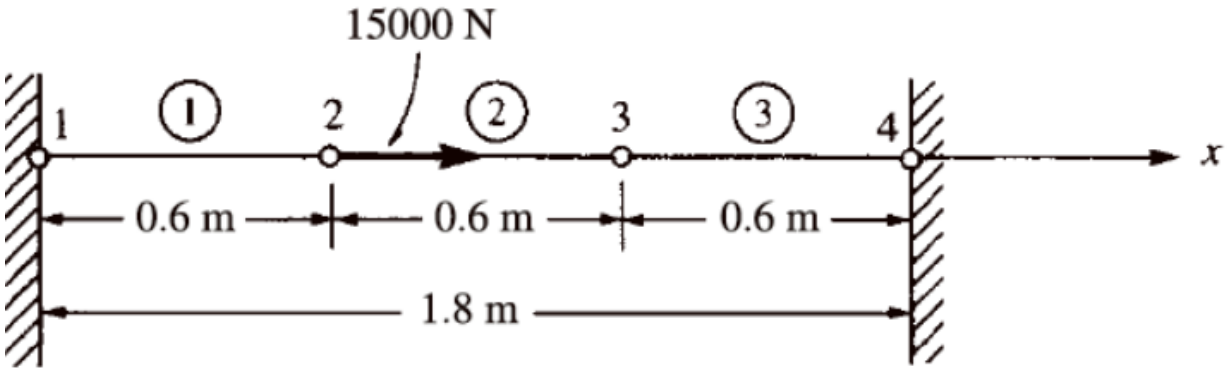
$$F_2 = -(EA/L)\, u_1 + (EA/L)\, u_2$$

In matrix form

$$\begin{matrix} F_1 \\ F_2 \end{matrix} = (EA/L) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Example:

For the three-bar assemblage shown in Figure 3–3 determine (a) the global stiffness matrix, (b) the displacements of nodes 2 and 3, and (c) the reactions at nodes 1 and 4. A force of 15,000 N is applied in the x direction at node 2. The length of each element is 0.6 m. Let $E = 2.0 \times 10^{11}$ Pa and $A = 6 \times 10^{-4}\text{m}^2$ for elements 1 and 2, and let $E = 1 \times 10^{11}$ Pa and $A = 12 \times 10^{-4}\text{m}^2$ for element 3. Nodes 1 and 4 are fixed.



■ **Figure 3–3** Three-bar assemblage

SOLUTION:

(a) Using Eq. (3.1.14), we find that the element stiffness matrices are

$$[k^{(1)}] = [k^{(2)}] = \frac{(6 \times 10^{-4})(2 \times 10^{11})}{0.6} \begin{matrix} 1 \\ 2 \end{matrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 & 2^{(1)} \\ 2 & 3^{(2)} \end{matrix} = 2 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{3.1.16}$$

$$[k^{(3)}] = \frac{(12 \times 10^{-4})(1 \times 10^{11})}{0.6} \begin{matrix} 3 \\ 4 \end{matrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 3 & 4 \end{matrix} = 2 \times 10^8 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{\text{N}}{\text{m}}$$

where, again, the numbers above the matrices in Eqs. (3.1.16) indicate the displacements associated with each matrix. Assembling the element stiffness matrices by the direct stiffness method, we obtain the global stiffness matrix as

$$[K] = 2 \times 10^8 \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{3.1.17}$$

(b) Equation (3.1.17) relates global nodal forces to global nodal displacements as follows:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 2 \times 10^8 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{3.1.18}$$

Invoking the boundary conditions, we have

$$u_1 = 0 \qquad u_4 = 0 \tag{3.1.19}$$

Using the boundary conditions, substituting known applied global forces into Eq. (3.1.18), and partitioning equations 1 and 4 of Eq. (3.1.18), we solve equations 2 and 3 of Eq. (3.1.18) to obtain

$$\begin{Bmatrix} 15000 \\ 0 \end{Bmatrix} = 2 \times 10^8 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \tag{3.1.20}$$

Solving Eq. (3.1.20) simultaneously for the displacements yields

$$u_2 = 5 \times 10^{-5} \text{ m} = 0.05 \text{ mm} \quad u_3 = 2.5 \times 10^{-5} \text{ m} = 0.025 \text{ mm} \tag{3.1.21}$$

(c) Back-substituting Eqs. (3.1.19) and (3.1.21) into Eq. (3.1.18), we obtain the global nodal forces, which include the reactions at nodes 1 and 4, as follows:

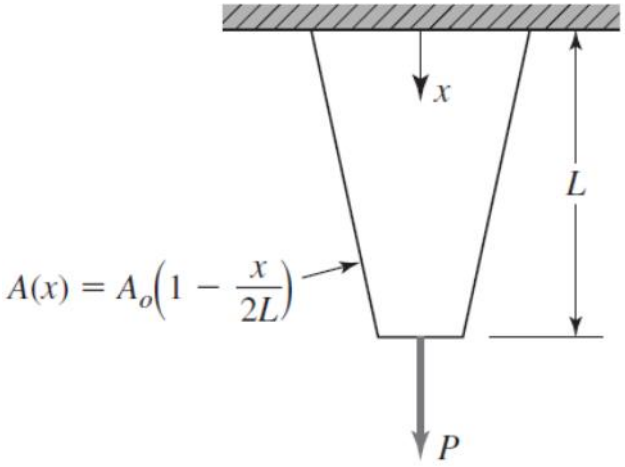
$$\begin{aligned} F_{1x} &= 2 \times 10^8(u_1 - u_2) = 2 \times 10^8(0 - 5 \times 10^{-5}) = -10{,}000 \text{ N} \\ F_{2x} &= 2 \times 10^8(-u_1 + 2u_2 - u_3) = 2 \times 10^8[0 + 2(5 \times 10^{-5}) - 2.5 \times 10^{-5}] = 15{,}000 \text{ N} \\ F_{3x} &= 2 \times 10^8(-u_2 + 2u_3 - u_4) = 2 \times 10^8[-5 \times 10^{-5} + 2(2.5 \times 10^{-5}) - 0] = 0 \\ F_{4x} &= 2 \times 10^8(-u_3 + u_4) = 2 \times 10^8(-2.5 \times 10^{-5} + 0) = -5000 \text{ N} \end{aligned} \tag{3.1.22}$$

Example: The figure below depicts a tapered elastic bar subjected to an applied tensile load P at one end and attached to a fixed support at the other end. The cross-sectional area varies linearly from A_0 at the fixed support at $x = 0$ to $A_0/2$ at $x = L$. Calculate the displacement of the end of the bar

(a) by modeling the bar as a single element having cross-sectional area equal to the area of the actual bar at its midpoint along the length,

(b) using two bar elements of equal length and similarly evaluating the area at the midpoint of each, and

(c) using integration to obtain the exact solution.



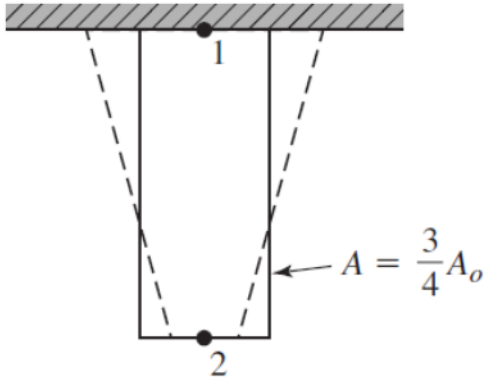
Solution:

(a) For a single element, the cross-sectional area is $3A_0/4$ and the element "spring constant" is

$$k = \frac{AE}{L} = \frac{3A_0E}{4L}$$

and the element equations are

$$\frac{3A_0E}{4L}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \end{Bmatrix}$$



The element and nodal displacements are as shown in Figure 2.7b. Applying the constraint condition $U_1 = 0$, we find

$$U_2 = \frac{4PL}{3A_0E} = 1.333\frac{PL}{A_0E}$$

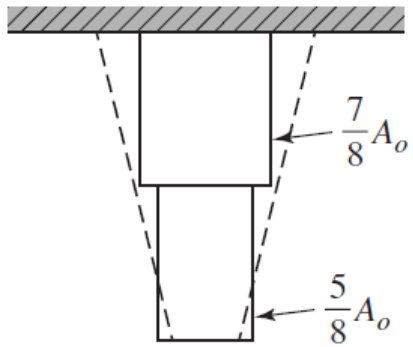
as the displacement at $x = L$.

(b) Two elements of equal length $L/2$ with associated nodal displacements are depicted in Figure 2.7c. For element 1, $A_1 = 7A_0/8$ so

$$k_1 = \frac{A_1 E}{L_1} = \frac{7A_0 E}{8(L/2)} = \frac{7A_0 E}{4L}$$

while for element 2, we have

$$A_1 = \frac{5A_0}{8} \quad \text{and} \quad k_2 = \frac{A_2 E}{L_2} = \frac{5A_0 E}{8(L/2)} = \frac{5A_0 E}{4L}$$



Since no load is applied at the center of the bar, the equilibrium equations for the system of two elements is

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ P \end{Bmatrix}$$

Applying the constraint condition $U_1 = 0$ results in

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Adding the two equations gives

$$U_2 = \frac{P}{k_1} = \frac{4PL}{7A_0 E}$$

and substituting this result into the first equation results in

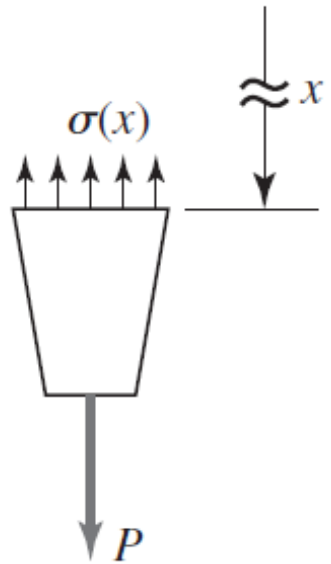
$$U_3 = \frac{k_1 + k_2}{k_2} = \frac{48PL}{35A_0 E} = 1.371\frac{PL}{A_0 E}$$

(c) To obtain the exact solution, we refer to Figure 2.7d, which is a free-body diagram of a section of the bar between an arbitrary position x and the end $x = L$. For equilibrium,

$$\sigma_x A = P \quad \text{and since} \quad A = A(x) = A_0\left(1 - \frac{x}{2L}\right)$$

the axial stress variation along the length of the bar is described by

$$\sigma_x = \frac{P}{A_0\left(1 - \frac{x}{2L}\right)}$$



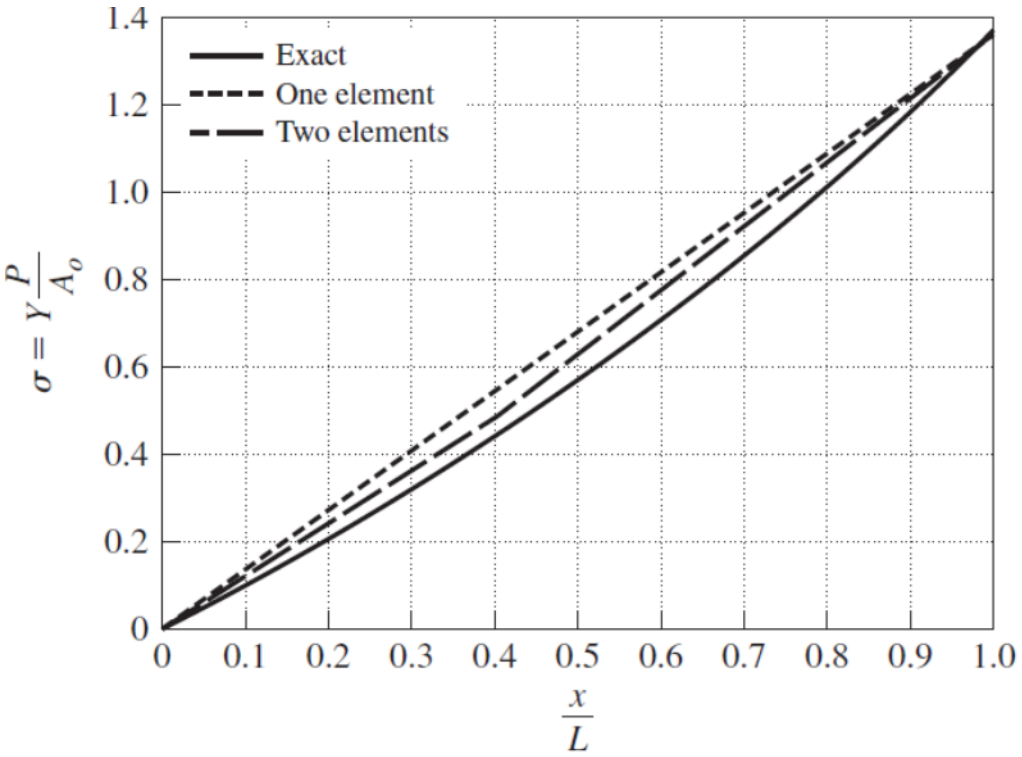
Therefore, the axial strain is

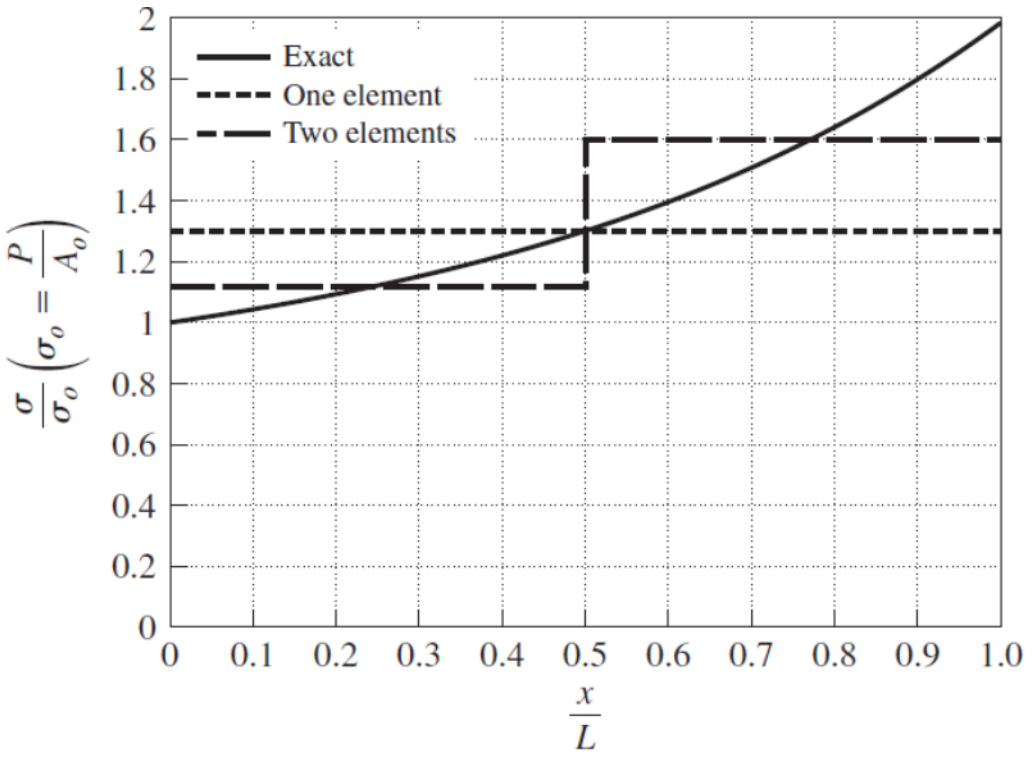
$$\varepsilon_x = \frac{\sigma_x}{E} = \frac{P}{EA_0\left(1 - \frac{x}{2L}\right)}$$

Since the bar is fixed at $x = 0$, the displacement at $x = L$ is given by

$$\delta = \int_0^L \varepsilon_x \, dx = \frac{P}{EA_0} \int_0^L \frac{dx}{\left(1 - \frac{x}{2L}\right)}$$

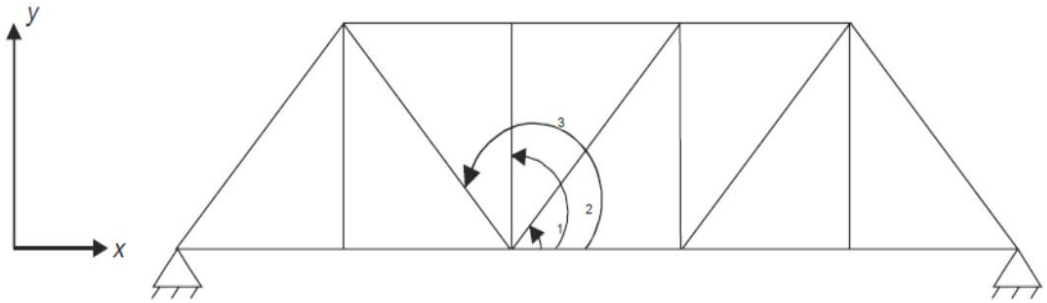
$$= \frac{2PL}{EA_0}[-\ln(2L - x)]\Big|_0^L = \frac{2PL}{EA_0}[\ln(2L) - \ln L] = \frac{2PL}{EA_0}\ln 2 = 1.386\frac{PL}{A_0E}$$


Exact
One element
Two elements
$\sigma = Y\frac{P}{A_o}$
$\frac{x}{L}$
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.1
0.3
0.5
0.7
0.9

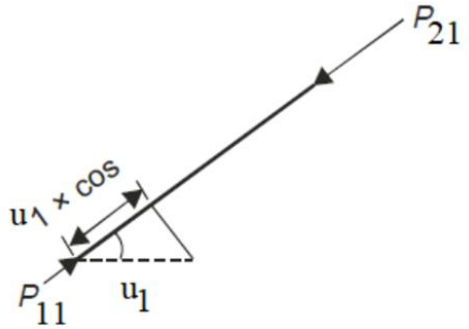

Exact
One element
Two elements
$\frac{\sigma}{\sigma_o}\left(\sigma_o = \frac{P}{A_o}\right)$
$\frac{x}{L}$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.1
0.3
0.5
0.7
0.9

3. Truss element:

- Members of the trusses are subjected to axial forces only,
- Their orientation in the plane may be at any angle to the coordinate directions selected.



Consider a typical member of the truss with Young's Modulus *E*, cross sectional area *A*, length *L* and at angle θ to x-axis



I. Displacements in the direction of x-axis

(a) If end 1 is subjected to a displacement u_1 in the direction of x-axis, the resulting displacement (elongation/shortening), along the member (element) 1-2 will be

$u_{1\theta} = u_1 \cos\theta$

The axial force (in local direction) at end 1 due to this displacement is

$P_{11} = (EA/L)\, u_1 \cos\theta$

This force can be resolved to two components in the direction of the global coordinates (x and y) to:

$(F_x)_{11} = P_{11} \cos\theta = (EA/L)\, u_1 (\cos\theta)^2$

and

$(F_y)_{11} = P_{11} \sin\theta = (EA/L)\, u_1 \sin\theta \cos\theta$

The axial force (in local direction) at end 2 due to u_1 is

$P_{21} = - (EA/L)\, u_1 \cos\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{21} = P_{21} \cos\theta = - (EA/L)\, u_1 (\cos\theta)^2$

and

$(F_y)_{21} = P_{21} \sin\theta = - (EA/L)\, u_1 \sin\theta \cos\theta$

(b) If end 2 is subjected to a displacement u_2 in the direction of x-axis, the resulting displacement (elongation/shortening), along the member (element) 1-2 will be

$u_{2\theta} = u_2 \cos\theta$

The axial force (in local direction) at end 1 due to this displacement is

$P_{12} = -(EA/L)\, u_2 \cos\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{12} = P_{12} \cos\theta = -(EA/L)\, u_2 (\cos\theta)^2$

and

$(F_y)_{12} = P_{12} \sin\theta = -(EA/L)\, u_2 \sin\theta \cos\theta$

The axial force (in local direction) at end 2 due to u_2 is

$P_{22} = (EA/L)\, u_2 \cos\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{22} = P_{22} \cos\theta = (EA/L)\, u_2 (\cos\theta)^2$

and

$(F_y)_{22} = P_{22} \sin\theta = (EA/L)\, u_2 \sin\theta \cos\theta$

II. Displacement in the direction of y-axis

(b) If end 1 is subjected to a displacement v_1 in the direction of y-axis, the resulting displacement (elongation/shortening), along the member (element) 1-2 will be

$$v_{1\theta} = v_1 \sin\theta$$

The axial force (in local direction) at end 1 due to this displacement is

$$P_{11} = (EA/L)\, v_1 \sin\theta$$

The two components in the direction of the global coordinates (x and y) are:

$$(F_x)_{11} = P_{11} \cos\theta = (EA/L)\, v_1 \sin\theta \cos\theta$$

and

$$(F_y)_{11} = P_{11} \sin\theta = (EA/L)\, v_1 (\sin\theta)^2$$

The axial force (in local direction) at end 2 due to v_1 is

$P_{21} = -(EA/L)\, v_1 \sin\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{21} = P_{21} \cos\theta = -(EA/L)\, v_1 \sin\theta \cos\theta$

and

$(F_y)_{21} = P_{21} \sin\theta = -(EA/L)\, v_1 (\sin\theta)^2$

(b). If end 2 is subjected to a displacement v_2 in the direction of y-axis, the resulting displacement (elongation/shortening), along the member (element) 1-2 will be

$v_{2\theta} = v_2 \sin\theta$

The axial force (in local direction) at end 1 due to this displacement is

$P_{12} = -(EA/L)\, v_2 \sin\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{12} = P_{12} \cos\theta = -(EA/L)\, v_2 \sin\theta \cos\theta$

and

$(F_y)_{12} = P_{12} \sin\theta = -(EA/L)\, v_2 (\sin\theta)^2$

The axial force (in local direction) at end 2 due to u_2 is

$P_{22} = (EA/L)\, v_2 \sin\theta$

Its components in the direction of the global coordinates (x and y) are:

$(F_x)_{22} = P_{22} \cos\theta = (EA/L)\, v_2 \sin\theta \cos\theta$

and

$(F_y)_{22} = P_{22} \sin\theta = (EA/L)\, v_2 (\sin\theta)^2$

- The resultant forces at 1 and 2 are the algebraic sum of all cases,

$F_{x1} = (EA/L)\, u_1 (\cos\theta)^2 + (EA/L)\, v_1 \sin\theta \cos\theta - (EA/L)\, u_2 (\cos\theta)^2 - (EA/L)\, v_2 \sin\theta \cos\theta$

And so on ….

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{3.3.9}$$

where $C = \cos\theta$ and $S = \sin\theta$.

Equation (3.3.9) relates the global displacement matrix $\{d\}$ to the local displacement $\{d'\}$ as

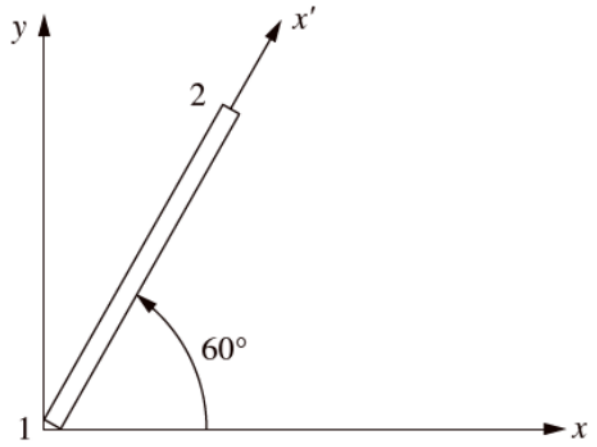
$$\{d'\} = [T]\{d\} \tag{3.3.10}$$

where

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \{d'\} = \begin{Bmatrix} u' \\ v' \end{Bmatrix}, \quad [T] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \tag{3.3.11}$$

The matrix $[T]$ is called the *transformation* (or *rotation*) *matrix*. For an additional description of this matrix, see Appendix A. It will be used in Section 3.4 to develop the global stiffness matrix for an arbitrarily oriented bar element and to transform global nodal displacements and forces to local ones.

Example: The global nodal displacements at node 2 have been determined to be u_2= 2.5 mm and v_2 = 5 mm for the bar element shown. Determine the local x displacement at node 2.

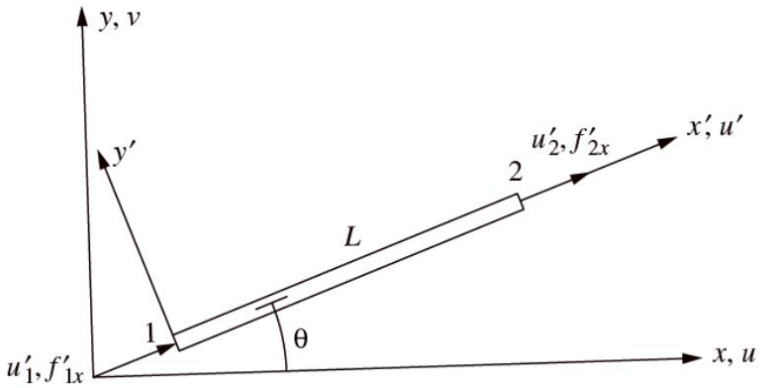


Solution:

$$u_2' = (\cos 60°)(2.5) + (\sin 60°)(5) = 5.58 \text{ mm}$$

Global Stiffness Matrix for Bar Arbitrarily Oriented in the Plane:

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \qquad (3.4.1)$$



or

$$\{f'\} = [k']\{d'\} \qquad (3.4.2)$$

We now want to relate the global element nodal forces $\{f\}$ to the global nodal displacements $\{d\}$ for a bar element arbitrarily oriented with respect to the global axes as shown in Figure 3–9. This relationship will yield the global stiffness matrix $[k]$ of the element. That is, we want to find a matrix $[k]$ such that

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = [k] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad (3.4.3)$$

or, in simplified matrix form, Eq. (3.4.3) becomes

$$\{f\} = [k]\{d\} \qquad (3.4.4)$$

$$u_1' = u_1 \cos\theta + v_1 \sin\theta$$
$$u_2' = u_2 \cos\theta + v_2 \sin\theta \qquad (3.4.5)$$

In matrix form, Eqs. (3.4.5) can be written as

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad (3.4.6)$$

or as

$$\{d'\} = [T^*]\{d\} \qquad (3.4.7)$$

where

$$[T^*] = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \qquad (3.4.8)$$

Similarly, because forces transform in the same manner as displacements, we replace local and global displacements in Eq. (3.4.6) with local and global forces and obtain

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \tag{3.4.9}$$

Similar to Eq. (3.4.7), we can write Eq. (3.4.9) as

$$\{f'\} = [T^*]\{f\} \tag{3.4.10}$$

Now, substituting Eq. (3.4.7) into Eq. (3.4.2), we obtain

$$\{f'\} = [k'][T^*]\{d\} \tag{3.4.11}$$

and using Eq. (3.4.10) in Eq. (3.4.11) yields

$$[T^*]\{f\} = [k'][T^*]\{d\} \tag{3.4.12}$$

However, to write the final expression relating global nodal forces to global nodal displacements for an element, we must invert $[T^*]$ in Eq. (3.4.12). This is not immediately possible because $[T^*]$ is not a square matrix. Therefore, we must expand $\{d'\}$, $\{f'\}$, and $[k']$ to the order that is consistent with the use of global coordinates even though f'_{1y} and v'_{2y} are zero. Using Eq. (3.3.9) for each nodal displacement, we thus obtain

$$\begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \tag{3.4.13}$$

or

$$\{d'\} = [T]\{d\} \tag{3.4.14}$$

where

$$[T] = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \tag{3.4.15}$$

Similarly, we can write

$$\{f'\} = [T]\{f\} \tag{3.4.16}$$

because forces are like displacements—both are vectors. Also, $[k']$ must be expanded to a 4×4 matrix. Therefore, Eq. (3.4.1) in expanded form becomes

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} \tag{3.4.17}$$

In Eq. (3.4.17), because f'_{1y} and f'_{2y} are zero, rows of zeros corresponding to the row numbers f'_{1y} and f'_{2y} appear in $[k']$. Now, using Eqs. (3.4.14) and (3.4.16) in Eq. (3.4.2), we obtain

$$[T]\{f\} = [k'][T]\{d\} \tag{3.4.18}$$

Equation (3.4.18) is Eq. (3.4.12) expanded. Premultiplying both sides of Eq. (3.4.18) by $[T]^{-1}$, we have

$$\{f\} = [T]^{-1}[k'][T]\{d\} \tag{3.4.19}$$

where $[T]^{-1}$ is the *inverse* of $[T]$. However, it can be shown (see Problem 3.28) that

$$[T]^{-1} = [T]^T \tag{3.4.20}$$

where $[T]^T$ is the *transpose* of $[T]$. The property of square matrices such as $[T]$ given by Eq. (3.4.20) defines $[T]$ to be an orthogonal matrix. For more about orthogonal matrices, see Appendix A. The transformation matrix $[T]$ between rectangular coordinate frames is orthogonal. This property of $[T]$ is used throughout this text. Substituting Eq. (3.4.20) into Eq. (3.4.19), we obtain

$$\{f\} = [T]^T [k'][T]\{d\} \tag{3.4.21}$$

Equating Eqs. (3.4.4) and (3.4.21), we obtain the global stiffness matrix for an element as

$$[k] = [T]^T [k'][T] \tag{3.4.22}$$

Substituting Eq. (3.4.15) for $[T]$ and the expanded form of $[k']$ given in Eq. (3.4.17) into Eq. (3.4.22), we obtain $[k]$ given in explicit form by

$$[k] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix} \tag{3.4.23}$$

Equation (3.4.23) is the explicit stiffness matrix for a bar arbitrarily oriented in the $x-y$ plane.

Now, because the trial displacement function Eq. (3.2.6) and Figure 3–5 was assumed piece-wise-continuous element by element, the stiffness matrix for each element can be summed by using the direct stiffness method to obtain

$$\sum_{e=1}^{N}\left[k^{(e)}\right] = [K] \tag{3.4.24}$$

where $[K]$ is the total stiffness matrix and N is the total number of elements. Similarly, each element global nodal force matrix can be summed such that

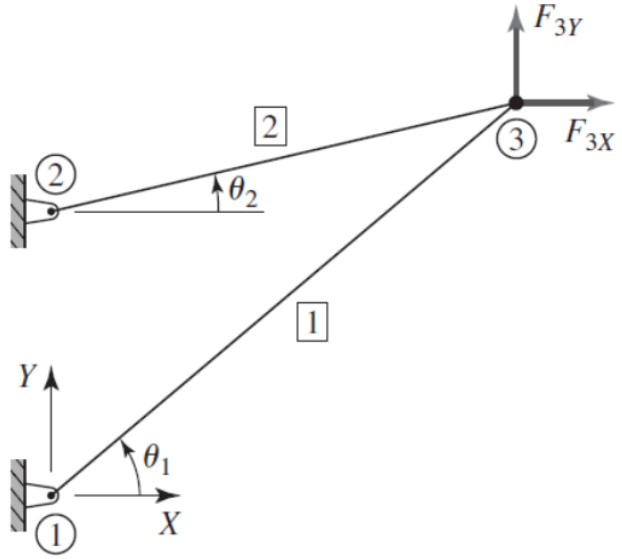
$$\sum_{e=1}^{N}\{f^{(e)}\} = \{F\} \tag{3.4.25}$$

$[K]$ now relates the global nodal forces $\{F\}$ to the global nodal displacements $\{d\}$ for the whole structure by

$$\{F\} = [K]\{d\} \tag{3.4.26}$$

- Example: For the truss shown, $\theta_1 = \pi/4$, $\theta_1 = 0$, and the element properties are such that $k_1 = A_1 E_1 / L_1$, $k_2 = A_2 E_2 / L_2$. Transform the element stiffness matrix of each element into the global reference frame and assemble the global stiffness matrix.

Solution:



$$[K^{(1)}] = \frac{k_1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

$$[K^{(2)}] = k_2\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K^{(1)}] = \begin{matrix} \\ 1 \\ 2 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} \\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{bmatrix} \end{matrix}$$

$$[K^{(2)}] = \begin{matrix} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix} \end{matrix} \begin{matrix} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

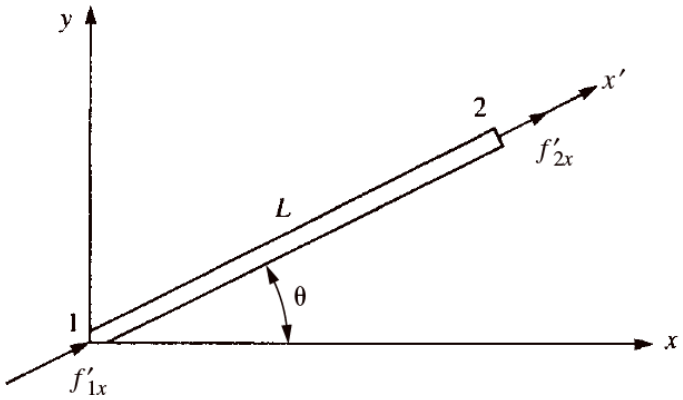
The complete global stiffness matrix is then

$$[K] = \begin{bmatrix} k_1/2 & k_1/2 & 0 & 0 & -k_1/2 & -k_1/2 \\ k_1/2 & k_1/2 & 0 & 0 & -k_1/2 & -k_1/2 \\ 0 & 0 & k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k_1/2 & -k_1/2 & -k_2 & 0 & k_1/2 + k_2 & k_1/2 \\ -k_1/2 & -k_1/2 & 0 & 0 & k_1/2 & k_1/2 \end{bmatrix}$$

Computation of Stress for a Bar in the *x* – *y* Plane:

We will now consider the determination of the stress in a bar element. For a bar, the local forces are related to the local displacements by Eq. (3.4.1) or Eq. (3.4.17). This equation is repeated here for convenience.

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \tag{3.5.1}$$



The usual definition of axial tensile stress is axial force divided by cross-sectional area. Therefore, axial stress is

$$\sigma = \frac{f'_{2x}}{A} \tag{3.5.2}$$

where f'_{2x} is used because it is the axial force that pulls on the bar as shown in Figure 3–11. By Eq. (3.5.1),

$$f'_{2x} = \frac{AE}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \tag{3.5.3}$$

Therefore, combining Eqs. (3.5.2) and (3.5.3) yields

$$\{\sigma\} = \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\{d'\} \tag{3.5.4}$$

Now, using Eq. (3.4.7), we obtain

$$\{\sigma\} = \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}[T^*]\{d\} \tag{3.5.5}$$

Equation (3.5.5) can be expressed in simpler form as

$$\{\sigma\} = [C']\{d\} \tag{3.5.6}$$

where, when we use Eq. (3.4.8) for $[T^*]$,

$$[C'] = \frac{E}{L}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}C & S & 0 & 0\\ 0 & 0 & C & S\end{bmatrix} \tag{3.5.7}$$

After multiplying the matrices in Eq. (3.5.7), we have

$$[C'] = \frac{E}{L}[-C \quad -S \quad C \quad S] \tag{3.5.8}$$

Derivation of Finite Element Equations by Variational Methods

Derivation of FEM Equations For One Dimensional Elements

Interpolation functions:

- One of the main ideas in the finite element method is to describe the variation of the field variable (e.g. the displacement) throughout the element by a trial approximate functions.
- This implies that since it is difficult to find a closed form or exact solution, we guess a solution shape or distribution of displacement by using an appropriate mathematical function.
- In choosing this function, we must follow the laws, principles, and constraints or boundary conditions inherent in the problem.
- The most common functions used are polynomials.

The Linear Element:

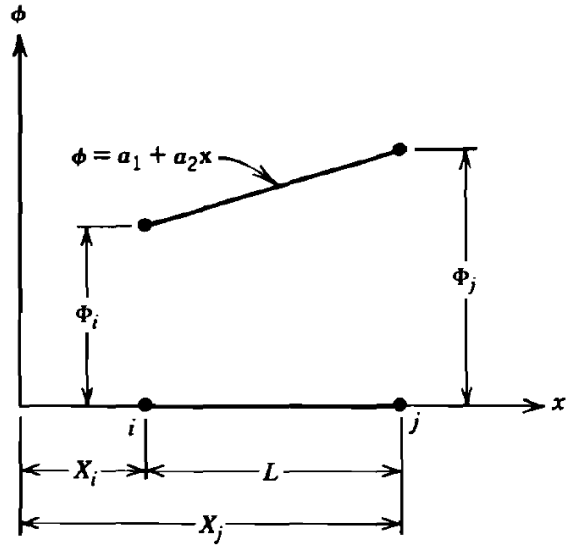
- The one – dimensional element is a line segment with a length L and two nodes, one at each end.
- The nodes are denoted by i and j and the nodal values (values of the field variable, e.g. displacement, heat, ..) are Φi and Φj.
- The origin of the coordinate is to the left of node i.
- The field variable ∅ varies linearly between the two nodes.
- The equation for ∅ is

$$\emptyset = a_1 + a_2 x \qquad (1)$$

The coefficients a_1 and a_2 can be determined by using the nodal conditions;

at x= X_i $\quad \emptyset = \Phi_i$

at x = X_j $\quad \emptyset = \Phi_j$



- from which

$$\Phi_i = a_1 + a_2 X_i$$

And $\Phi_j = a_1 + a_2 X_j$

Solving for a_1 and a_2 yields,

$$a_1 = \frac{\Phi_i X_j - \Phi_j X_i}{X_j - X_i} = \frac{\Phi_i X_j - \Phi_j X_i}{L} \quad \text{and} \quad a_2 = \frac{\Phi_j - \Phi_j}{X_j - X_i} = \frac{\Phi_j - \Phi_j}{L}$$

Substituting for a_1 and a_2 in the equation of ∅ results;

$$\emptyset = \frac{\Phi_i X_j - \Phi_j X_i}{L} + \frac{\Phi_j - \Phi_i}{L} x \qquad (2)$$

- Re-arranging the terms gives;

$$\emptyset = \left(\frac{X_j - x}{L}\right) \Phi_i + \left(\frac{x - X_i}{L}\right) \Phi_j \qquad (3)$$

- Equation (3) is the standard finite element form.
- The nodal values are multiplied by a linear functions of x.
- These functions are called shape functions or interpolation functions.

- The shape functions are usually denoted by N with a subscript to indicate the node with which a specific shape function is associated, i.e.

$$N_i = \frac{X_j - x}{L} \qquad \text{and} \qquad N_j = \frac{x - X_i}{L}$$

- Thus, equation (3) will be

$$\emptyset = \mathrm{N}_i\, \Phi_i + \mathrm{N}_j\, \Phi_j \qquad (4)$$

or in matrix form as

$$\emptyset = [N]\{\Phi\} \qquad (5)$$

where $[\mathrm{N}] = [\mathrm{N}_i \quad \mathrm{N}_j]$ is a row vector of shape functions, and

and $\{\Phi\} = \begin{Bmatrix} \Phi_i \\ \Phi_j \end{Bmatrix}$ usually written as $\{\Phi\}^{\mathrm{T}} = \{\, \Phi_i \quad \Phi_j \,\}$ is a column vector of element nodal values.

Properties of shape functions:

- At $x = X_i$ $N_i = 1.0$ and $N_j = 0$
- At $x = X_j$ $N_i = 0$ and $N_j = 1.0$
- At any x within the element $N_i + N_j = 1.0$
- The shape functions are polynomials of the same type as the original interpolation equation, i.e. eq. (1).
- $\frac{d(N_i)}{dx} + \frac{d(N_j)}{dx} = 0$

Example: A one dimensional linear element is used to approximate the temperature distribution on a steel rod. The solution indicates that the temperature at node i= 120 and at node j= 90^{o}C. Determine the temperature at a point 4 cm from the origin and the temperature gradient within the element. Nodes i and j are located at 1.5 cm and 6 cm from the origin, respectively.

Solution:

The field variable ø is the temperature.

$$\emptyset = N_i \Phi_i + N_j \Phi_j$$

$$N_i = \frac{X_j - x}{L} = \frac{6-x}{4.5} \quad \text{and} \quad N_j = \frac{x - X_i}{L} = \frac{x-1.5}{4.5}$$

$$\Phi_i = 120 \quad \text{and} \quad \Phi_j = 90$$

Thus, $\emptyset = \frac{6-x}{4.5} \times 120 + \frac{x-1.5}{4.5} \times 90$

At x=4.0 cm, $\varphi = \frac{6-4}{4.5} \times (120) + \frac{4-1.5}{4.5} \times 90 = 103.333^{o}C$

The temperature gradient is the first derivative of ø

$$d\emptyset/dx = \frac{dN_i}{dx} \Phi_i + \frac{dN_j}{dx} \Phi_j = (-1/L)\ \Phi_i + (1/L)\ \Phi_j = (\Phi_j - \Phi_i)/L = (90-120)/4.5 = -6.667^{o}C/cm$$

A continuous piecewise smooth equation:

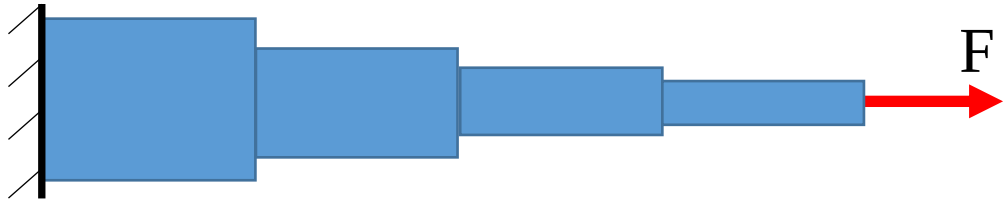
- This equation can be constructed by connecting several linear equations of the elements.
- Each equation can be written as

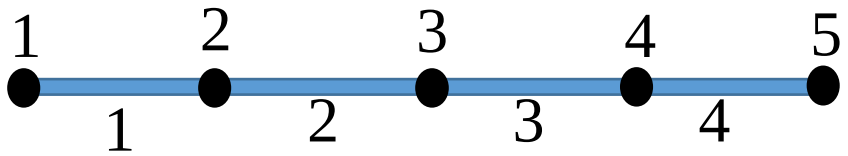
$$\emptyset^{(e)} = N_i^{(e)}\ \Phi_i + N_j^{(e)} \Phi_j \qquad (6)$$

- The superscript (e) indicates that the term is for an element (element term).
- Then, substitute the values of i and j for each element from the grid.
- Node i is the left hand of each element.

For example:

- The bar shown is discretized to four elements.
- The information of the grid is listed in the table.
- The equations for the element are:





$$\emptyset^{(1)} = N_1^{(1)}\,\Phi_1 + N_2^{(1)}\Phi_2$$

$$\emptyset^{(2)} = N_2^{(2)}\,\Phi_2 + N_3^{(2)}\Phi_3$$

$$\emptyset^{(3)} = N_3^{(3)}\,\Phi_3 + N_4^{(3)}\Phi_4$$

$$\emptyset^{(4)} = N_4^{(4)}\,\Phi_4 + N_5^{(4)}\Phi_5$$

e	i	j
1	1	2
2	2	3
3	3	4
4	4	5

- Note that $N_2^{(1)}$ and $N_2^{(2)}$ are different equations, even though both involve node two, their equations are

$$N_2^{(1)} = \frac{x - X_1}{X_2 - X_1} \quad \text{and} \qquad N_2^{(2)} = \frac{X_3 - x}{X_3 - X_2}$$

- Each one of the equations of $\emptyset$ is applicable for one element only.
- For example, the first equation should be considered as

$$\emptyset^{(1)} = N_1^{(1)}\,\Phi_1 + N_2^{(1)}\Phi_2 \qquad X_1 \le x \le X_2$$

- The superscript (e) usually does not have to be placed on each coefficient.
- When a brackets or parentheses have a superscript (e), i.e. $(G\,\emptyset + Q)^{(e)}$, then every term within the parentheses should be interpreted as an element term.
- A quantity on the left hand side of an equal sign with a superscript (e) implies that the quantities on the right hand side are element quantities, i.e.

$$\emptyset^{(e)} = N_i\,\Phi_i + N_j\,\Phi_j$$

implies that N_i and N_j are really $N_i^{(e)}$ and $N_j^{(e)}$, respectively, and Φ_i and Φ_j are the element nodal values.

Shape Functions in Local Coordinates System:

- The shape functions in global coordinate system are:

$$N_i = \frac{X_j - x}{L} \qquad \text{and} \qquad N_j = \frac{x - X_i}{L}$$

- In local coordinates system, $X_i = 0$, and $X_j = L$
- This yields, $N_i = \frac{L-x}{L} = 1 - \frac{x}{L}$ and $N_j = \frac{x}{L}$
- Using the non-dimensional local coordinate, s = x/L then

$$N_i = 1\text{- s} \qquad \text{and} \qquad N_j = \text{s}$$

- In natural coordinate, ξ in which

ξ = -1 at node i and ξ = +1 at node j

$N_i = \frac{1}{2}(1\text{- }\xi)$ and $N_j = \frac{1}{2}(1\text{+ }\xi)$

Finite Element Formulation of Axial Force Members:

- Finite element method is applicable to the analysis of both discrete and continuous structures.
- Discrete structures are those with individual members such as trusses, beams, and rigid frames.
- Continuous structures are plate and shell type structures as well as structural components that must be analyzed using the theory of elasticity.
- The analysis of both discrete and continuous structures can be approached from several points of view, such as the weighted residual and variational methods.

One dimensional model:

- The finite element grid for a system of axial force members consists of a straight line segment with nodes wherever there is:

a) A change in the material properties,

b) A change in cross section, or

c) There is an external force.

- Nodes are added at points of application of external forces to simplify the calculation of the work term in the potential energy equation.
- The work will be a force times the displacement.

Finite element grid:

- A major difference between the grids for axial force members and other one dimensional problems (such as heat transfer) is the concept of grid refinement.
- A finite element solution for the displacement in a discrete structures yields the correct values.
- No improvement is obtained by subdividing each member into several smaller elements.
- Each member is represented by a single element except when there are applied load or change in properties between end points.
- The quantities calculated in a FEA of discrete or continuous structure are displacements.
- The node displacements and externally applied forces are often indicated using arrows.

Finite Element Formulation:

- Few elements can be formulated using the direct method, as applied to bars and beams.
- In general, formulation of elements for structural mechanics relies on long established tools of stress analysis, including stress – strain relations, strain-displacement relations, and energy equations.

Stress-Strain Relations. Let $\{\boldsymbol{\sigma}\}$ be the array of stresses and $\{\boldsymbol{\varepsilon}\}$ the array of strains. Subscripts zero indicate initial values. Constitutive matrix $[\mathbf{E}]$ contains elastic constants. For linearly elastic conditions, stress-strain relations can be stated in the matrix forms

$$\{\boldsymbol{\sigma}\} = [\mathbf{E}]\{\boldsymbol{\varepsilon}\} + \{\boldsymbol{\sigma}_0\} \quad \text{or} \quad \{\boldsymbol{\sigma}\} = [\mathbf{E}](\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}_0\})$$
$$\text{where} \qquad \{\boldsymbol{\sigma}_0\} = -[\mathbf{E}]\{\boldsymbol{\varepsilon}_0\} \qquad (3.1\text{-}1)$$

This relation is valid in one, two, or three dimensions. For a uniaxial stress state, with no initial stress, it is simply $\sigma = E\varepsilon$, where E is the elastic modulus. In two dimensions, with x and y as the in-plane coordinates, Eq. 3.1-1 is

Stresses = Constitutive matrix × Strains + Initial stresses

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} + \begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \tau_{xy0} \end{Bmatrix} \tag{3.1-2}$$

Constitutive matrix [**E**] is symmetric; $E_{ij} = E_{ji}$. [**E**] can represent isotropic or anisotropic material properties. For isotropy and plane stress conditions ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$), [**E**] and its inverse are

$$[\mathbf{E}] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \qquad [\mathbf{E}]^{-1} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \tag{3.1-3}$$

where ν is Poisson's ratio and $G = 0.5E/(1+\nu)$ is the shear modulus.

Inverting eq. (3.1-1) yields,

General form:

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{E}]^{-1}\{\boldsymbol{\sigma}\} + \{\boldsymbol{\varepsilon}_0\}$$

For isotropy and plane stress conditions:

$$\begin{aligned} \varepsilon_x &= \sigma_x/E - \nu\sigma_y/E + \varepsilon_{x0} \\ \varepsilon_y &= -\nu\sigma_x/E + \sigma_y/E + \varepsilon_{y0} \\ \gamma_{xy} &= \tau_{xy}/G + \gamma_{xy0} \end{aligned} \qquad (3.1\text{-}4)$$

Initial strains $\{\boldsymbol{\varepsilon}_0\}$ may have various causes, including temperature change and swelling due to moisture or radiation. If convenient, in order to account for initial effects from the simultaneous action of two or more sources, $\{\boldsymbol{\varepsilon}_0\}$ and $\{\boldsymbol{\sigma}_0\}$ can both appear in the stress-strain relation. If the material is isotropic and initial strains are produced by temperature change T, then $\varepsilon_{x0} = \varepsilon_{y0} = \alpha T$ and $\gamma_{xy0} = 0$, where α is the coefficient of thermal expansion, here assumed to be independent of temperature. Temperature T is measured relative to a reference temperature, perhaps room temperature, at which the body may be regarded as free of stress.

In three dimensions, [**E**] is a symmetric 6 by 6 array that relates stresses $\{\boldsymbol{\sigma}\} = \lfloor \sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \rfloor^T$ and strains $\{\boldsymbol{\varepsilon}\} = \lfloor \varepsilon_x \quad \varepsilon_y \quad \varepsilon_z \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \rfloor^T$. For the case of isotropy and initial strains caused by temperature change T, nonzero entries in [**E**] and $\{\boldsymbol{\varepsilon}_0\}$ are

$$E_{11} = E_{22} = E_{33} = (1-\nu)c \qquad \varepsilon_{x0} = \alpha T$$

$$E_{44} = E_{55} = E_{66} = G \qquad \varepsilon_{y0} = \alpha T \qquad \text{(3.1-5a)}$$

$$E_{12} = E_{21} = E_{13} = E_{31} = E_{23} = E_{32} = \nu c \qquad \varepsilon_{z0} = \alpha T$$

$$\text{where} \quad c = \frac{E}{(1+\nu)(1-2\nu)} \quad \text{and} \quad G = \frac{E}{2(1+\nu)} \qquad \text{(3.1-5b)}$$

Strain-Displacement Relations. A displacement field describes how a body deforms as well as how it displaces. Strain-displacement relations extract the strain field contained in a displacement field and play a prominent role in formulating commonly used elements.

To obtain formulas, we use engineering definitions of strain. *Normal strain* is change in length divided by original length; *shear strain* is the amount of change in a right angle. Deformations shown in **Figure below** provide formulas shown for strains ε_x, ε_y, and γ_{xy} in the xy plane.

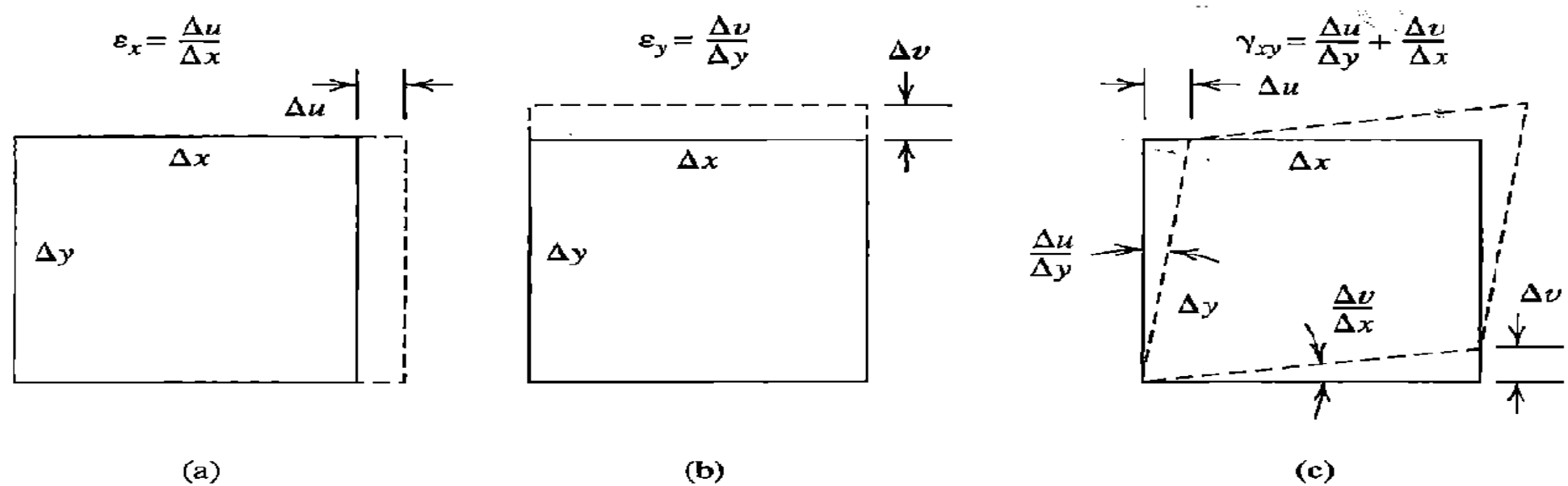


Figure 3.1-1. An infinitesimal rectangle, subjected to (a) x-direction normal strain, (b) y-direction normal strain, and (c) shear strain.

In general, x-direction displacement u and y-direction displacement v are functions of the coordinates; $u = u(x,y)$ and $v = v(x,y)$. Therefore, we must use partial derivatives. Doing so, and letting Δx and Δy approach zero, we obtain the *two-dimensional strain-displacement relations*

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.1-6}$$

Subsequently it will be convenient to use a comma to denote partial differentiation with respect to the subscript that follows. In this notation, Eqs. 3.1-6 are

$$\varepsilon_x = u_{,x} \qquad \varepsilon_y = v_{,y} \qquad \gamma_{xy} = u_{,y} + v_{,x} \tag{3.1-7}$$

In *three* dimensions, displacements in coordinate directions x, y, and z are $u = u(x,y,z)$, $v = v(x,y,z)$, and $w = w(x,y,z)$, and Eqs. 3.1-7 are supplemented by the relations

$$\varepsilon_z = w_{,z} \qquad \gamma_{yz} = v_{,z} + w_{,y} \qquad \gamma_{zx} = w_{,x} + u_{,z} \tag{3.1-8}$$

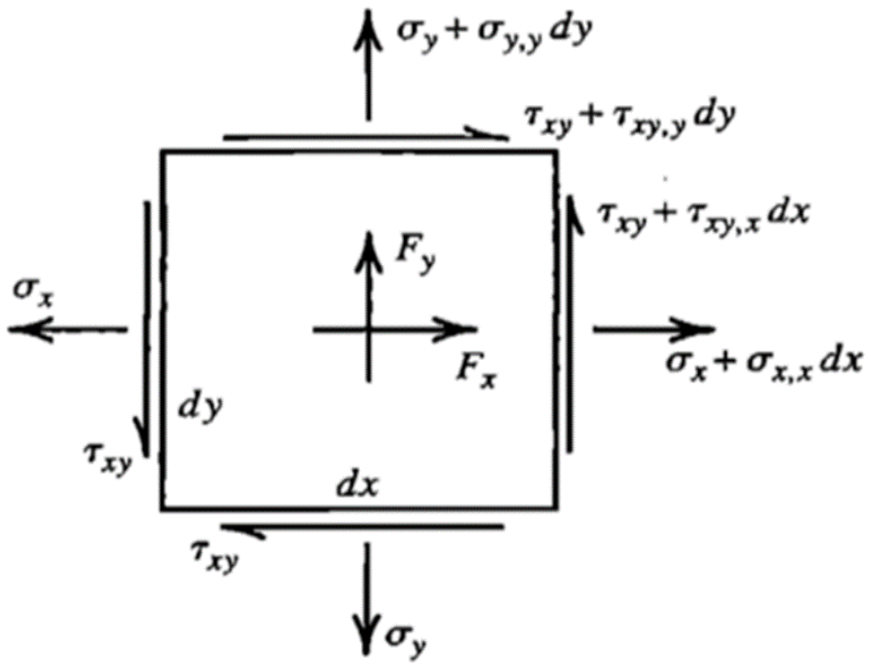
In matrix operator format, for 2D and 3D cases respectively, the strain-displacement relations are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \tag{3.1-9}$$

Symbolically, for both of Eqs. 3.1-9 and for strain-displacement relations in general, we write

$$\{\boldsymbol{\varepsilon}\} = [\partial]\{\mathbf{u}\} \tag{3.1-10}$$

Equilibrium Equations. Figure **below** shows stresses that act on a differential element in a two-dimensional problem. In rectangular Cartesian coordinates, we now develop equations stating that the differential element is in equilibrium under forces applied to it. Forces come from stresses on the sides and from body forces.



Body forces, F_x and F_y in x and y directions respectively, are defined as forces per unit volume, positive when acting in positive coordinate directions. Body forces can be produced by gravity, acceleration, a magnetic field, and so on. On each differential element of volume ($dV = t\,dx\,dy$, where t = thickness), F_x and F_y produce differential forces $F_x\,dV$ and $F_y\,dV$. In general, body forces and stresses are functions of the coordinates. Thus, for example, $\sigma_{x,x}$ is the *rate* of change and $\sigma_{x,x}\,dx$ is the *amount* of change in σ_x over distance dx. For uniform thickness t, static equilibrium of forces in the x direction requires that

$$-\sigma_x t\,dy - \tau_{xy} t\,dx + (\sigma_x + \sigma_{x,x}\,dx)t\,dy + (\tau_{xy} + \tau_{xy,y}\,dy)t\,dx + F_x t\,dx\,dy = 0 \qquad (3.1\text{-}11)$$

where $\sigma_{x,x} = \dfrac{\partial \sigma_x}{\partial x}$, and $\tau_{xy,x} = \dfrac{\partial \tau_{xy}}{\partial x}$

There is a corresponding y-direction equation of equilibrium. After simplification, the *differential equations of equilibrium* for a plane (2D) problem are as follows. For reference, analogous equations for a solid (3D) problem are also stated here.

	2D:	3D:
x direction:	$\sigma_{x,x} + \tau_{xy,y} + F_x = 0$	$\sigma_{x,x} + \tau_{xy,y} + \tau_{zx,z} + F_x = 0$
y direction:	$\tau_{xy,x} + \sigma_{y,y} + F_y = 0$	$\tau_{xy,x} + \sigma_{y,y} + \tau_{yz,z} + F_y = 0$
z direction:	(not used)	$\tau_{zx,x} + \tau_{yz,y} + \sigma_{z,z} + F_z = 0$

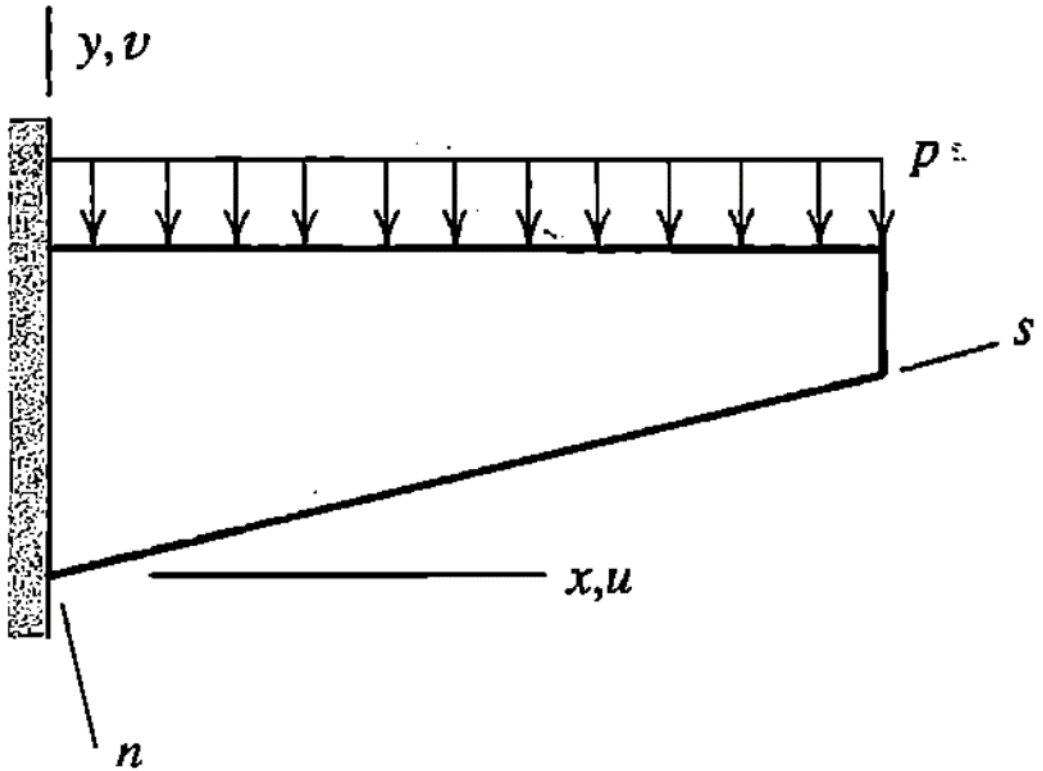
(3.1-12)

Although derived for static conditions, Eqs. 3.1-12 can also be used if acceleration is present, provided that F_x, F_y, and F_z include d'Alembert or "effective" body forces per unit volume. For example, if there is x-direction acceleration a_x, then F_x must include the inertial force term $-\rho a_x$, where ρ is the mass density. Whether in two dimensions or three, equilibrium equations can be symbolized as

$$[\partial]^T\{\boldsymbol{\sigma}\} + \{\mathbf{F}\} = \{\mathbf{0}\} \tag{3.1-13}$$

where $[\partial]$ is given in Eqs. 3.1-9 and, in 2D and 3D rectangular Cartesian coordinates respectively, $\{\mathbf{F}\}$ is $\lfloor F_x \quad F_y \rfloor^T$ or $\lfloor F_x \quad F_y \quad F_z \rfloor^T$.

Boundary Conditions. Boundary conditions include prescriptions of displacements or stresses on sides or surfaces of a body. For example, in the plane problem of **figure below** the rigid support implies that $u = v = 0$ along the left side. Stress boundary conditions prevail along the remaining sides: $\tau_{xy} = 0$ and $\sigma_y = -p$ along the top side, $\sigma_x = 0$ and $\tau_{xy} = 0$ along the right side, and $\sigma_n = 0$ and $\tau_{ns} = 0$ along the bottom side.



In general, distributed load can act tangent to a boundary as well as normal to it. On any boundary, including one not perpendicular to a coordinate axis, normal and tangential loads can be expressed as *surface tractions,* which are forces per unit of surface area, directed parallel to the coordinate axes. In rectangular Cartesian coordinates *xyz*, surface tractions $\{\Phi\}$ are

$$\{\Phi\} = \begin{Bmatrix} \Phi_x \\ \Phi_y \\ \Phi_z \end{Bmatrix} \qquad \text{where} \qquad \begin{aligned} \Phi_x &= l\sigma_x + m\tau_{xy} + n\tau_{zx} \\ \Phi_y &= l\tau_{xy} + m\sigma_y + n\tau_{yz} \\ \Phi_z &= l\tau_{zx} + m\tau_{yz} + n\sigma_z \end{aligned} \tag{3.1-14}$$

in which l, m, and n are direction cosines of a vector normal to the surface. When Eqs. 3.1-14 are satisfied, each differential element of the surface is in equilibrium under the action of surface tractions and internal stresses (evaluated at the surface). Such is also the case on a portion of the boundary where displacements rather than tractions are prescribed, but tractions applied by a support are not known *a priori* and are usually not calculated in the course of a solution.

Exact and Approximate Solutions. An exact solution must satisfy compatibility, equilibrium, and boundary conditions. For example, if we begin with a compatible displacement field, we can obtain strains from Eq. 3.1-10 and then stresses from Eq. 3.1-1. If these stresses satisfy Eq. 3.1-13 at every point throughout the volume of a body, and all boundary conditions are satisfied, then we have obtained the exact solution of the mathematical model (which is subject to basic assumptions such as linearity of the stress-strain relation and smallness of displacements). This is easy to say but difficult to do. Exact solutions are known only for simple combinations of geometry, loading, and support conditions.

Finite element based on displacement fields do not satisfy equilibrium conditions at every material point. Instead, displacement-based elements satisfy eqs. 3.1-13 and 3.1-14 in an integral or average sense.

Other Problems. The foregoing formulas of structural mechanics have counterparts in other areas. Details appear as needed in subsequent chapters. Here we briefly compare basic equations of structural mechanics and heat conduction. First we note that heat conduction is a *scalar* problem because the field quantity, temperature T, has no direction associated with it. In contrast, the displacement field of structural mechanics is a vector field having components in coordinate directions. The following list is for static (steady-state) conditions:

Quantity	Structural mechanics	Heat conduction
Independent variables	Coordinates x, y, z	Coordinates x, y, z
Dependent variable(s)	Displacements u, v, w	Temperature T
Field gradient	Strains $\varepsilon_x, \varepsilon_y, \gamma_{xy}$, etc.	$\{\nabla T\} = \lfloor T_{,x} \quad T_{,y} \quad T_{,z} \rfloor^T$
Constitutive matrix	Elastic constants $[\mathbf{E}]$	Thermal conductivities $[\boldsymbol{\kappa}]$
Induced field	Stresses $\{\boldsymbol{\sigma}\} = [\mathbf{E}]\{\boldsymbol{\varepsilon}\}$	Heat fluxes $\{\mathbf{f}\} = -[\boldsymbol{\kappa}]\{\nabla T\}$
Surface load	Tractions $\{\boldsymbol{\Phi}\}$ on boundary	Normal flux f_n at a boundary
Internal load	Body forces F_x, F_y, F_z	Internal heat generation Q
Equilibrium equation	$[\partial]^T\{\boldsymbol{\sigma}\} + \{\mathbf{F}\} = \{\mathbf{0}\}$	$f_{x,x} + f_{y,y} + f_{z,z} - Q = 0$

Interpolation and Shape Functions:

- The interpolation is to devise a continuous function that satisfies prescribed conditions at finite number of points.
- In FEA, the points are nodes of an element, and the prescribed conditions are nodal values of a field quantity (and perhaps its derivatives as well).
- Nodal values are rarely exact, and even when they are, interpolation generally provides approximate values at other locations.
- In FEA, the interpolating function is almost always a polynomial, which automatically provides a single-valued and continuous field.
- In terms of generalized DOF a_i, interpolating polynomial with dependent variable $\emptyset$ and independent variable x can be written in the form:

$$\phi = \sum_{i=0}^{n} a_i x^i \qquad \text{or} \qquad \phi = \lfloor \mathbf{X} \rfloor \{\mathbf{a}\} \qquad (3.2\text{-}1a)$$

in which

$$\lfloor \mathbf{X} \rfloor = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \end{bmatrix} \quad \text{and} \quad \{\mathbf{a}\} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \end{bmatrix}^T \qquad (3.2\text{-}1b)$$

where $n = 1$ for linear interpolation, $n = 2$ for quadratic interpolation, and so on. The a_i can be expressed in terms of nodal values of ϕ, which appear at known values of x. The relation between nodal values $\{\boldsymbol{\phi}_e\}$ and the a_i is symbolized as

$$\{\boldsymbol{\phi}_e\} = [\mathbf{A}]\{\mathbf{a}\} \qquad (3.2\text{-}2)$$

where each row of $[\mathbf{A}]$ is $\lfloor \mathbf{X} \rfloor$ evaluated at the appropriate nodal location (examples follow). From Eqs. 3.2-1 and 3.2-2 we obtain $\{a\}=[A]^{-1}\{\emptyset_e\}$ $\{\emptyset\}= [X]\,[A]^{-1}\{\emptyset_e\}$

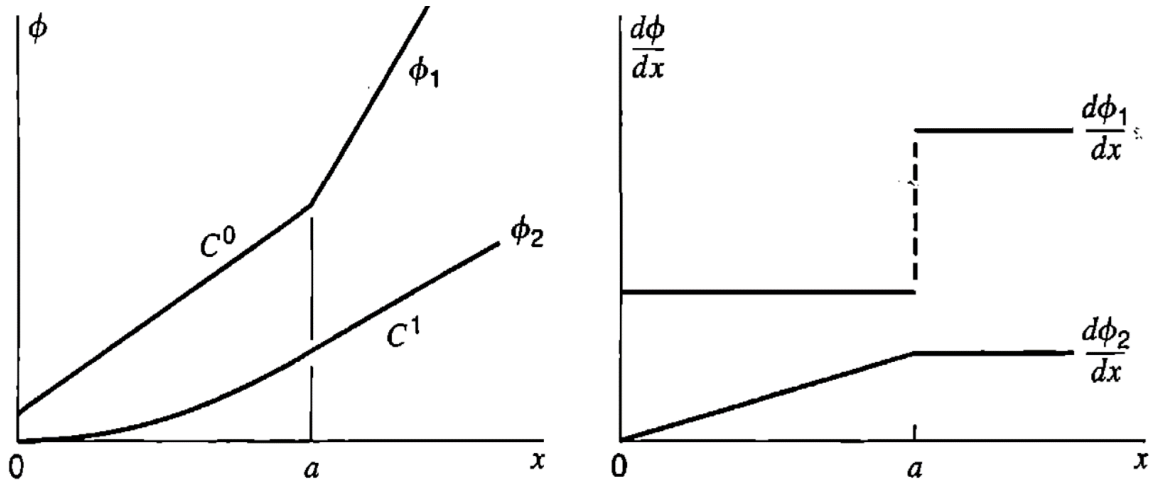
$$\phi = \lfloor \mathbf{N} \rfloor \{\boldsymbol{\phi}_e\} \quad \text{where} \quad \lfloor \mathbf{N} \rfloor = \lfloor \mathbf{X} \rfloor [\mathbf{A}]^{-1} = \begin{bmatrix} N_1 & N_2 & \cdots \end{bmatrix} \qquad (3.2\text{-}3)$$

An individual N_i in matrix $\lfloor \mathbf{N} \rfloor$ is called a *shape function.* The name *basis function* is sometimes used instead. Each N_i states the interpolated $\phi = \phi(x)$ when the corresponding ϕ_i is unity and all other ϕ_i are zero. In FEA, assembly of elements causes element nodal values $\{\boldsymbol{\phi}_e\}$ to appear in $\{\mathbf{D}\}$, the global vector of d.o.f. Thus, in FEA, $\{\boldsymbol{\phi}_e\}$ for each element is determined by solving global equations $[\mathbf{K}]\{\mathbf{D}\} = \{\mathbf{R}\}$.

Degree of Continuity. Field quantity ϕ is interpolated in piecewise fashion over an FE mesh. That is, each "interpolation piece" is defined only within its element. So while ϕ can be guaranteed to vary smoothly *within* each element, the transition *between* elements may not be smooth. The symbol C^m is used to describe the continuity of a piecewise field. A field is C^m continuous if its derivatives up to and including degree m are interelement-continuous. Thus, in one dimension, $\phi = \phi(x)$ is C^0 continuous if ϕ is continuous but $\phi_{,x}$ is not, and $\phi = \phi(x)$ is C^1 continuous if both ϕ and $\phi_{,x}$ are continuous but $\phi_{,xx}$ is not.

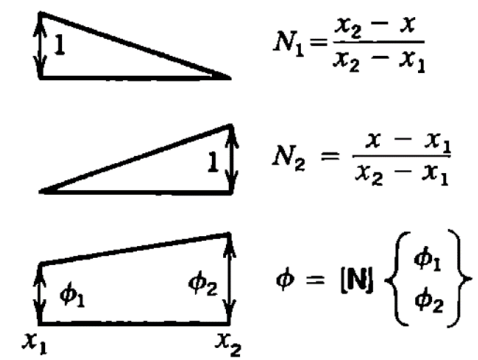
- In general, it is necessary that derivatives of $\emptyset$ of degree m be included as nodal DOF if field $\emptyset$ is to be C^m continuous.

- The C^m terminology is also applied to element type.
- Usually, C^0 elements are used to model plane and solid bodies.
- C^1 elements are used to model beams, plates, shells, thus providing inter-element continuity of slope.



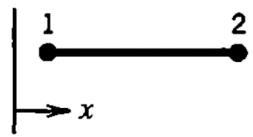
C^0 Interpolation. We begin with linear interpolation between points (x_1,ϕ_1) and (x_2,ϕ_2), for which $\lfloor \mathbf{X} \rfloor = \lfloor 1 \quad x \rfloor$ in Eq. 3.2-1. Evaluating $\lfloor \mathbf{X} \rfloor$ at points 1 and 2, we obtain

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = [\mathbf{A}] \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \qquad \text{where} \qquad [\mathbf{A}] = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \tag{3.2-4}$$



Inverting $[\mathbf{A}]$ and using Eq. 3.2-3, we obtain

$$[\mathbf{A}]^{-1} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \lfloor \mathbf{N} \rfloor = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix} \tag{3.2-5}$$

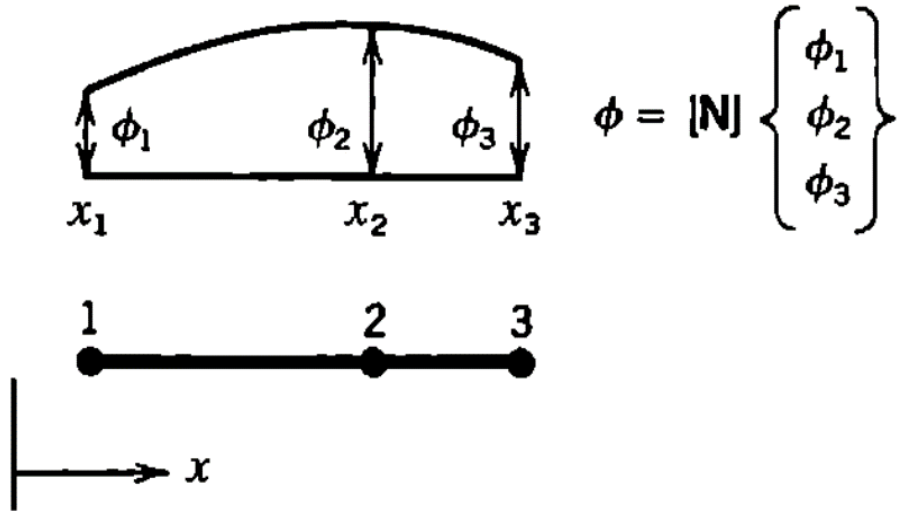


The two linear shape functions N_1 and N_2 are shown in the figure. This example displays the simplest interpolation used in FEA. In formulating properties of a two-node element of length L, we will use $x_1 = 0, x_2 = L$, and nodal d.o.f. ϕ_1 and ϕ_2.

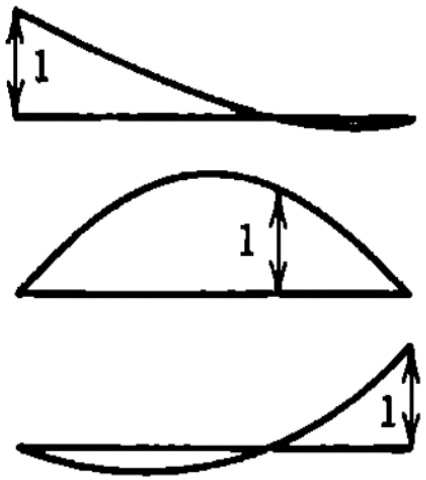
Quadratic interpolation fits a parabola to the points (x_1,ϕ_1), (x_2,ϕ_2), and (x_3,ϕ_3). These points need not be equidistant. Now $\lfloor \mathbf{X} \rfloor = \lfloor 1 \quad x \quad x^2 \rfloor$, and Eq. 3.2-2 becomes

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [\mathbf{A}] \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \qquad \text{where} \qquad [\mathbf{A}] = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \tag{3.2-6}$$

Equation 3.2-3 yield [N], whose individual shape function are shown below.



$$\phi = [\mathbf{N}] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$



$$N_1 = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2 = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3 = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

Results shown in Fig. 3.2-2 can be regarded as particular instances of *Lagrange's interpolation formula,* which provides the following shape functions for a curve fitted to ordinates at n points.

$$N_1 = \frac{(x_2 - x)(x_3 - x)\cdots(x_n - x)}{(x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)}, \qquad N_2 = \frac{(x_1 - x)(x_3 - x)\cdots(x_n - x)}{(x_1 - x_2)(x_3 - x_2)\cdots(x_n - x_2)}, \qquad \text{etc.} \qquad (3.2\text{-}7a)$$

or more generally

$$N_k = \frac{(x_1 - x)(x_2 - x)\cdots[x_k - x]\cdots(x_n - x)}{(x_1 - x_k)(x_2 - x_k)\cdots[x_k - x_k]\cdots(x_n - x_k)} \qquad (3.2\text{-}7b)$$

- In which the bracketed terms are omitted to obtain the kth shape function.
- For linear interpolation, N's and x's having subscripts greater than 2 do not appear.
- For quadratic interpolation, N's and x's having subscripts greater than 3 do not appear; and so on.

- The foregoing shape functions have the following characteristics:

- All shape functions N_i, along with function ϕ itself, are polynomials of the same degree.
- For any shape function N_i, $N_i = 1$ when $x = x_i$ and $N_i = 0$ when $x = x_j$ for any integer $j \neq i$. That is, N_i is unity at its own node but is zero at other nodes.
- C^0 shape functions sum to unity; that is, $\sum N_i = 1$. This conclusion is implied by Eq. 3.2-3, because we must obtain $\phi = 1$ when $\{\phi_e\}$ is a column of 1's.

- ❑ Lagrange's interpolation formula uses only ordinates $\emptyset_i$ in fitting a curve.
- ❑ Slope information is not used, so Lagrange interpolation may display slopes other than those desired.

C^1 Interpolation: Use of both ordinates and slopes (Hermitian interpolation) is C^1 interpolation.

- Consider a cubic curve $\emptyset = \emptyset(x)$, whose shape is determined by four data items.
- These items are ordinates $\emptyset_i$ and small slopes $(d\emptyset/dx)_i$at either end of a line of length L.

$[X] = [1 \quad x \quad x^2 \quad x^3]$, and upon evaluating $\emptyset$ and $\emptyset_{,x}$ at $x= 0$ and $x= L$, leads to:

$$\emptyset = a_o + a_1 x + a_2 x^2 + a_3 x^3$$

To determine the coefficients a_i, the values of $\emptyset$ and $\emptyset_{,x}$ at two ends are used.

1. At $x = x_1 , \emptyset = \emptyset_1$ thus,

$$\emptyset_1 = a_o + a_1 x_1 + a_2 {x_1}^2 + a_1 {x_1}^3 \qquad \text{--- (A1)}$$

2. At $x = x_1$, $\frac{\partial \emptyset}{\partial x} = (\frac{\partial \emptyset}{\partial x})_1$ thus,

$$\frac{\partial \emptyset}{\partial x} = a_1 + 2a_2x + 3a_3x^2$$

$$(\frac{\partial \emptyset}{\partial x})_1 = a_1 + 2a_2x_1 + 3a_3{x_1}^2 \quad \text{---(A2)}$$

3. At $x = x_2$, $\emptyset = \emptyset_2$ thus,

$$\emptyset_2 = a_o + a_1x_2 + a_2{x_2}^2 + a_1{x_2}^3 \quad \text{--- (A3)}$$

2. At $x = x_2$, $\frac{\partial \emptyset}{\partial x} = (\frac{\partial \emptyset}{\partial x})_2$ thus,

$$(\frac{\partial \emptyset}{\partial x})_2 = a_1 + 2a_2x_2 + 3a_3{x_2}^2 \quad \text{---(A4)}$$

To simplify the equations, use the local coordinates,

$x_1 = 0$, and $x_2 = L$, this reduces the equations to

$$\emptyset_1 = a_o \quad \text{--- (B1)}$$

$$\left(\frac{\partial \emptyset}{\partial x}\right)_1 = a_1 \quad \text{---(B2)}$$

$$\emptyset_2 = a_o + a_1 L + a_2 L^2 + a_3 L^3 \quad \text{--- (B3)}$$

$$\left(\frac{\partial \emptyset}{\partial x}\right)_2 = a_1 + 2a_2 L + 3a_3 L^2 \quad \text{---(B4)}$$

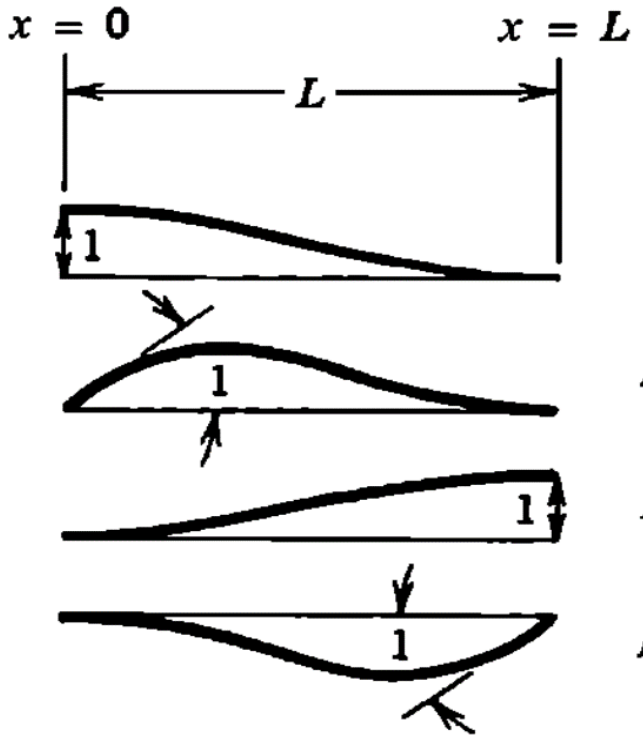
In matrix form:

$$\begin{Bmatrix} \emptyset_1 \\ \left(\frac{\partial \emptyset}{\partial x}\right)_1 \\ \emptyset_2 \\ \left(\frac{\partial \emptyset}{\partial x}\right)_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (3.2\text{-}8)$$

- The shape functions can be found by using Eq. 3.2-3,

$$\phi = \lfloor \mathbf{N} \rfloor \{ \boldsymbol{\phi}_e \} \qquad \text{where} \qquad \lfloor \mathbf{N} \rfloor = \lfloor \mathbf{X} \rfloor [\mathbf{A}]^{-1} = \lfloor N_1 \quad N_2 \quad \cdots \rfloor \tag{3.2-3}$$

- These produced four shape functions turn out to be the four lateral displacement of a beam element.



		At $x = 0$		At $x = L$	
		N_i	$N_{i,x}$	N_i	$N_{i,x}$
N_1	$= 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}$	1	0	0	0
N_2	$= x - \frac{2x^2}{L} + \frac{x^3}{L^2}$	0	1	0	0
N_3	$= \frac{3x^2}{L^2} - \frac{2x^3}{L^3}$	0	0	1	0
N_4	$= -\frac{x^2}{L} + \frac{x^3}{L^2}$	0	0	0	1

Formulation of Finite Element Equations:

Strong and weak forms:

- Differential equations, are *strong forms* of the governing system of equations.
- The strong form, in contrast to *weak form*, requires strong continuity on the dependent field variables (the displacements, temperature, ..)
- Whatever functions that define these field variables, they have to be differentiable up to the order of the differential equations that exists in the strong form of the system equations.
- Obtaining the exact solution for a strong form of the system equation is usually very difficult for many engineering problems.
- The finite difference method can be used to solve equations of the strong form to obtain an approximate solution.
- However, the method works well for problems with simple and regular geometry and boundary conditions.

- A weak form of the system equations is usually created using one of the following widely used methods; energy principle, and weighted residual method.
- The energy principle can be categorized as a special form of the variational principle which is particularly suited for problems of the mechanics of solids and structures.
- The weighted residual method is a more general mathematical tool applicable, in principle, for solving all kinds of partial differential equations.
- The weak form is often an integral form and requires a weaker continuity on the field variables.
- Due to the weaker requirements on the field variables, and the integral operation, a formulation based on a weak form usually produce a set of discretized system of equations that give much more accurate results, especially for problems of complex geometry.
- Hence, the weak form is preferred by many for obtaining an approximate solution.
- The finite element method is a typical example of successfully using of weak form formulation.

Energy Methods to Derive Element Equations:

- One of the methods often used to derive the element equations and the stiffness matrix for an element is based on the principle of *minimum potential energy*.
- This method has the advantage of being more general than the direct equilibrium method which involves nodal and element equilibrium equations along with the stress/strain laws for the element.
- Thus the principle of minimum potential energy is more adaptable to the determination of element equations for complicated elements.
- Complicated elements are those with large numbers of degrees of freedom such as the plane stress/strain element, the axisymmetric stress element, the plate bending element, and the three-dimensional solid stress element.

- The principle of virtual work is applicable for any material behavior, whereas the principle of minimum potential energy is applicable only for elastic materials.
- However, both principles yield the same element equations for linear-elastic materials.
- The principle of minimum potential energy, being included in the general category of *variational methods* (as is the principle of virtual work), leads to other variational functions (or functionals) similar to potential energy that can be formulated for other classes of problems, primarily of the nonstructural type.
- These other problems are generally classified as *field problems* and include, among others, torsion of a bar, heat transfer, fluid flow, and electric potential.
- Still other classes of problems, for which a variational formulation is not clearly definable, can be formulated by ***weighted residual methods***.
- These methods include Galerkin's method, collocation, least squares, and the subdomain weighted residual methods.

Formulation of Finite Element Matrices Using Castigliano's First Theorem

Castigliano's First Theorem:

"*For an elastic system in equilibrium, the partial derivative of total strain energy with respect to deflection at a point is equal to the applied force in the direction of the deflection at that point.*"

$$\frac{\Delta U_e}{\Delta \delta_i} = F_i$$

- The first theorem of Castigliano is a powerful tool for finite element formulation.
- For example, the total strain energy for the bar element is given by

$$U_e = \frac{1}{2}\sigma_x \varepsilon_x V = \frac{1}{2}E\left(\frac{u_2 - u_1}{L}\right)^2 AL$$

- Applying Castigliano's theorem with respect to each displacement yields

$$\frac{\partial U_e}{\partial u_1} = \frac{AE}{L}(u_1 - u_2) = f_1$$

$$\frac{\partial U_e}{\partial u_2} = \frac{AE}{L}(u_2 - u_1) = f_2$$

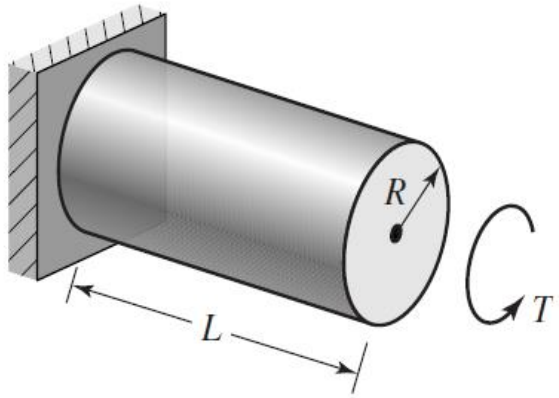
Example: A solid circular shaft of radius R and length L is subjected to constant torque T. The shaft is fixed at one end, as shown in the figure below. Formulate the elastic strain energy in terms of the angle of twist θ at x = L and show that Castigliano's first theorem gives the correct expression for the applied torque.

Solution:

From strength of materials theory,

the shear stress at any cross section along the

length of the member is given by $\tau = \frac{Tr}{J}$



where r is radial distance from the axis of the member and J is polar moment of inertia of the cross section.

For elastic behavior, we have $\gamma = \frac{\tau}{G} = \frac{Tr}{JG}$

where G is the shear modulus of the material, and the strain energy is then

$$U_e = \frac{1}{2}\int_V \tau\gamma \, dV = \frac{1}{2}\int_0^L \left[\int_A \left(\frac{Tr}{J}\right)\left(\frac{Tr}{JG}\right) dA\right] dx$$

$$= \frac{T^2}{2J^2G}\int_0^L \int_A r^2 \, dA \, dx = \frac{T^2 L}{2JG}$$

where we have used the definition of the polar moment of inertia

$$J = \int_A r^2 \, dA$$

The angle of twist at the end of the member is known to be

so the strain energy can be written as

$$\theta = \frac{TL}{JG}$$

so the strain energy can be written as

$$U_e = \frac{1}{2}\frac{L}{JG}\left(\frac{JG\theta}{L}\right)^2 = \frac{JG}{2L}\theta^2$$

And applying Castangliano's first theorem,

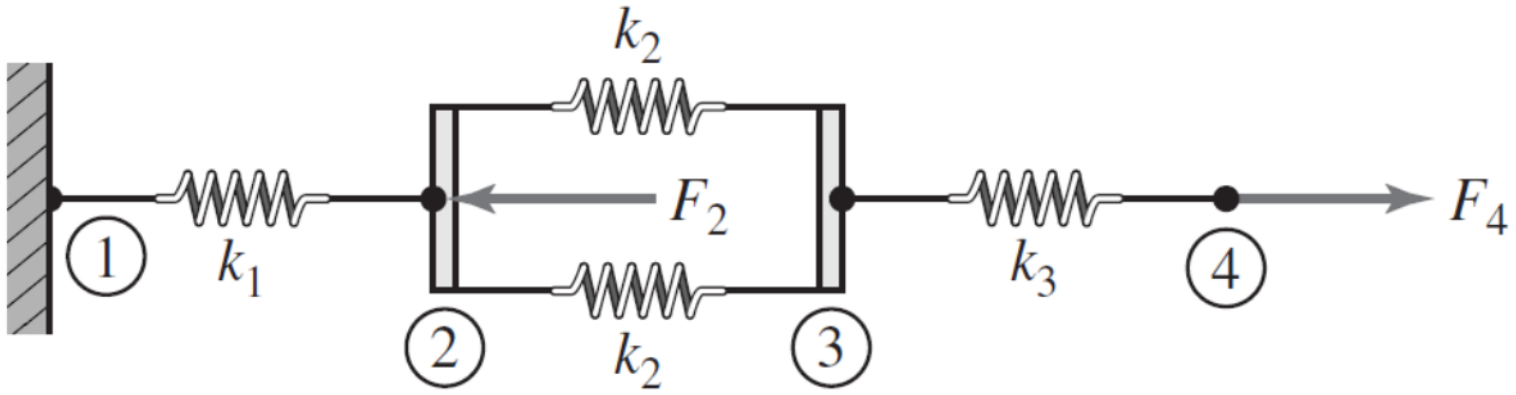
$$\frac{\partial U_e}{\partial \theta} = T = \frac{JG\theta}{L}$$

which is exactly the relation shown by strength of materials theory.

Example: (a) Apply Castigliano's first theorem to the system of four spring elements depicted in figure to obtain the system stiffness matrix. The vertical members at nodes 2 and 3 are to be considered rigid.

(b) Solve for the displacements and the reaction force at node 1 if:

$k_1 = 4$ N/mm $\quad k_2 = 6$ N/mm $\quad k_3 = 3$ N/mm $\qquad F_2 = -30$ N $\quad F_3 = 0 \quad F_4 = 50$ N



■ Solution

(a) The total strain energy of the system of four springs is expressed in terms of the nodal displacements and spring constants as

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2$$

Applying Castigliano's theorem, using each nodal displacement in turn,

$$\frac{\partial U_e}{\partial U_1} = F_1 = k_1(U_2 - U_1)(-1) = k_1(U_1 - U_2)$$

$$\frac{\partial U_e}{\partial U_2} = F_2 = k_1(U_2 - U_1) + 2k_2(U_3 - U_2)(-1) = -k_1U_1 + (k_1 + 2k_2)U_2 - 2k_2U_3$$

$$\frac{\partial U_e}{\partial U_3} = F_3 = 2k_2(U_3 - U_2) + k_3(U_4 - U_3)(-1) = -2k_2U_2 + (2k_2 + k_3)U_3 - k_3U_4$$

$$\frac{\partial U_e}{\partial U_4} = F_4 = k_3(U_4 - U_3) = -k_3U_3 + k_3U_4$$

which can be written in matrix form as

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

and the system stiffness matrix is thus obtained via Castigliano's theorem.

(b) Substituting the specified numerical values, the system equations become

$$\begin{bmatrix} 4 & -4 & 0 & 0 \\ -4 & 16 & -12 & 0 \\ 0 & -12 & 15 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ -30 \\ 0 \\ 50 \end{Bmatrix}$$

Eliminating the constraint equation, the active displacements are governed by

$$\begin{bmatrix} 16 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ 0 \\ 50 \end{Bmatrix}$$

which we solve by manipulating the equations to convert the coefficient matrix (the stiffness matrix) to upper-triangular form; that is, all terms below the main diagonal become zero.

Step 1. Multiply the first equation (row) by 12, multiply the second equation (row) by 16, add the two and replace the second equation with the resulting equation to obtain:

$$\begin{bmatrix} 16 & -12 & 0 \\ 0 & 96 & -48 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -360 \\ 50 \end{Bmatrix}$$

Step 2. Multiply the third equation by 32, add it to the second equation, and replace the third equation with the result. This gives the triangularized form desired:

$$\begin{bmatrix} 16 & -12 & 0 \\ 0 & 96 & -48 \\ 0 & 0 & 48 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -360 \\ 1240 \end{Bmatrix}.$$

In this form, the equations can now be solved from the “bottom to the top,” and it will be found that, at each step, there is only one unknown. In this case, the sequence is:

$U_1 = \frac{1240}{48} = 25.83\ mm$

$U_3 = \frac{1}{96}\left[-360 + 48(25.83)\right] = 8.17\ mm$

$U_4 = \frac{1}{16}\left[-30 + 12(9.17)\right] = 5.0\ mm$

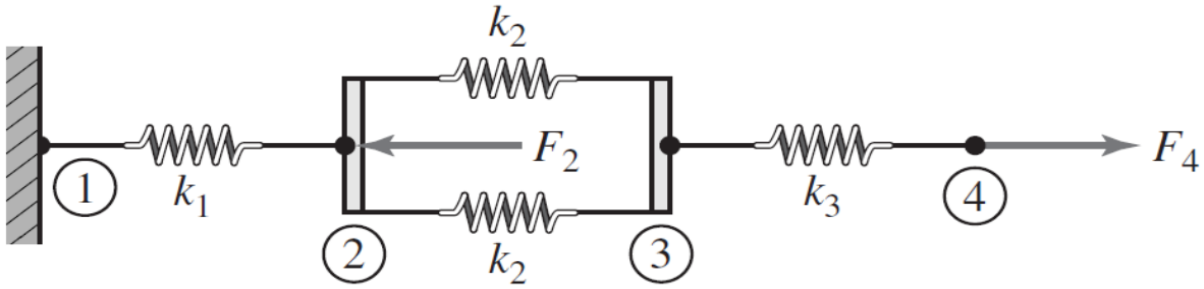
The reaction force at node 1 is obtained from the constraint equation

$F_1 = -4U_2 = -4(5.0) = -20$ N,

and we observe system equilibrium since the external forces sum to zero as required.

Example: Repeat the solution of the previous example by using the principle of minimum potential energy.

$k_1 = 4$ N/mm $k_2 = 6$ N/mm $k_3 = 3$ N/mm $F_2 = -30$ N $F_3 = 0$ $F_4 = 50$ N



Solution:

The total potential energy is given by the equation:

$$\Pi = U_e - W$$

The elastic strain energy is

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2$$

and the potential energy of applied forces is

$$U_F = -W = -F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

Hence, the total potential energy is expressed as

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2 - F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

The principle of minimum potential energy requires that

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad i = 1,2,3,4$$

the resulting algebraic equations are

$$\frac{\partial \Pi}{\partial U_1} = \quad k_1(U_2 - U_1)(-1) - F_1 = k_1U_1 - k_1U_2 - F_1 = 0$$

$$\frac{\partial \Pi}{\partial U_2} = k_1(U_2 - U_1) + 2k_2(U_3 - U_2)(-1) - F_2 = -k_1U_1 + (k_1 + 2k_2)U_2 - 2k_2U_3 - F_2 = 0$$

$$\frac{\partial \Pi}{\partial U_3} = 2k_2(U_3 - U_2) + k_3\,(U_4 - U_3)(-1) - F_3 = -2k_2U_2 + (2k_2 + k_3)\,U_3 - \,k_3U_4 - F_3 = 0$$

$$\frac{\partial \Pi}{\partial U_4} = \quad k_3(U_4 - U_3)(-1) - F_1 = -k_3U_3 + k_3U_4 - F_4 = 0$$

which, when written in matrix form, are

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

and can be seen to be identical to the previous result.

Formulation of Finite Element Method using Principle of Minimum Potential Energy

Principle of Minimum Potential Energy:

- The principle of minimum potential energy is stated as follows:

" Of all displacement states of a body or structure, subjected to external loading, that satisfy the geometric boundary conditions (imposed displacements), the displacement state that also satisfies the equilibrium equations is such that the total potential energy is a minimum for stable equilibrium."

- In simple words, if a loaded elastic body is in equilibrium under given geometric constraints or boundary conditions, the potential energy of the deformed body assumes a stationary value.
- In the case of linear elastic bodies in equilibrium, the value is a minimum.

For methods in this category, the knowledge of governing differential equation of the structure is not necessary. However, it is necessary to know the functional to be minimised. For problems in structural engineering (in which displacements are primary unknowns), the functional is the total potential energy of the structure.

The total potential energy of the structure π is expressed either in terms of unknown mathematical constants or in terms of nodal displacements. The conditions of minimising π with respect to constants or nodal displacements result in a set of simultaneous equations. The solution of these equations decides the deformed state of the structure. Other dependant quantities can then be found at various locations in the structure.

1. Rayleigh Method:

In Rayleigh method, applied to beams and plates, the field variable (w) is assumed as

$$w = A\phi$$

where, A is a mathematical constant and does not have any physical meaning, and ϕ is a function (polynomial or trigonometric) which satisfies kinematic boundary conditions. The total potential energy of the structure is minimised with respect to unknown constant A.

Example 1.9. A cantilever beam AB of span L, flexural rigidity EI carries uniformly distributed load q/unit length on entire span (Fig. 1.10). Use Rayleigh method to analyse and comment on results.

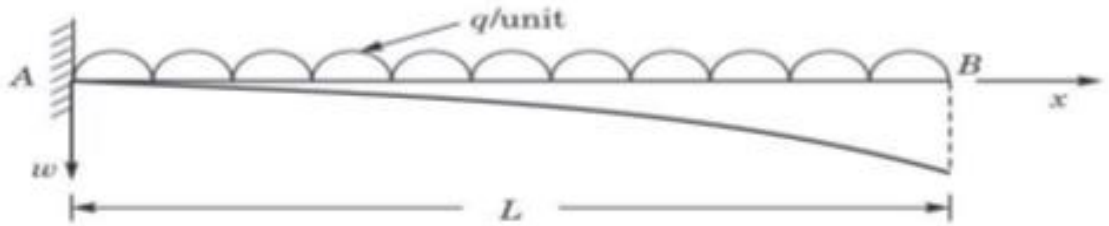


Fig. 1.10. Cantilever beam.

Assume,

$$\left.\begin{aligned} w(x) &= Ax^2 \\ \frac{dw}{dx} &= 2Ax \\ \frac{d^2w}{dx} &= 2A \end{aligned}\right\} \quad \ldots(1.55)$$

Assumed function satisfies kinematic boundary conditions as

(*i*) $w = 0$ at $x = 0$ (*ii*) $\frac{dw}{dx} = 0$ at $x = 0$

Hence, assumed function is acceptable. The strain energy in the beam is

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L (2A)^2\, dx$$

$$U = 2EIA^2L$$

The potential of the distributed load is

$$V = -\int_0^L q\, w\, dx = -q\int_0^L Ax^2\, dx$$

$$V = -qAL^3/3$$

Total potential energy of the beam is

$$\pi = U + V = 2EIA^2 - qAL^3/3$$

Applying minimum π theorem

$$\frac{\partial \pi}{\partial A} = 4EIAL - qL^3/3 = 0$$

$$A = \frac{qL^2}{12EI}$$

Hence,
$$w(x) = \frac{qL^2x^2}{12EI} \qquad ...(1.59)$$

The deflection at free end ($x = L$) using Eq. 1.59 is $w(x = L) = qL^4/(12EI)$ as against exact value of $qL^4/(8EI)$.

In general, the results obtained by Rayleigh method are not satisfactory. We seek to find that deflected shape of the structure which gives absolute minimum total potential energy. For this, it is necessary to examine all possible kinematically admissible deflected shapes. But, Rayleigh method examines only those deflected shapes which are generated by assumed function by changing associated constant A. Obviously, the search is limited to the type of shapes generated by assumed function and the one giving minimum π is treated as true solution.

2. Rayleigh – Ritz Method

Example 1.10 In Rayleigh-Ritz method, the field variable is expressed as a linear combination of more than one functions. Thus,

$$w(x) = \sum A_i \phi_i \qquad \ldots(1.60)$$

Each function ϕ_i must satisfy kinematic boundary conditions of the problem. The cantilever beam shown in Fig. 1.10 is analysed.

Assume

$$\left.\begin{aligned} w(x) &= A_1 x^2 + A_2 x^3 + A_3 x^4 \\ \frac{dw}{dx} &= 2A_1 x + 3A_2 x^2 + 4A_3 x^3 \\ \frac{d^2 w}{dx^2} &= 2A_1 + 6A_2 x + 12A_3 x^2 \end{aligned}\right\} \qquad \ldots(1.61)$$

The assumed trial function satisfies boundary conditions as

At $x = 0, \quad w = 0 + 0 + 0$

At $x = 0, \quad \frac{dw}{dx} = 0 + 0 + 0$

The strain energy in beam is

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^L (2A_1 + 6A_2x + 12A_3x^2)^2\, dx$$

$$U = \frac{EI}{2}\left[4A_1^2L + 12A_1A_2L^2 + 16A_1A_3L^3 + 12A_2^2L^3 + 36A_2A_3L^4 + \frac{144}{5}A_3^2L^5\right] \quad \ldots$$

Potential of distributed load is

$$V = -\int_0^L q(A_1x^2 + A_2x^3 + A_3x^4)dx$$

$$V = -\frac{qA_1L^3}{3} - \frac{qA_2L^4}{4} - \frac{qA_3L^5}{5} \quad \ldots$$

$$\pi = U + V$$

Applying minimum π theorem

$$\frac{d\pi}{dA_1} = 0, \quad \frac{d\pi}{dA_2} = 0, \quad \frac{d\pi}{dA_3} = 0$$

Resulting equations in matrix form are

$$\frac{EI}{2}\begin{bmatrix} 8L & 12L^2 & 16L^3 \\ 12L^2 & 24L^3 & 36L^4 \\ 16L^3 & 36L^4 & 288L^5/5 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} qL^3/3 \\ qL^4/4 \\ qL^5/5 \end{Bmatrix}$$

Solution gives

$$A_1 = \frac{qL^2}{4EI}, \quad A_2 = \frac{-qL}{6EI}, \quad A_3 = \frac{q}{24EI}$$

Hence, the solution for deflection $w(x)$ is

$$w(x) = \frac{1}{EI}\left[\frac{qL^2x^2}{4} - \frac{qLx^3}{6} + \frac{qx^4}{24}\right]$$

This is the exact solution for beam in Fig. 1.10. This was possible because trial function included all terms of the polynomial (e.g. x^2, x^3, x^4) which are present in exact solution. Exact solution will not be obtained if trial function is chosen as $w(x) = A_1x^2 + A_2x^3$.

Potential Energy Approach to Derive Spring Element Equations:

- One of the alternative methods often used to derive the element equations and the stiffness matrix for an element is based on the principle of *minimum potential energy*.
- This method has the advantage of being more general than the direct stiffness method.
- The principle of minimum potential energy is more adaptable to the determination of element equations for complicated elements (those with large numbers of degrees of freedom) such as the plane stress/strain element, the axisymmetric stress element, the plate bending element, and the three-dimensional solid stress element.
- Again, we state that the principle of virtual work is applicable for any material behavior, whereas the principle of minimum potential energy is applicable only for elastic materials.
- However, both principles yield the same element equations for linear-elastic materials.

- Moreover, the principle of minimum potential energy, being included in the general category of *variational methods* (as is the principle of virtual work), leads to other variational functions (or functionals) similar to potential energy that can be formulated for other classes of problems, primarily of the nonstructural type.
- These other problems are generally classified as *field problems* and include, among others, torsion of a bar, heat transfer, fluid flow, and electric potential.
- Still other classes of problems, for which a variational formulation is not clearly definable, can be formulated by *weighted residual methods*. We will describe Galerkin's method
- In the following the principle of minimum potential energy will be presented as used to derive the spring element equations.
- This concept will be illustrated by applying it to the simplest of elements in hopes that the reader will then be more comfortable when applying it to handle more complicated element types. in subsequent chapters.

- The total potential energy π_p of a structure is expressed in terms of displacements.
- In the finite element formulation, these will generally be nodal displacements such that

$$\pi_p = \pi_p (d_1, d_2, d_3, \dots)$$

- When π_p is minimized with respect to these displacements, equilibrium equations result.
- For the spring element, we will show that the same nodal equilibrium equations $[k]\{d\} = \{f\}$ result as previously derived in direct stiffness method.
- The principle of minimum potential energy can be stated as follows:

" Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the satisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy."

The concept of Potential Energy and stationary value of a function:

- **Total potential energy** is defined as *the sum of the internal strain energy U and the potential energy of the external forces* Ω ; that is,

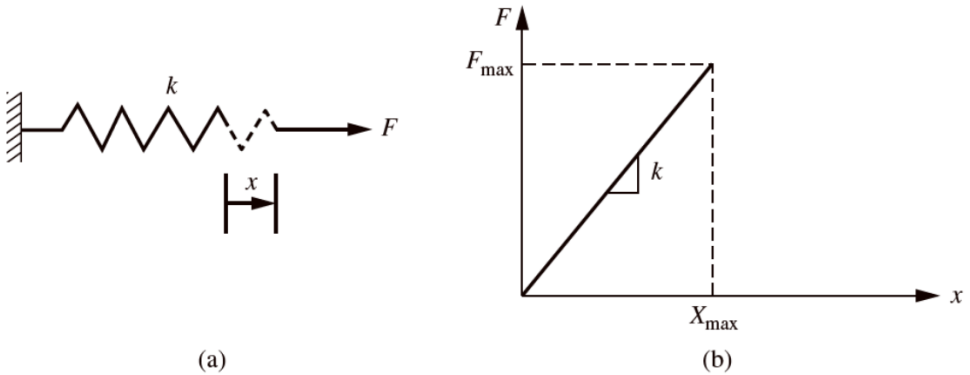
$$\pi_p = U + \Omega \qquad (1)$$

- U is the capacity of internal forces (or stresses) to do work through deformations (strains) in the structure.
- Ω is the capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through deformation of the structure.
- To understand the concept of internal strain energy, we first describe the concept of external work.
- In this section, we consider only the external work due to an applied nodal force.
- External work can also be due to body forces (typically self weight) and surface tractions (distributed forces).

- External work is done on a linear-elastic behaving member [here we consider an elastic spring shown in figure below, by applying a gradually increasing magnitude force *F* to the end of the spring up to some maximum value F_{max} less than that which would cause permanent deformation in the spring.

(a) Spring subjected to gradually increasing force *F*

(b) Force/deformation curve for linear spring



- The maximum deformation X_{max} occurs when the maximum force occurs as shown in figure (b).
- The external work is given by the area under the force-deformation curve.
- The slope of the straight line is equal to the spring constant *k*.
- The external work W_e then given from basic mechanics principles as the integral of the dot product of vector force **F** with the differential displacement *dx*,

$$W_e = \int F \cdot dx = \int_0^{x_{max}} F_{max}\left(\frac{x}{x_{max}}\right) dx = F_{max}x_{max}/2 \qquad (2)$$

- Where F is given as

$$F = F_{\max}(x/x_{\max}) \qquad (3)$$

- In Eq. (2), we note that *F* and *dx* are in the same direction when expressing the second integral on the right side of Eq. (2).
- By the conservation of mechanical energy principle, the external work due to the applied force F is transformed into the internal strain energy *U* of the spring.
- This strain energy is then given by

$$W_e = U = F_{\max} x_{\max}/2 \qquad (4)$$

- Upon gradual reduction of the force to zero, the spring returns to its original undeformed state.
- This returned energy that is stored in the deformed elastic spring is called *internal strain energy* or just *strain energy*.
- Also

$$F_{\max} = kx_{\max} \qquad (5)$$

- By substituting Eq. (5) into Eq. (4), we can express the strain energy as

$$U = kx_{\max}^2/2 \qquad (6)$$

- The potential energy of the external force, being opposite in sign from the external work expression because the potential energy of the external force is lost when the work is done by the external force, is given by

$$\Omega = -F_{\max}x_{\max} \qquad (7)$$

- Therefore, substituting Eqs. (6) and (7) into Eq. (1), yields the total potential energy as

$$\pi_p = \frac{1}{2}kx_{\max}^2 - F_{\max}x_{\max} \qquad (8)$$

- In general for any deformation x of the spring corresponding to force F, we replace $x_{\max}$ with x and $F_{\max}$ with F and express U and as

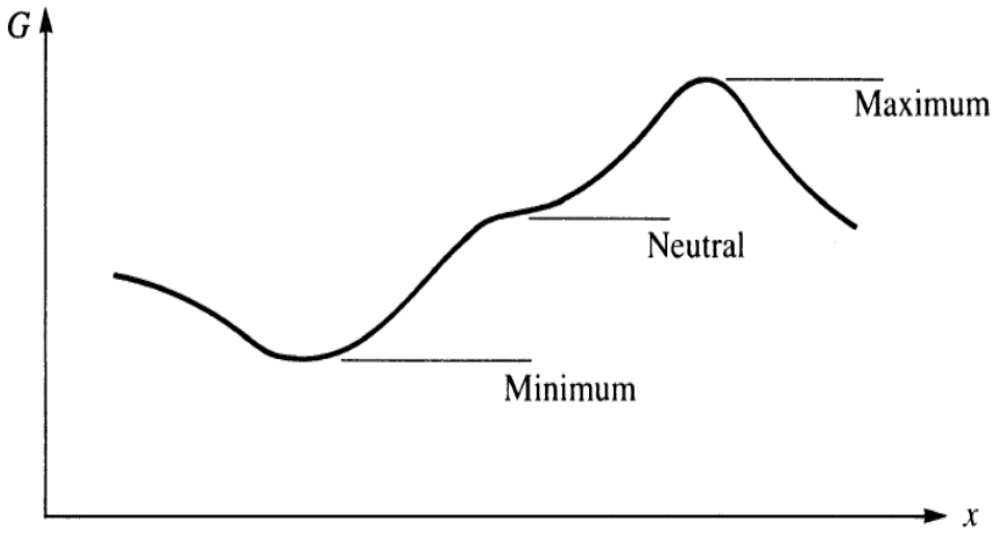
$$U(x) = kx^2/2 \qquad (8 - a)$$

$$\Omega(x) = -Fx \qquad (8 - b)$$

- Substituting Eq. (8a) and (8b) into Eq. (1), the total potential energy will be

$$\pi_p(x) = \frac{1}{2}kx^2 - Fx \qquad (9)$$

- The concept of a *stationary value* of a function G (used in the definition of the principle of minimum potential energy) is shown in figure. Here G is expressed as a function of the variable x. The stationary value can be a maximum, a minimum, or a neutral point of $G(x)$.



- To find a value of x yielding a stationary value of $G(x)$, we use differential calculus to differentiate G with respect to x and set the expression equal to zero, as follows:

$$\frac{dG}{dx} = 0 \qquad (10)$$

- An analogous process will subsequently be used to replace G with π_p and x with discrete values (nodal displacements) d_i .
- With an understanding of variational calculus we could use the first variation of π_p (denoted by $\delta\pi_p$) to minimize π_p.
- However, we will avoid the details of variational calculus and show that we can really use the familiar differential calculus to perform the minimization of π_p .
- To apply the principle of minimum potential energy that is, to minimize π_p we take the *variation* of π_p , which is a function of nodal displacements *di* defined in general as:

$$\delta\pi_p = \frac{\delta\pi_p}{\delta d_1}\delta d_1 + \frac{\delta\pi_p}{\delta d_2}\delta d_2 + \cdots + \frac{\delta\pi_p}{\delta d_n}\delta d_n \qquad (11)$$

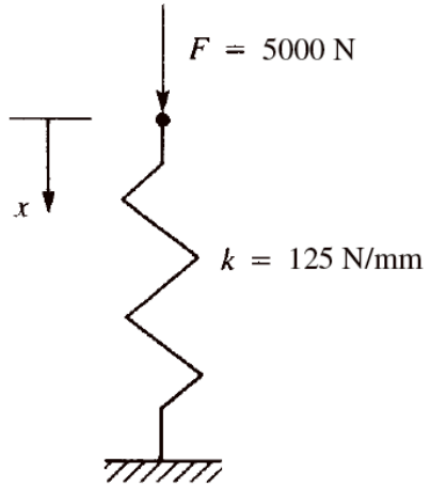
- The principle states that equilibrium exists when the *di* define a structure state such that $\delta\pi_p = 0$ (change in potential energy = 0) for arbitrary admissible variations in displacement δdi from the equilibrium state.
- An *admissible variation* is one in which the displacement field still satisfies the boundary conditions and interelement continuity.

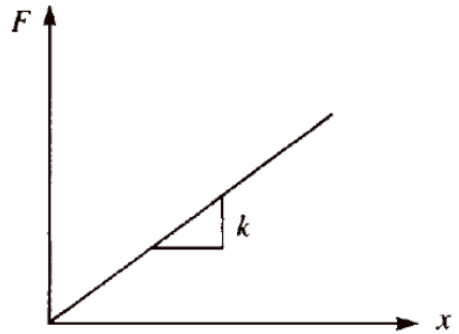
- To satisfy $\delta\pi_p$=0, all coefficients associated with the δd_i must be zero independently. Thus,

$$\frac{\partial \pi_p}{\partial d_i} = 0 \qquad (i = 1, 2, 3, \ldots, n) \qquad \text{or} \qquad \frac{\partial \pi_p}{\partial \{d\}} = 0 \tag{12}$$

- where n equations must be solved for the n values of d_i that define the static equilibrium state of the structure.

Example: For the linear-elastic spring subjected to a force of 5000 N shown, evaluate the potential energy for various displacement values and show that the minimum potential energy also corresponds to the equilibrium position of the spring.





Solution:

We evaluate the total potential energy as

$$\pi_p = U + \Omega$$

where $U = \frac{1}{2}(kx)x$ and $\Omega = -Fx$

$$\pi_p = \frac{1}{2}\, kx^2 - Fx$$

$$\pi_p = \frac{1}{2}\, 125x^2 - 5000x$$

We now illustrate the minimization of π_p through standard mathematics. Taking the variation of π_p with respect to x, or, equivalently, taking the derivative of π_p with respect to x (as π_p is a function of only one displacement x), we have

$$\delta\pi_p = \frac{\partial \pi_p}{\partial x}\delta x = 0$$

or, because δx is arbitrary and might not be zero, $\frac{\partial \pi_p}{\partial x} = 0$

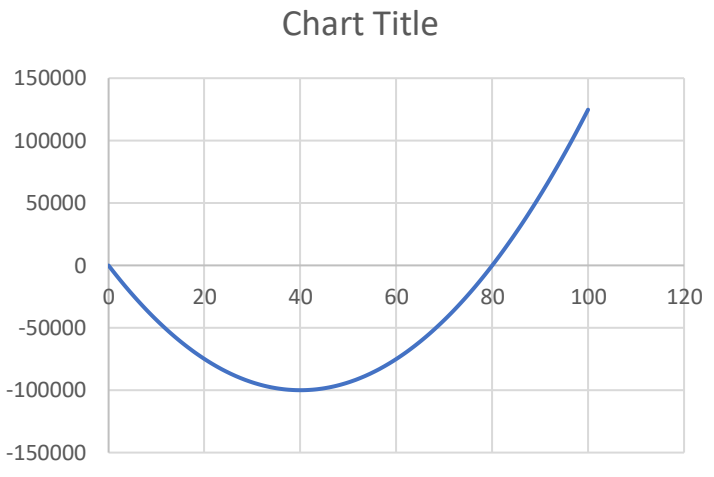
$$\frac{d\pi_p}{dx} = 125x - 5000$$

Thus, $125x - 5000 = 0$

$x = 40\ mm$

This value for x is then back-substituted into π_p to yield,

$$\pi_p = \frac{1}{2}\ 125(4)^2 - 5000(4) = -100{,}000\ \text{N.mm}$$



Derivation of Spring element equations using Principle of minimum potential energy:

We now derive the spring element equations and stiffness matrix using the principle of minimum potential energy. Consider the linear spring subjected to nodal forces shown in Figure 2–22. Using Eq. (2.6.9) reveals that the total potential energy is

$$\pi_p = \frac{1}{2}k(u_2 - u_1)^2 - f_{1x}u_1 - f_{2x}u_2 \tag{2.6.13}$$

where $u_2 - u_1$ is the deformation of the spring in Eq. (2.6.9). The first term on the right in Eq. (2.6.13) is the strain energy in the spring. Simplifying Eq. (2.6.13), we obtain

$$\pi_p = \frac{1}{2}k(u_2^2 - 2u_2u_1 + u_1^2) - f_{1x}u_1 - f_{2x}u_2 \tag{2.6.14}$$

■ **Figure 2–22** Linear spring subjected to nodal forces

The minimization of π_p with respect to each nodal displacement requires taking partial derivatives of π_p with respect to each nodal displacement such that

$$\begin{aligned} \frac{\partial \pi_p}{\partial u_1} &= \frac{1}{2}k(-2u_2 + 2u_1) - f_{1x} = 0 \\ \frac{\partial \pi_p}{\partial u_2} &= \frac{1}{2}k(2u_2 - 2u_1) - f_{2x} = 0 \end{aligned} \tag{2.6.15}$$

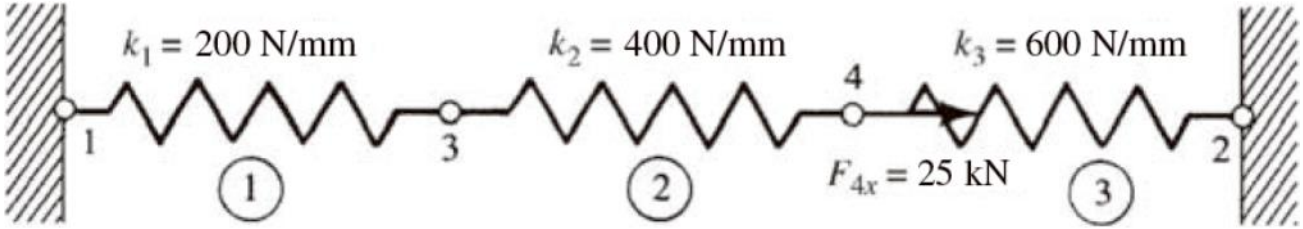
Simplifying Eqs. (2.6.15), we have

$$\begin{aligned} k(-u_2 + u_1) &= f_{1x} \\ k(u_2 - u_1) &= f_{2x} \end{aligned} \tag{2.6.16}$$

In matrix form, we express Eq. (2.6.16) as

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} \tag{2.6.17}$$

- Example: Obtain the total potential energy of the spring assemblage shown and find its minimum value. The procedure of assembling element equations can then be seen to be obtained from the minimization of the total potential energy.



- Solution:

Using Eq. (2.6.8a), the strain energy stored in spring 1 is given by

$$U^{(1)} = k_1(u_3 - u_1)^2/2 \tag{2.6.19}$$

where the difference in nodal displacements $u_3 - u_1$ is the deformation x in spring 1. Eq. (2.6.19) can be written in matrix form as

$$U^{(1)} = \frac{1}{2}[u_3 \; u_1]\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_1 \end{Bmatrix} = \frac{1}{2}\{d\}^{\mathrm{T}}[K]\{d\} \tag{2.6.20}$$

We observe from Eq. (2.6.20) that the strain energy U is a quadratic function of the nodal displacements.

Similar strain energy expressions for springs 2 and 3 are given by

$$U^{(2)} = k_2(u_4 - u_3)^2/2 \qquad \text{and} \qquad U^{(3)} = k_3(u_2 - u_4)^2/2 \tag{2.6.21}$$

with similar matrix expressions as given by Eq. (2.6.20) for spring 1.

Since the strain energy is a scalar quantity, we can add the energy in each spring to obtain the total strain energy in the system as

$$U = \sum_{i=1}^{3} U^{(e)} \tag{2.6.22}$$

The potential energy of the external nodal forces given in the order of the node numbering for the spring assemblage is

$$\Omega = -(F_{1x}u_1 + F_{3x}u_3 + F_{4x}u_4 + F_{2x}u_2) \tag{2.6.23}$$

Equation (2.6.23) can be expressed in matrix form as

$$\Omega = -[u_1\ u_2\ u_3\ u_4]\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{2.6.24}$$

The total potential of the assemblage is the sum of the strain energy and the potential energy of the external forces given by adding Eqs. (2.6.19), (2.6.21) and (2.6.23) together as

$$\Pi_p = U + \Omega = \frac{1}{2}k_1(u_3 - u_1)^2 + \frac{1}{2}k_2(u_4 - u_3)^2 + \frac{1}{2}k_3(u_2 - u_4)^2 - F_{1x}u_1 - F_{2x}u_2 - F_{3x}u_3 - F_{4x}u_4 \tag{2.6.25}$$

Upon minimizing π_p with respect to each nodal displacement, we obtain

$$\begin{aligned}
\frac{\partial \pi_p}{\partial u_1} &= -k_1u_3 + k_1u_1 - F_{1x} = 0 \\
\frac{\partial \pi_p}{\partial u_2} &= k_3u_2 - k_3u_4 - F_{2x} = 0 \\
\frac{\partial \pi_p}{\partial u_3} &= k_1u_3 - k_1u_1 - k_2u_4 + k_2u_3 - F_{3x} = 0 \\
\frac{\partial \pi_p}{\partial u_4} &= k_2u_4 - k_2u_3 - k_3u_2 + k_3u_4 - F_{4x} = 0
\end{aligned} \tag{2.6.26}$$

In matrix form, Eqs. (2.6.26) become

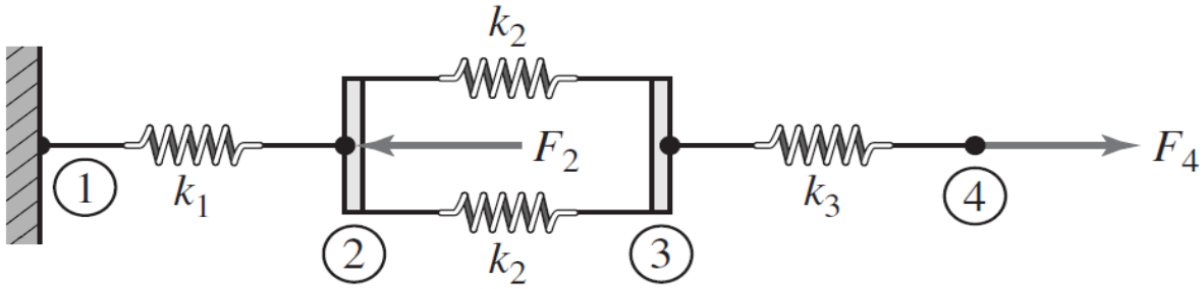
$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{2.6.27}$$

Substituting numerical values for k_1, k_2, and k_3 into Eq. (2.6.27), we obtain

$$\begin{bmatrix} 200 & 0 & -200 & 0 \\ 0 & 600 & 0 & -600 \\ -200 & 0 & 6000 & -400 \\ 0 & -600 & -400 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} \tag{2.6.28}$$

Example: Repeat the solution of the previous example by using the principle of minimum potential energy.

$k_1 = 4$ N/mm $k_2 = 6$ N/mm $k_3 = 3$ N/mm $F_2 = -30$ N $F_3 = 0$ $F_4 = 50$ N



Solution:

The total potential energy is given by the equation:

$$\Pi = U_e - W$$

The elastic strain energy is

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2$$

and the potential energy of applied forces is

$$U_F = -W = -F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

Hence, the total potential energy is expressed as

$$U_e = \frac{1}{2}k_1(U_2 - U_1)^2 + 2\left[\frac{1}{2}k_2(U_3 - U_2)^2\right] + \frac{1}{2}k_3(U_4 - U_3)^2 - F_1U_1 - F_2U_2 - F_3U_3 - F_4U_4$$

The principle of minimum potential energy requires that

$$\frac{\partial \Pi}{\partial U_i} = 0 \qquad i = 1,2,3,4$$

the resulting algebraic equations are

$$\frac{\partial \Pi}{\partial U_1} = \quad k_1(U_2 - U_1)(-1) - F_1 = k_1U_1 - k_1U_2 - F_1 = 0$$

$$\frac{\partial \Pi}{\partial U_2} = k_1(U_2 - U_1) + 2k_2(U_3 - U_2)(-1) - F_2 = -k_1U_1 + (k_1 + 2k_2)U_2 - 2k_2U_3 - F_2 = 0$$

$$\frac{\partial \Pi}{\partial U_3} = 2k_2(U_3 - U_2) + k_3\,(U_4 - U_3)(-1) - F_3 = -2k_2U_2 + (2k_2 + k_3)\,U_3 - \,k_3U_4 - F_3 = 0$$

$$\frac{\partial \Pi}{\partial U_4} = \quad k_3(U_4 - U_3)(-1) - F_1 = -k_3U_3 + k_3U_4 - F_4 = 0$$

which, when written in matrix form, are

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + 2k_2 & -2k_2 & 0 \\ 0 & -2k_2 & 2k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

U_2=25.83 mm, U_3= 8.17 mm, U_4= 5.0 mm

$F_1 = -4U_2 = -4(5.0) = -20$ N

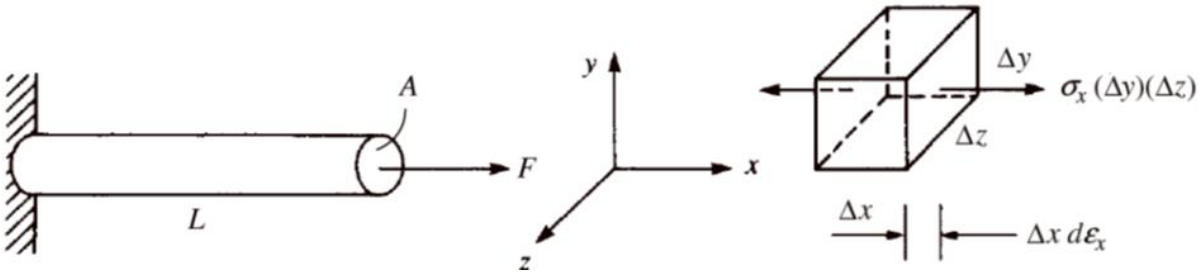
Potential Energy Approach to Derive Bar Element Equations:

We now present the principle of minimum potential energy to derive the bar element equations. Recall from Section 2.6 that the total potential energy π_p was defined as the sum of the internal strain energy U and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega \tag{3.10.1}$$

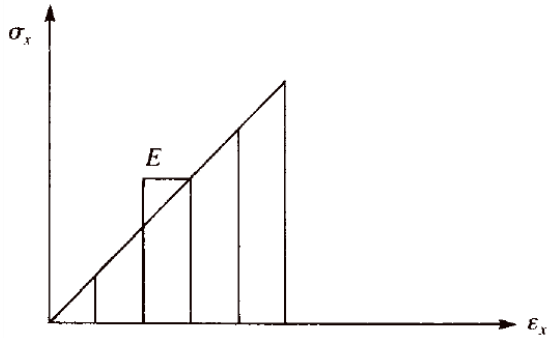
To evaluate the strain energy for a bar,we consider only the work done by the internal forces during deformation. Because we are dealing with a one-dimensional bar, the internal force doing work on a differential element of sides $\Delta x, \Delta y, \Delta z$, is given in Figure 3–24 as $\sigma_x(\Delta y)(\Delta z)$, due only to normal stress σ_x. The displacement of the x face of the element is $\Delta x(\varepsilon_x)$; the displacement of the $x + \Delta x$ face is $\Delta x(\varepsilon_x + d\varepsilon_x)$. The change in displacement is then $\Delta x\, d\varepsilon_x$, where $d\varepsilon_x$ is the differential change in strain occurring over length Δx. The differential internal work (or strain energy) dU is the internal force multiplied by the displacement through which the force moves, given by

$$dU = \sigma_x(\Delta y)(\Delta z)(\Delta x)d\varepsilon_x \tag{3.10.2}$$



Rearranging and letting the volume of the element approach zero, we obtain, from Eq. (3.10.2),

$$dU = \sigma_x d\varepsilon_x dV \tag{3.10.3}$$



For the whole bar, we then have

$$U = \iiint_V \left\{ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \right\} dV \tag{3.10.4}$$

Now, for a linear-elastic (Hooke's law) material as shown in Figure 3–25, we see that $\sigma_x = E\varepsilon_x$. Hence substituting this relationship into Eq. (3.10.4), integrating with respect to ε_x, and then resubstituting σ_x for $E\varepsilon_x$, we have

$$U = \frac{1}{2} \iiint_V \sigma_x \varepsilon_x dV \tag{3.10.5a}$$

as the expression for the strain energy for one-dimensional stress.

For a uniform cross-sectional area A of a bar with stress and strain dependent only on the x coordinate, Eq. (3.10.5a) can be simplified to

$$U = \frac{A}{2} \int_x \sigma_x \varepsilon_x dx \tag{3.10.5b}$$

We observe from the integral in Eq. (3.10.5b) that the strain energy is described as the area under the stress/strain curve.

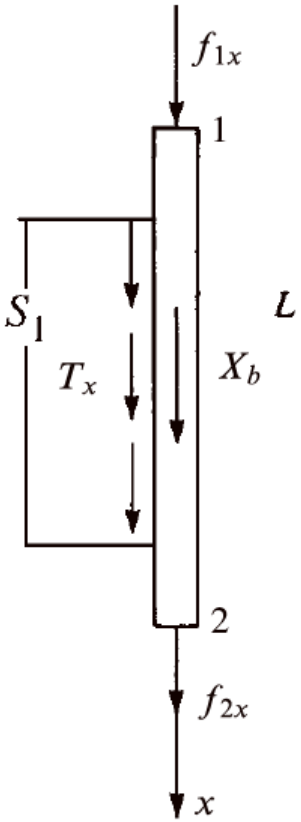
The potential energy of the external forces, being opposite in sign from the external work expression because the potential energy of external forces is lost when the work is done by the external forces, is given by

$$\Omega = -\iiint_{v} X_b u \, dV - \iint_{S_1} T_x u_s \, dS - \sum_{i=1}^{M} f_{ix} u_i \tag{3.10.6}$$

where the first, second, and third terms on the right side of Eq. (3.10.6) represent the potential energy of (1) body forces X_b, typically from the self-weight of the bar (in units of force per unit volume) moving through displacement function u, (2) surface loading or traction T_x, typically from distributed loading acting along the surface of the element (in units of force per unit surface area) moving through displacements u_s, where u_s are the displacements occurring over surface S_1, and (3) nodal concentrated forces f_{ix} moving through nodal displacements u_i. The forces X_b, T_x, and f_{ix} are considered to act in the local x direction of the bar as shown in Figure 3–26. In Eqs. (3.10.5) and (3.10.6), V is the volume of the body and S_1 is the part of the surface S on which surface loading acts. For a bar element with two nodes and one degree of freedom per node, $M = 2$.

Consider the bar element of length L, with constant cross-sectional area A, shown in Figure 3–26. Using Eqs. (3.10.5) and (3.10.6), we find that the total potential energy, Eq. (3.10.1), becomes

$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, d_x - f_{1x}u_1 - f_{2x}u_2 - \iint_{S_1} u_s T_x \, dS - \iiint_v u X_b \, dV \qquad (3.10.7)$$



because A is a constant and variables σ_x and ε_x at most vary with x.

From Eqs. (3.2.8) and (3.2.9), we have the axial displacement function expressed in terms of the shape functions and nodal displacements by

$$u = [N]\{d\} \quad u_s = [N_s]\{d\} \qquad (3.10.8)$$

where

$$[N] = \left[1 - \frac{x}{L} \qquad \frac{x}{L}\right] \qquad (3.10.9)$$

$[N_s]$ is the shape function matrix evaluated over the surface that the distributed surface traction acts and

$$\{d\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (3.10.10)$$

Then, using the strain/displacement relationship $\varepsilon_x = du/dx$, we can write the axial strain in matrix form as

$$\{\varepsilon_x\} = \left[-\frac{1}{L}\frac{1}{L}\right]\{d\} \tag{3.10.11}$$

or

$$\{\varepsilon_x\} = [B]\{d\} \tag{3.10.12}$$

where we define $[B]$ as the gradient matrix

$$[B] = \left[-\frac{1}{L}\frac{1}{L}\right] \tag{3.10.13}$$

The axial stress/strain relationship in matrix form is given by

$$\{\sigma_x\} = [D]\{\varepsilon_x\} \tag{3.10.14}$$

where

$$[D] = [E] \tag{3.10.15}$$

for the one-dimensional stress/strain relationship matrix and E is the modulus of elasticity. Now, by Eq. (3.10.12), we can express Eq. (3.10.14) as

$$\{\sigma_x\} = [D][B]\{d\} \tag{3.10.16}$$

Using Eq. (3.10.7) expressed in matrix notation form, we have the total potential energy given by

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T \{\varepsilon_x\} dx - \{d\}^T \{P\} - \iint_{S_1} \{u_s\}^T \{T_x\} dS - \iiint_v \{u\}^T \{X_b\} dV \qquad (3.10.17)$$

where $\{P\}$ now represents the concentrated nodal loads and where in general both $\{\sigma_x\}$ and $\{\varepsilon_x\}$ are column matrices. For proper matrix multiplication, we must place the transpose on $\{\sigma_x\}$. Similarly, $\{u\}$ and $\{T_x\}$ in general are column matrices, so for proper matrix multiplication, $\{u\}$ is transposed in Eq. (3.10.17).

Using Eqs. (3.10.8), (3.10.12), and (3.10.16) in Eq. 3.10.17, we obtain

$$\begin{aligned}\pi_p = \frac{A}{2}\int_0^L \{d\}^T [B]^T [D]^T [B]\{d\} dx - \{d\}^T \{P\} \\ -\iint_{S_1} \{d\}^T [N_s]^T \{T_x\} dS - \iiint_V \{d\}^T [N]^T \{X_b\} dV\end{aligned} \qquad (3.10.18)$$

In Eq. (3.10.18), π_p is seen to be a function of $\{d\}$; that is, $\pi_p = \pi_p(u_1, u_2)$. However, $[B]$ and $[D]$, Eqs. (3.10.13) and (3.10.15), and the nodal degrees of freedom u_1 and u_2 are not functions of x. Therefore, integrating the first integral in Eq. (3.10.18) with respect to x yields

$$\pi_p = \frac{AL}{2}\{d\}^T [B]^T [D]^T [B]\{d\} - \{d\}^T \{f\} \qquad (3.10.19)$$

where

$$\{f\} = \{P\} + \iint_{s_1} [N_s]^T \{T_x\} dS + \iiint_V [N]^T \{X_b\} dV \qquad (3.10.20)$$

From Eq. (3.10.20), we observe three separate types of load contributions from concentrated nodal forces, surface tractions, and body forces, respectively. We define these surface tractions and body-force matrices as

$$\{f_s\} = \iint_{s_1} [N_s]^T \{T_x\} dS \tag{3.10.20a}$$

$$\{f_b\} = \iiint_{v} [N]^T \{X_b\} dV \tag{3.10.20b}$$

The expression for $[f]$ given by Eq. (3.10.20) then describes how certain loads can be considered to best advantage.

Loads calculated by Eqs. (3.10.20a) and (3.10.20b) are called consistent because they are based on the same shape functions $[N]$ used to calculate the element stiffness matrix.

The loads calculated by Eq. (3.10.20a) and (3.10.20b) are also statically equivalent to the original loading; that is, both $\{f_s\}$ and $\{f_b\}$ and the original loads yield the same resultant force and same moment about an arbitrarily chosen point.

The minimization of π_p with respect to each nodal displacement requires that

$$\frac{\partial \pi_p}{\partial u_1} = 0 \quad \text{and} \quad \frac{\partial \pi_P}{\partial u_2} = 0 \tag{3.10.21}$$

Now we explicitly evaluate π_p given by Eq. (3.10.19) to apply Eq. (3.10.21). We define the following for convenience:

$$\{U^*\} = \{d\}^T [B]^T [D]^T [B]\{d\} \tag{3.10.22}$$

Using Eqs. (3.10.10), (3.10.13), and (3.10.15) in Eq. (3.10.22) yields

$$\{U^*\} = [u_1 \quad u_2] \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{3.10.23}$$

Simplifying Eq. (3.10.23), we obtain

$$U^* = \frac{E}{L^2}\left(u_1^2 - 2u_1u_2 + u_2^2\right) \tag{3.10.24}$$

Also, the explicit expression for $\{d\}^T\{f\}$ is

$$\{d\}^T\{f\} = u_1 f_{1x} + u_2 f_{2x} \tag{3.10.25}$$

Therefore, using Eqs. (3.10.24) and (3.10.25) in Eq. (3.10.19) and then applying Eqs. (3.10.21), we obtain

$$\frac{\partial \pi_p}{\partial u_1} = \frac{AL}{2}\left[\frac{E}{L^2}(2u_1 - 2u_2)\right] - f_{1x} = 0$$

and (3.10.26)

$$\frac{\partial \pi_p}{\partial u_2} = \frac{AL}{2}\left[\frac{E}{L^2}(-2u_1 + 2u_2)\right] - f_{2x} = 0$$

In matrix form, we express Eqs. (3.10.26) as

$$\frac{\partial \pi_p}{\partial \{d\}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} - \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.10.27}$$

or, because $\{f\} = [k]\{d\}$, we have the stiffness matrix for the bar element obtained from Eq. (3.10.27) as

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3.10.28a}$$

As expected, Eq. (3.10.28a) is identical to the stiffness matrix Eq. (3.1.14) obtained in Section 3.1.

Now that we have derived the bar stiffness matrix by using the theorem of minimum potential energy, we can observe that the strain energy U [the first term on the right side of Eq. (3.10.18)] can also be expressed in the quadratic form $U = 1/2\{d\}^T[k]\{d\}$ as follows:

$$U = \frac{1}{2}\{d\}^T[k]\{d\} = \frac{1}{2}[u_1 u_2]\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{AE}{2L}[u_1^2 - 2u_1u_2 + u_2^2] \tag{3.10.28b}$$

Finally, instead of the cumbersome process of explicitly evaluating π_p, we can use the matrix differentiation as given by Eq. (2.6.12) and apply it directly to Eq. (3.10.19) to obtain

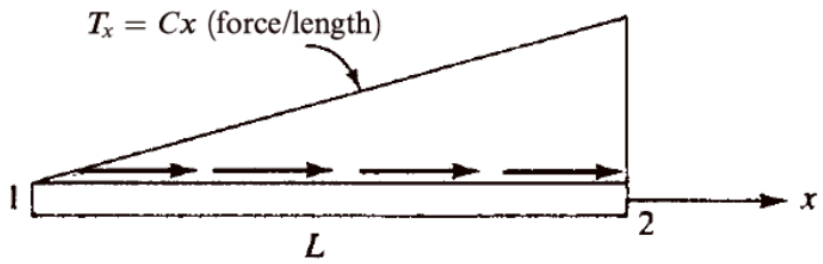
$$\frac{\partial \pi_p}{\partial \{d\}} = AL[B]^T[D][B]\{d\} - \{f\} = 0 \tag{3.10.29}$$

where $[D]^T = [D]$ has been used in writing Eq. (3.10.29). The result of the evaluation of $AL[B]^T[D][B]$ is then equal to $[k]$ given by Eq. (3.10.28a). Throughout this text, we will use this matrix differentiation concept (also see Appendix A), which greatly simplifies the task of evaluating $[k]$.

To illustrate the use of Eq. (3.10.20a) to evaluate the equivalent nodal loads for a bar subjected to axial loading traction T_x, we now solve Example 3.12.

Example:

A bar of length L is subjected to a linearly distributed axial line loading that varies from zero at node 1 to a maximum of CL at node 2 (Figure 3–27). Determine the energy equivalent nodal loads.



■ **Figure 3–27** Element subjected to linearly varying axial line load

SOLUTION:

Using Eq. (3.10.20a) and shape functions from Eq. (3.10.9), we solve for the energy equivalent nodal forces of the distributed loading as follows:

$$\{f_0\} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \int_{S_1} [N]^T \{T_x\} dS = \int_0^L \begin{Bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \{Cx\} dx \qquad (3.10.30)$$

$$= \begin{Bmatrix} \frac{Cx^2}{2} - \frac{Cx^3}{3L} \\ \frac{Cx^3}{3L} \end{Bmatrix}_0^L$$

$$= \begin{Bmatrix} \frac{CL^2}{6} \\ \frac{CL^2}{3} \end{Bmatrix} \tag{3.10.31}$$

where the integration was carried out over the length of the bar, because T_x is in units of force/length.

Note that the total load is the area under the load distribution given by

$$F = \frac{1}{2}(L)(CL) = \frac{CL^2}{2} \tag{3.10.32}$$

Therefore, comparing Eq. (3.10.31) with (3.10.32), we find that the equivalent nodal loads for a linearly varying load are

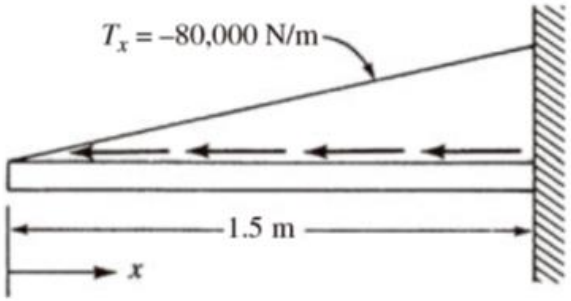
$$\begin{aligned} f_{1x} &= \frac{1}{3}F = \text{ one-third of the total load} \\ f_{2x} &= \frac{2}{3}F = \text{ two-thirds of the total load} \end{aligned} \tag{3.10.33}$$

Note:

For the simple two-noded bar element subjected to a linearly varying load (triangular loading), place one-third of the total load at the node where the distributed loading begins (zero end of the load) and two-thirds of the total load at the node where the peak value of the distributed load ends.

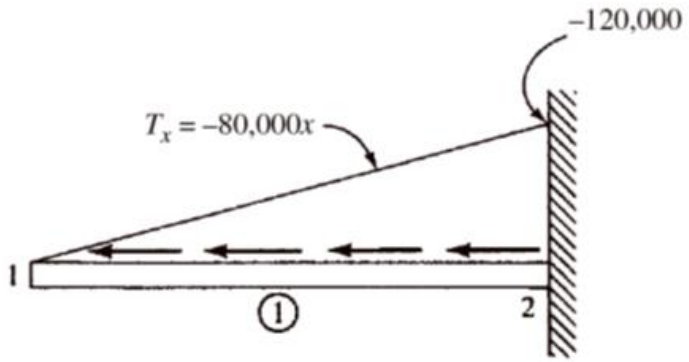
Example:

For the rod loaded axially as shown in Figure 3–28, determine the axial displacement and axial stress. Let $E = 2 \times 10^{11}$ N/m^2, $A = 12.5 \times 10^{-4}$ m^2, and $L = 1.5$ m Use (a) one and (b) two elements in the finite element solutions. In Section 3.11, one-, two-, four-, and eight-element solutions will be presented from the computer program Autodesk [9].



■ **Figure 3–28** Rod subjected to triangular load distribution

(a) One-element solution (Figure 3–31).



■ **Figure 3–29** One-element model

SOLUTION:

From Eq. (3.10.20a), the distributed load matrix is evaluated as follows:

$$\{F_0\} = \int_0^L [N]^T \{T_x\} dx \tag{3.10.34}$$

where T_x is a line load in units of newtons per meter and $\{f_0\} = \{F_0\}$. Therefore, using Eq. (3.2.9) for $[N]$ in Eq. (3.10.34), we obtain

$$\{f_0\} = \int_0^L \begin{Bmatrix} 1 - \dfrac{x}{L} \\ \dfrac{x}{L} \end{Bmatrix} \{-80{,}000x\} dx \tag{3.10.35}$$

or

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-80{,}000L^2}{2} + \dfrac{80{,}000L^2}{3} \\ \dfrac{-80{,}000L^2}{3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-80{,}000L^2}{6} \\ \dfrac{-80{,}000L^2}{3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{-80{,}000(1.5)^2}{6} \\ \dfrac{-80{,}000(1.5)^2}{3} \end{Bmatrix}$$

or

$$F_{1x} = -30{,}000 \text{ N} \qquad F_{2x} = -60{,}000 \text{ N} \tag{3.10.36}$$

Using Eq. (3.10.33), we could have determined the same forces at nodes 1 and 2—that is, one-third of the total load is at node 1 and two-thirds of the total load is at node 2.

Using Eq. (3.10.28), we find that the stiffness matrix is given by

$$[k^{(1)}] = 16.67 \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The element equations are then

$$16.67 \times 10^7 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -30,000 \\ R_{2x} - 60,000 \end{Bmatrix} \tag{3.10.37}$$

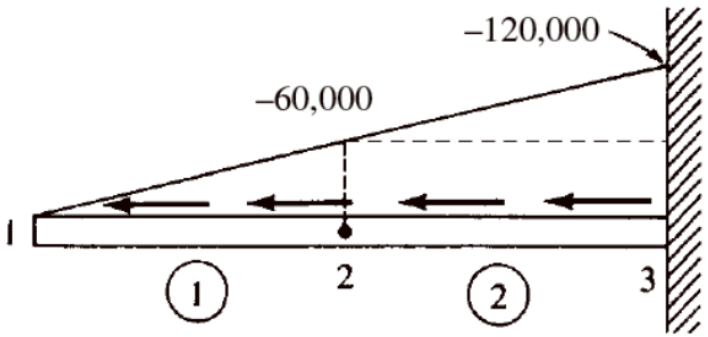
Solving Eq. 1 of Eq. (3.10.37), we obtain

$$u_1 = -0.18 \text{ mm} \tag{3.10.38}$$

The stress is obtained from Eq. (3.10.14) as

$$\begin{aligned} \{\sigma_x\} &= [D]\{\varepsilon_x\} \\ &= E[B]\{d\} \\ &= E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= E\left(\frac{u_2 - u_1}{L}\right) \\ &= 2 \times 10^{11}\left(\frac{0 + 0.00018}{1.5}\right) \\ &= 24 \text{ Mpa } (T) \end{aligned} \tag{3.10.39}$$

(b) Two-element solution (Figure 3–30).



■ **Figure 3–30** Two-element model

We first obtain the element forces. For element 2, we divide the load into a uniform part and a triangular part as shown in Figure 3–30. For the uniform part, half the total uniform load is placed at each node associated with the element. Therefore, the total uniform part is

$$(0.75\text{ m})(-60{,}000\text{ N/m}) = -45{,}000\text{ N}$$

and using Eq. (3.10.33) for the triangular part of the load, we have, for element 2,

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -[\frac{1}{2}(45{,}000) + \frac{1}{3}(22{,}500)] \\ -[\frac{1}{2}(45{,}000) + \frac{2}{3}(22{,}500)] \end{Bmatrix} = \begin{Bmatrix} -30{,}000\text{ N} \\ -37{,}500\text{ N} \end{Bmatrix} \qquad (3.10.40)$$

For element 1, the total force is from the triangle-shaped distributed load only and is given by

$$\frac{1}{2}(0.75\text{ m})(-60{,}000\text{ N/m}) = -22{,}500\text{ N}$$

On the basis of Eq. (3.10.33), this load is separated into nodal forces as shown:

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{3}(-22{,}500) \\ \frac{2}{3}(-22{,}500) \end{Bmatrix} = \begin{Bmatrix} -7500\ \text{N} \\ -15{,}000\ \text{N} \end{Bmatrix} \tag{3.10.41}$$

The final nodal force matrix is then

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{Bmatrix} -7500 \\ -30{,}000 - 15{,}000 \\ R_{3x} - 37{,}500 \end{Bmatrix} \tag{3.10.42}$$

The element stiffness matrices are now

$$[k^{(1)}] = [k^{(2)}] = \frac{AE}{L/2} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{matrix} 2 & 3 \end{matrix} \end{array} = (33.34 \times 10^7) \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \begin{matrix} 2 & 3 \end{matrix} \end{array} \tag{3.10.43}$$

The assembled global stiffness matrix is

$$[K] = (33.34 \times 10^7) \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{3.10.44}$$

The assembled global equations are then

$$(33.34 \times 10^7)\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} -7500 \\ -45{,}000 \\ R_{3x} - 37{,}500 \end{Bmatrix} \tag{3.10.45}$$

where the boundary condition $u_3 = 0$ has been substituted into Eq. (3.10.45). Now, solving equations 1 and 2 of Eq. (3.10.45), we obtain

$$\begin{aligned} u_1 &= -0.18 \text{ mm} \\ u_2 &= -0.1575 \text{ mm} \end{aligned} \tag{3.10.46}$$

The element stresses are as follows:

Element 1

$$\begin{aligned} \sigma_x &= E\begin{bmatrix} -\dfrac{1}{0.75} & \dfrac{1}{0.75} \end{bmatrix}\begin{Bmatrix} u_1 = -0.00018 \\ u_2 = -0.0001575 \end{Bmatrix} \\ &= 6 \text{ Mpa } (T) \end{aligned} \tag{3.10.47}$$

Element 2

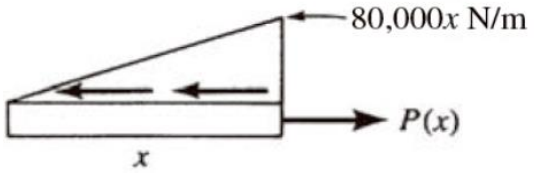
$$\begin{aligned} \sigma_x &= E\begin{bmatrix} -\dfrac{1}{0.75} & \dfrac{1}{0.75} \end{bmatrix}\begin{Bmatrix} u_2 = -0.0001575 \\ u_3 = 0 \end{Bmatrix} \\ &= 42 \text{ Mpa } (T) \end{aligned} \tag{3.10.48}$$

Comparison of Finite Element Solution to Exact Solution for Bar:

- We will now compare the finite element solutions for Example 3.13 using one, two, four, and eight elements to model the bar element and the exact solution.
- The exact solution for displacement is obtained by solving the equation

$$u = \frac{1}{AE}\int_0^x P(x)\,dx \qquad (3.11.1)$$

where, using the following free-body diagram,



we have $$P(x) = \tfrac{1}{2}x(80{,}000x) = 40{,}000x^2 \text{ N} \qquad (3.11.2)$$

Therefore, substituting Eq. (3.11.2) into Eq. (3.11.1), we have

$$u = \frac{1}{AE}\int_0^x 40{,}000x^2\,dx = \frac{40{,}000x^3}{3AE} + C_1 \qquad (3.11.3)$$

or

$$C_1 = -\frac{40{,}000L^3}{3AE} \tag{3.11.4}$$

Substituting Eq. (3.11.4) into Eq. (3.11.3) makes the final expression for displacement

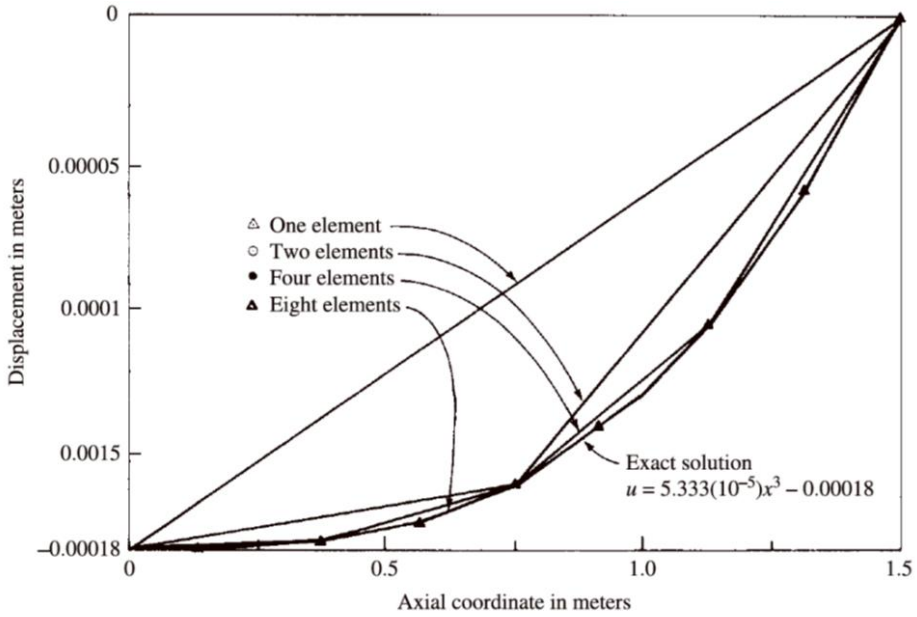
$$u = \frac{40{,}000}{3AE}(x^3 - L^3) \tag{3.11.5}$$

Substituting $A = 12.5 \times 10^{-4}\,\text{m}^2$, $E = 2 \times 10^{11}\ \text{N/m}^2$, and $L = 1.5$ m into Eq. (3.11.5), we obtain

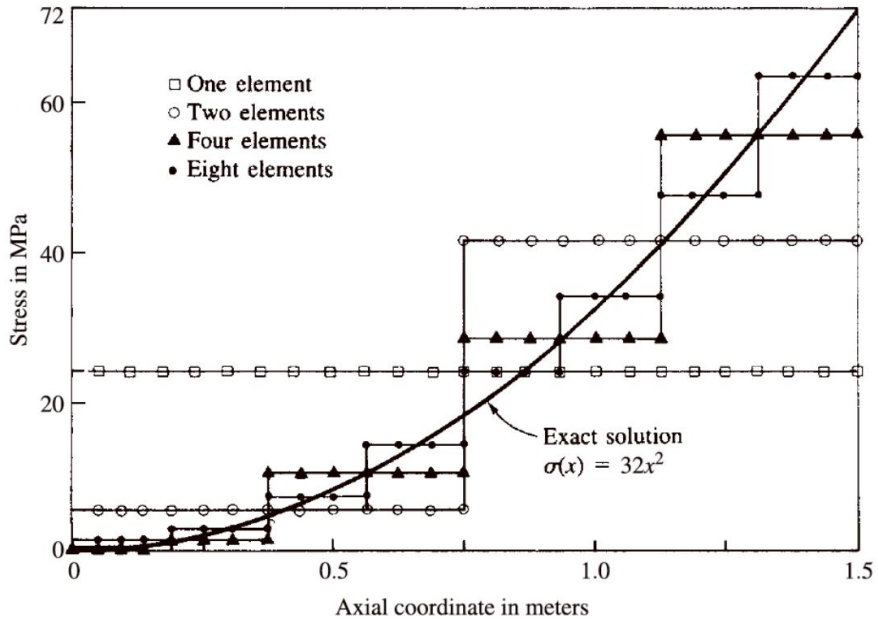
$$u = 5.333 \times 10^{-5}x^3 - 0.00018 \tag{3.11.6}$$

The exact solution for axial stress is obtained by solving the equation

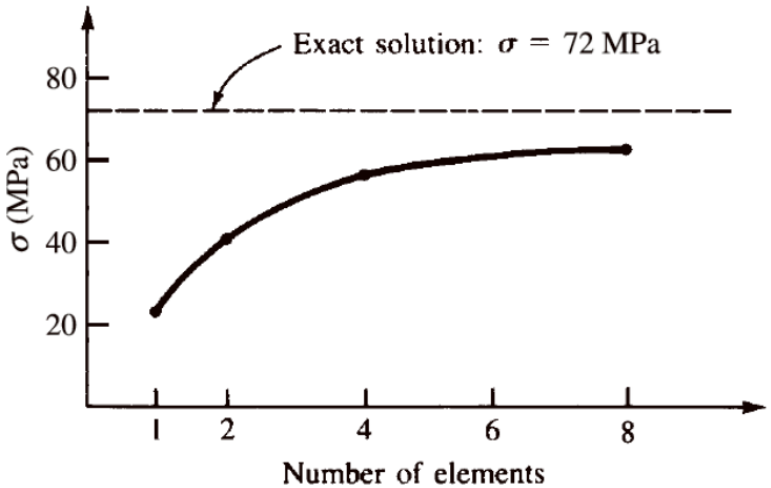
$$\sigma(x) = \frac{P(x)}{A} = \frac{40{,}000x^2}{12.5 \times 10^{-4}\ \text{m}^2} = 32x^2\ \text{Pa} \tag{3.11.7}$$



■ **Figure 3–31** Comparison of exact and finite element solutions for axial displacement (along length of bar)



■ **Figure 3–32** Comparison of exact and finite element solutions for axial stress (along length of bar)

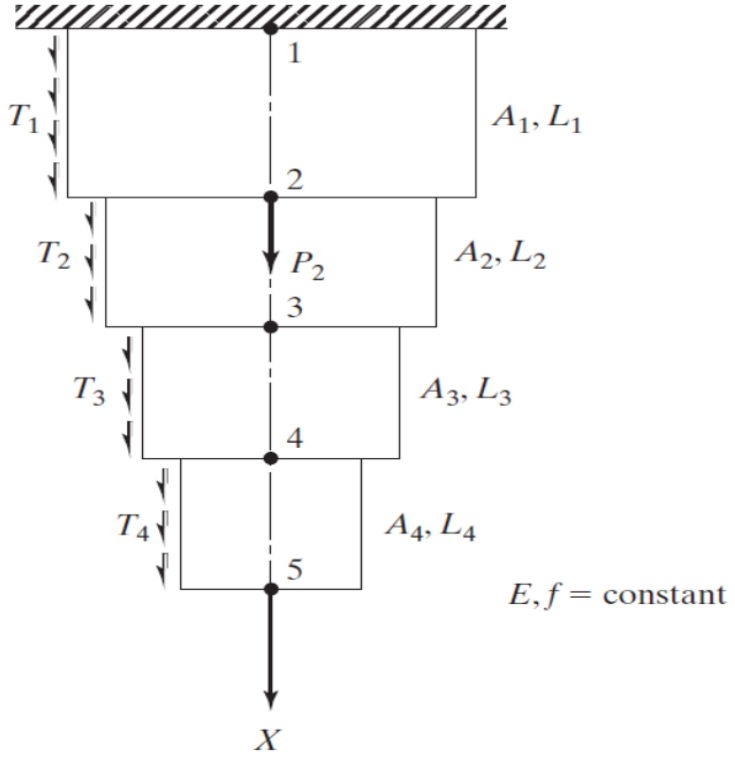


■ **Figure 3–34** Axial stress at fixed end as number of elements increases

Some conclusions from these comparison of results:

1) The finite element solutions match the exact solution at the node points.
2) Although the node values for displacement match the exact solution, the values at locations between the nodes are poor because we used a linear displacement function within each element, whereas the exact solution is a cubic function.
3) The stress is derived from the slope of the displacement curve as $\sigma = E\varepsilon = E(\frac{du}{dx})$. Therefore, by the finite element solution, because u is a linear function in each element, axial stress is constant in each element.
4) The best approximation of the stress occurs at the midpoint of the element, not at the nodes. This is because the derivative of displacement is better predicted between the nodes than at the nodes.
5) The stress is not continuous across element boundaries. Therefore, equilibrium is not satisfied across element boundaries. Also, equilibrium within each element is, in general, not satisfied.
6) The axial stress at the fixed end (x=L) converges as the number of elements increases.

Example: The properties of elements of the bar shown below are as indicated in the figure. Each element is subjected to a traction force T_i per unit length and a body force f per unit volume. The units of T_i , f , A_i , and so on are assumed to be consistent. The Young's modulus of the material is E. A concentrated load P_2 is applied at node 2. Determine the structural stiffness matrix and the nodal load vector.



- The element stiffness matrix for each element *i* is obtained from Eq. (23) as:

$$[k^{(i)}] = \frac{EA_i}{\ell_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The element connectivity table is given below:

Element	1	2
1	1	2
2	2	3
3	3	4
4	4	5

The element stiffness matrices can be "expanded" using the connectivity table and then summed (or assembled) to obtain the structural stiffness matrix as follows:

$$\mathbf{K} = \frac{EA_1}{\ell_1}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA_2}{\ell_2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA_3}{\ell_3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{EA_4}{\ell_4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

which gives

$$\mathbf{K} = E\begin{bmatrix} \frac{A_1}{\ell_1} & -\frac{A_1}{\ell_1} & 0 & 0 & 0 \\ -\frac{A_1}{\ell_1} & \left(\frac{A_1}{\ell_1}+\frac{A_2}{\ell_2}\right) & -\frac{A_2}{\ell_2} & 0 & 0 \\ 0 & -\frac{A_2}{\ell_2} & \left(\frac{A_2}{\ell_2}+\frac{A_3}{\ell_3}\right) & -\frac{A_3}{\ell_3} & 0 \\ 0 & 0 & -\frac{A_3}{\ell_3} & \left(\frac{A_3}{\ell_3}+\frac{A_4}{\ell_4}\right) & -\frac{A_4}{\ell_4} \\ 0 & 0 & 0 & -\frac{A_4}{\ell_4} & \frac{A_4}{\ell_4} \end{bmatrix}$$

$$\{f_b\} = \iiint_v [N]^T \{X_b\} dV$$

The load vector for each element is determined from Eq. (20) as

$$\{f\} = \{P\} + \iint_{s_1} [N_s]^T \{T_x\} dS + \iiint_V [N]^T \{X_b\} dV$$

1. Traction: $\{f_s\} = \iint_{s_1} [N_s]^T \{T_x\} dS$

$$\{f_s\}^{(1)} = \int_0^{L_1} \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} (T_1) dx = T_1 \begin{bmatrix} L_1 - \frac{L_1}{2} \\ \frac{L_1}{2} \end{bmatrix} = \frac{T_1 L_1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\{f_s\}^{(2)} = \frac{T_2 L_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \{f_s\}^{(3)} = \frac{T_3 L_3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \{f_s\}^{(4)} = \frac{T_4 L_4}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. Body forces: $\{f_b\} = \iiint_v [N]^T \{X_b\} dV$

$$\{f_b\}^{(1)} = \int_0^{L_1} \begin{bmatrix} 1 - \frac{x}{L} \\ \frac{x}{L} \end{bmatrix} (f) A_1 dx = A_1 f \begin{bmatrix} L_1 - \frac{L_1}{2} \\ \frac{L_1}{2} \end{bmatrix} = \frac{A_1 L_1 f}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\{f_b\}^{(2)} = \frac{A_2 L_2 f}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \{f_b\}^{(3)} = \frac{A_3 L_3 f}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \{f_b\}^{(4)} = \frac{A_4 L_4 f}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

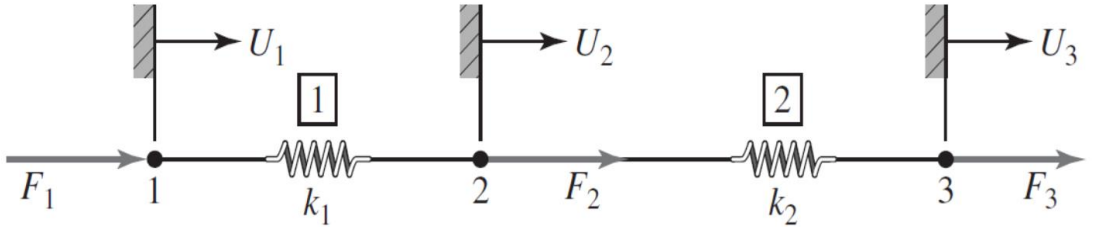
Assembling the traction and body forces matrices with the external load vector

$$\{f\} = \{f_s\} + \{f_b\} + \{P\} = \begin{Bmatrix} \frac{A_1 L_1 f}{2} + \frac{T_1 L_1}{2} \\ \frac{A_1 L_1 f}{2} + \frac{T_1 L_1}{2} + \frac{A_2 L_2 f}{2} + \frac{T_2 L_2}{2} \\ \frac{A_2 L_2 f}{2} + \frac{T_2 L_2}{2} + \frac{A_3 L_3 f}{2} + \frac{T_3 L_3}{2} \\ \frac{A_3 L_3 f}{2} + \frac{T_3 L_3}{2} + \frac{A_4 L_4 f}{2} + \frac{T_4 L_4}{2} \\ \frac{A_4 L_4 f}{2} + \frac{T_4 L_4}{2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ P_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

System Assembly in Global Coordinates:

Notes on element loads at node between two element:

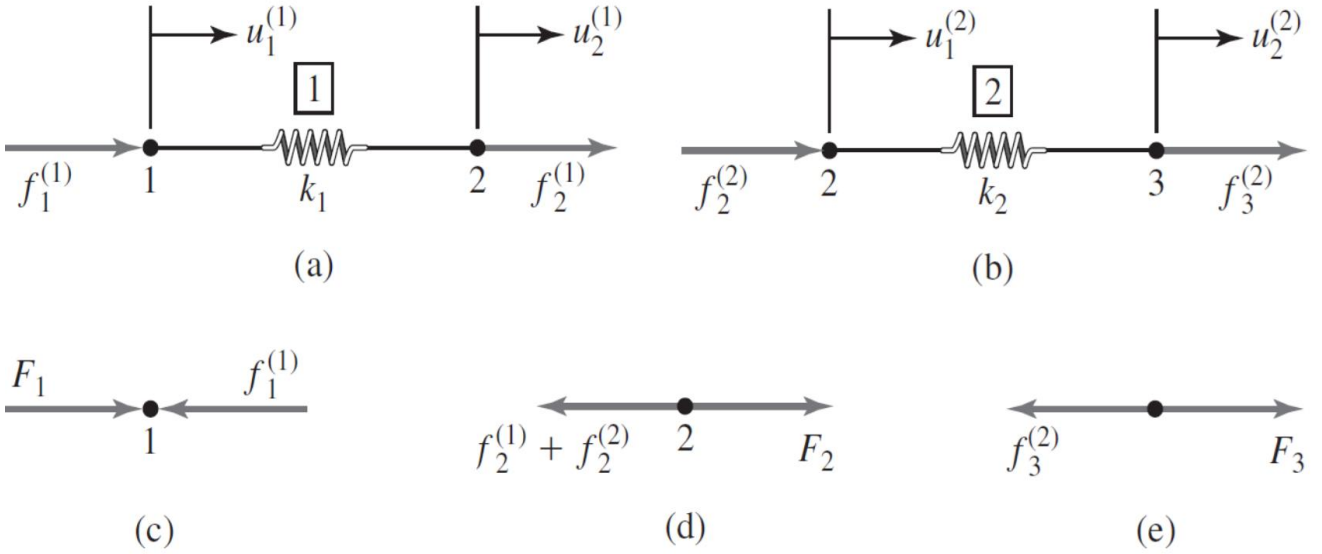
Consider the system which has three nodes and two elements.



The two elements are subjected to three point loads.

Force F_2 at node 2 is common for the two elements.

The free body diagram for the two elements shows that the force at node 2 is divided by the two elements.



- Element matrices are:

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Express the element DOF in terms of the global system DOF.

$$u_1^{(1)} = U_1 \qquad u_2^{(1)} = U_2 \qquad u_1^{(2)} = U_2 \qquad u_2^{(2)} = U_3$$

Substitute the global DOF in element matrices.

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

Assembling element matrices to construct the global system matrices.

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

From the free body diagram it can be found that

$$f_1^{(1)} = F_1 \qquad f_2^{(1)} + f_2^{(2)} = F_2 \qquad f_3^{(2)} = F_3$$

The global matrices will be

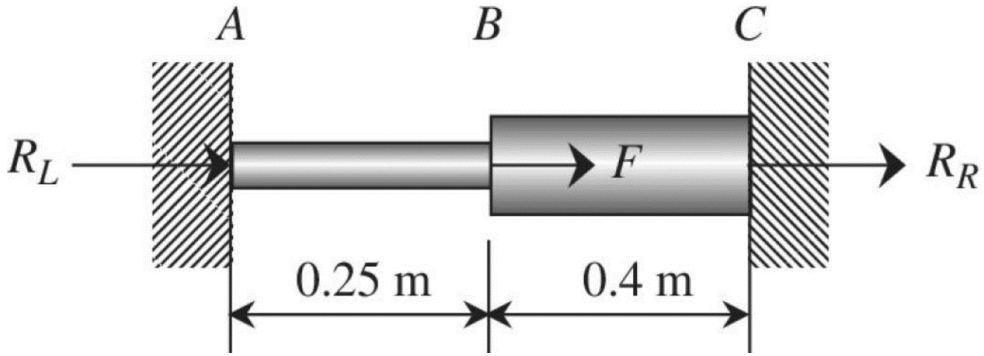
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Where the global stiffness matrix is usually written in upper case K as

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

Q1: Use FEM to determine the reactions at the supports. E= 100 GPa, A(AB)= 0.0001m^2 , A(BC)= 0.0002m^2, and F= 10 kN.

Ans. R_L= -4.444 kN, R_R= 5.556 kN

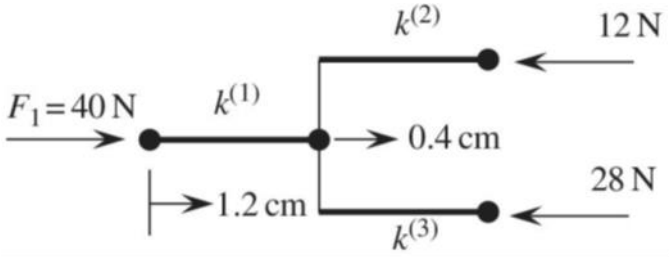


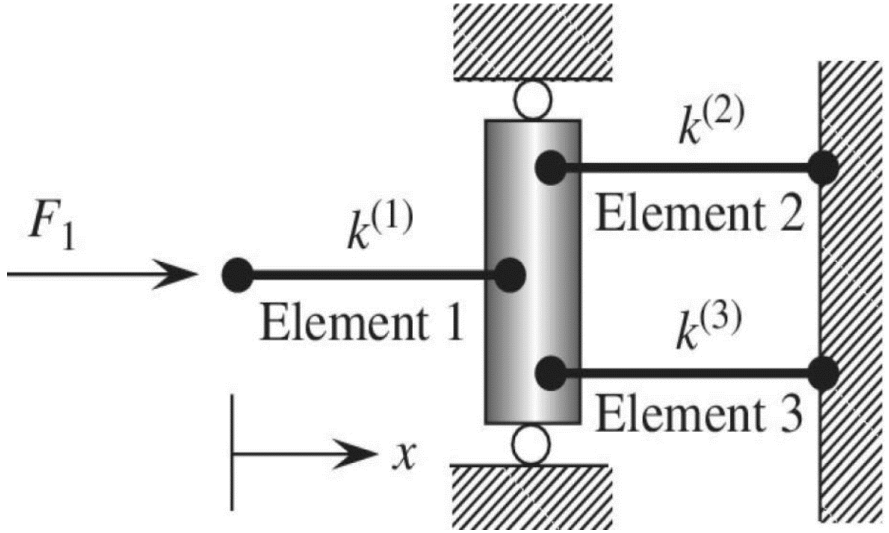
Q2: Determine the displacement of rigid member, element forces an reactions at wall.

$k^{(1)}$= 50N/cm, $k^{(1)}$= 30N/cm, $k^{(1)}$= 70N/cm, and

F_1= 40 N.

Ans.

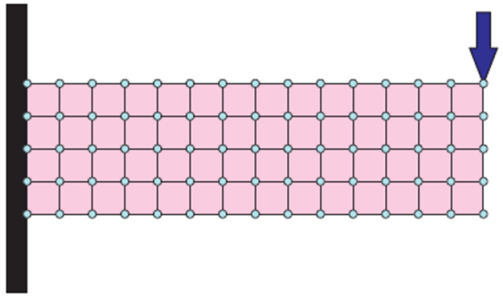


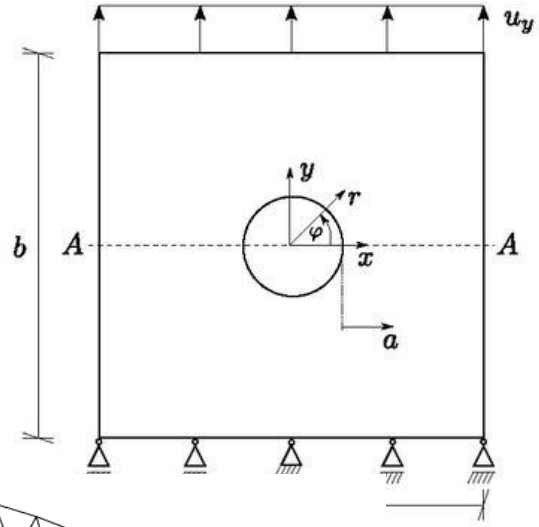


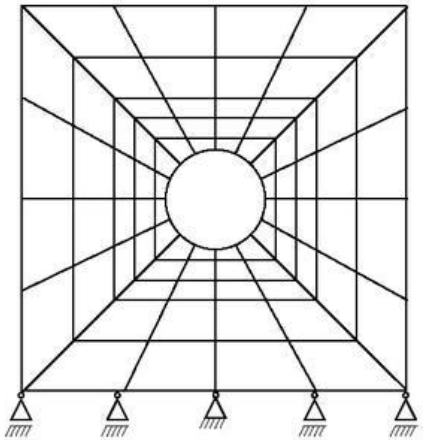
Derivation of Finite Element Equations for Two Dimensional Elements

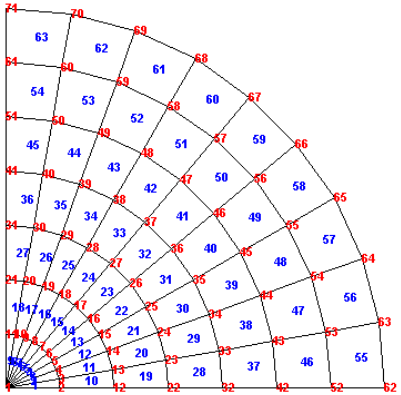
Plane Stress and Plane Strain Stiffness Equations

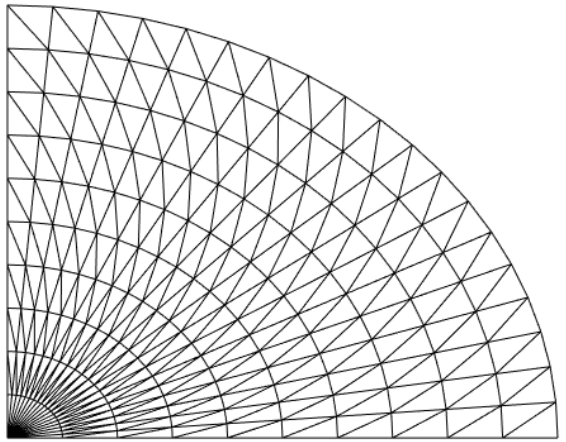
- Two-dimensional (planar) elements are defined by three or more nodes in a two-dimensional plane (that is, $x - y$).
- The elements are connected at common nodes and/or along common edges to form continuous structures.











(b)

- Nodal displacement compatibility is then enforced during the formulation of the nodal equilibrium equations for two-dimensional elements.
- If proper displacement functions are chosen, compatibility along common edges is also obtained.
- The two-dimensional element is extremely important for:

(1) plane stress analysis, which includes problems such as plates with holes, fillets, or other changes in geometry that are loaded in their plane resulting in local stress concentrations, and

(2) plane strain analysis, which includes problems such as a long underground box culvert subjected to a uniform load acting constantly over its length, a long, cylindrical control rod subjected to a load that remains constant over the rod length (or depth), and dams and pipes subjected to loads that remain constant over their lengths.

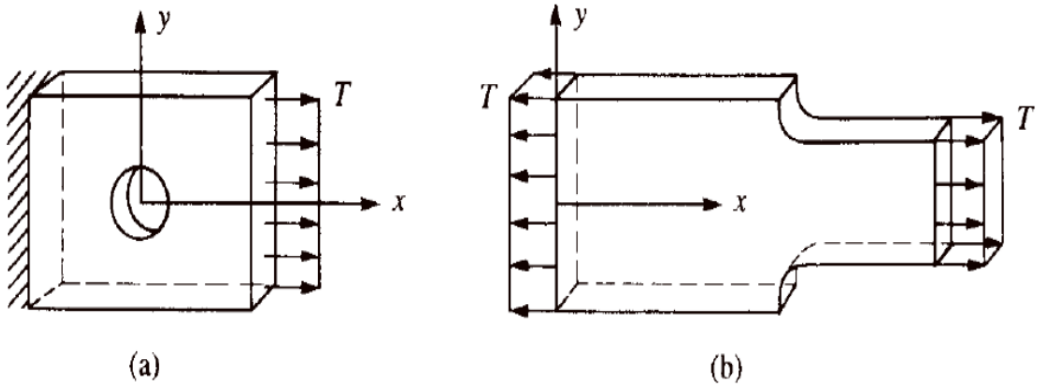
- We begin with the development of the stiffness matrix for a basic two-dimensional or plane finite element, called the *constant-strain triangular element*.
- The constant-strain triangle (CST) stiffness matrix is considered first because its derivation is the simplest among the available two-dimensional elements.
- The element is called a CST because it has a constant strain throughout it.
- We will derive the CST stiffness matrix by using the principle of minimum potential energy because the energy formulation is the most feasible for the development of the equations for both two- and three-dimensional finite elements.
- Finally, the development of the stiffness matrix for the simple four-noded rectangular (Q4) element will be considered and compared with the finite element solution to a beam bending problem modeled using the CST and Q4 elements.

Basic Concepts of Plane Stress and Plane Strain:

- These concepts are important because the developments in this chapter are directly applicable only to systems assumed to behave in a plane stress or plane strain manner.

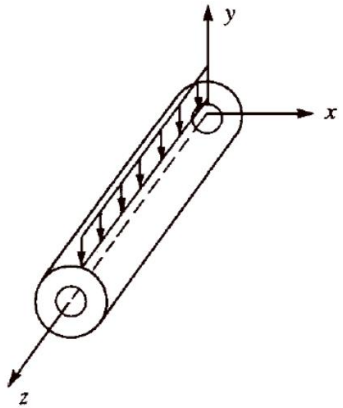
I- Plane Stress:

- **Plane stress** *is defined to be a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero*.
- For instance the plates in the $x - y$ plane shown subjected to surface tractions T (pressure acting on the surface edge or face of a member in units of force/area) in the plane are under a state of plane stress; that is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.
- Generally, members that are thin (those with a small z dimension compared to the in-plane x and y dimensions) and whose loads act only in the $x - y$ plane can be considered to be under plane stress.



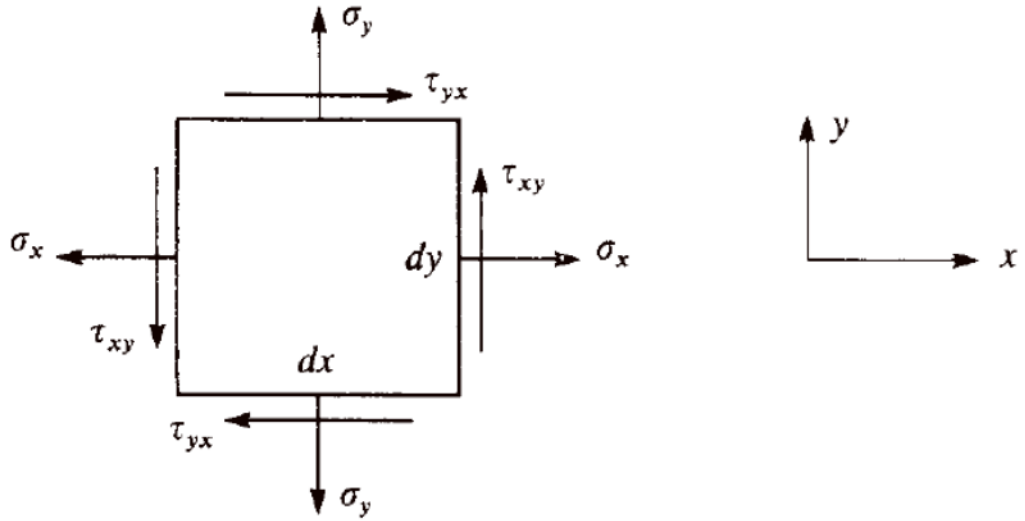
II- Plane Strain:

- Plane strain is defined to be a state of strain in which the strain normal to the x – y plane ε_z and the shear strains γ_{xz} and γ_{yz} are assumed to be zero.
- The assumptions of plane strain are realistic for long bodies (say, in the z direction) with constant cross-sectional area subjected to loads that act only in the x and/or y directions and do not vary in the z direction.
- Some plane strain examples are shown in figure.
- In these examples, only a unit thickness (1 mm or 1 m) of the structure is considered because each unit thickness behaves identically (except near the ends).
- The finite element models of the structures in figure consist of appropriately discretized cross sections in the *x* – *y* plane with the loads acting over unit thicknesses in the *x* and/or *y* directions only.



Two-Dimensional State of Stress and Strain:

- The infinitesimal element with sides d_x and d_y has normal stresses σ_x and σ_y acting in the *x* and *y* directions (here on the vertical and horizontal faces), respectively.
- The shear stress τ_{xy} acts on the *x* edge (vertical face) in the *y* direction.
- The shear stress τ_{yx} acts on the *y* edge (horizontal face) in the *x* direction.
- Moment equilibrium of the element results in τ_{xy} being equal in magnitude to τ_{yx}.



- Hence, three independent stresses exist and are represented by the vector column matrix

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{6.1.1}$$

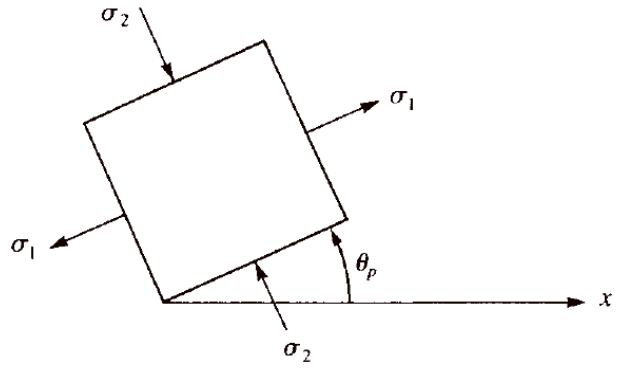
The stresses given by Eq. (6.1.1) will be expressed in terms of the nodal displacement degrees of freedom. Hence, once the nodal displacements are determined, these stresses can be evaluated directly.

Recall from strength of materials [2] that the **principal stresses**, which are the maximum and minimum normal stresses in the two-dimensional plane, can be obtained from the following expressions:

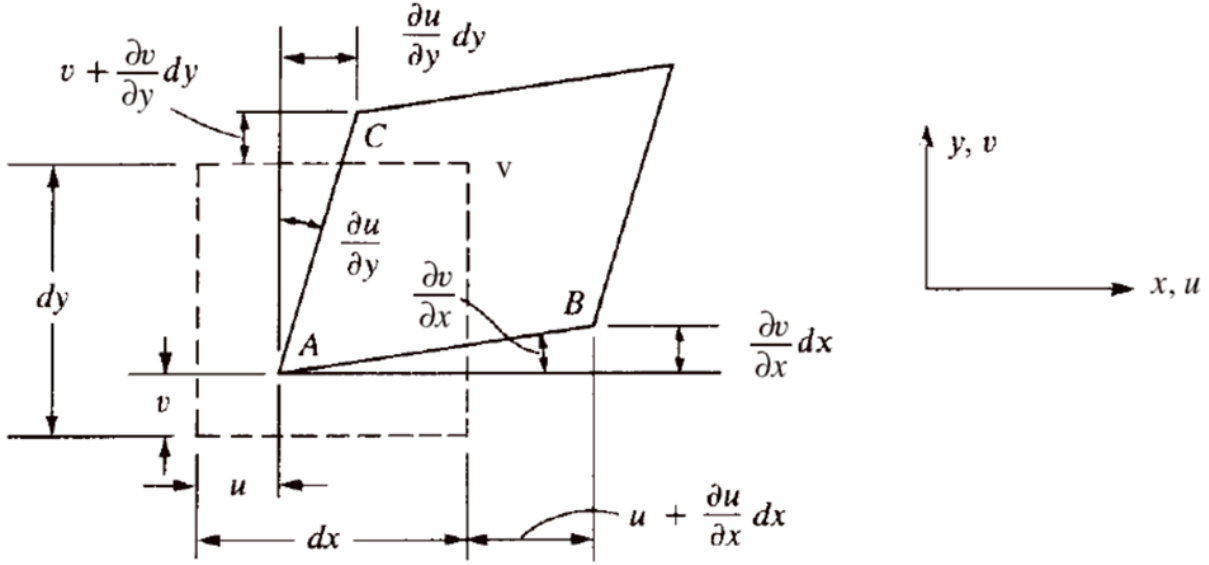
$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\max}$$
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\min} \tag{6.1.2}$$

Also, the **principal angle** θ_p, which defines the normal whose direction is perpendicular to the plane on which the maximum or minimum principal stress acts, is defined by

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{6.1.3}$$



- From the general definitions of normal and shear strains, we obtain



$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{6.1.4}$$

The strains are generally represented by the vector column matrix

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{6.1.5}$$

For plane stress

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \qquad (6.1.6)$$

$$\gamma_{xz} = \gamma_{yz} = 0, \text{ but } \varepsilon_z \neq 0.$$

$$\{\sigma\} = [D]\{\varepsilon\} \qquad (6.1.7)$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \qquad (6.1.8)$$

[D] is called the *stress*/strain matrix (or *constitutive matrix*)

- For plane strain, we assume the following strains to be zero:

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \qquad (6.1.9)$$

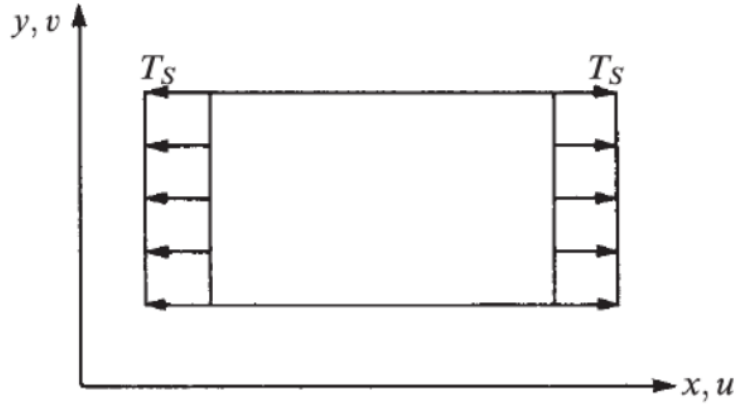
$$\tau_{xz} = \tau_{yz} = 0, \text{ but } \sigma_z \neq 0.$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \qquad (6.1.10)$$

The $\{\sigma\}$ and $\{\varepsilon\}$ matrices remain the same as for the plane stress case.

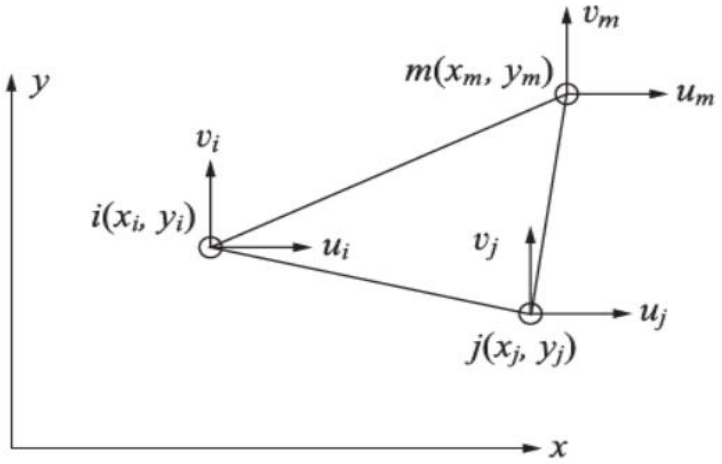
Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations:

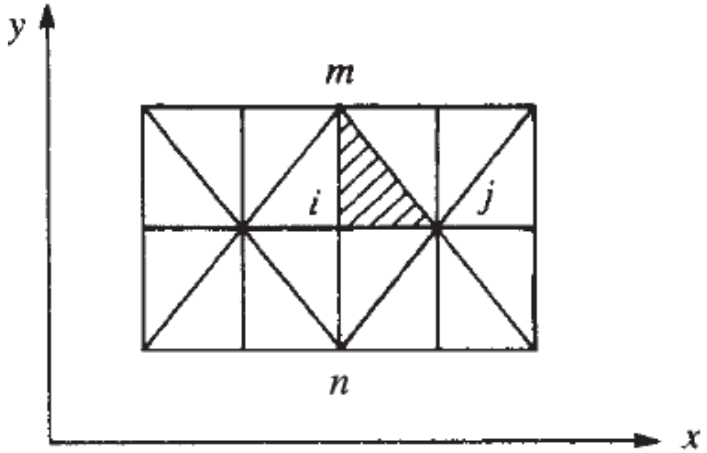
- Consider the thin plate subjected to tensile surface traction loads T_S in the figure.



Step 1: Select Element Type

- The basic element is the linear displacement triangular (LDT) element.
- The discretized plate will be as shown.



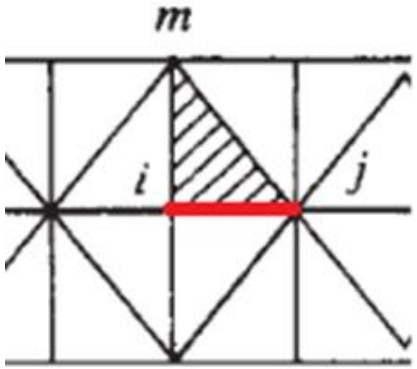


- The triangular elements is used because boundaries of irregularly shaped bodies can be closely approximated, and because the expressions related to the triangular element are comparatively simple.
- This discretization is called a *coarse-mesh generation* if a few large elements are used.
- Each node has two degrees of freedom; an x and a y displacement.
- Let u_i and v_i represent the node i displacement components in the x and y directions, respectively.
- Here all formulations are based on this counterclockwise system of labeling of nodes, although a formulation based on a clockwise system of labeling could be used.

- Remember that a consistent labeling procedure for the whole body is necessary to avoid problems in the calculations such as negative element areas.
- Here (x_i, y_i), (x_j, y_j), and (x_m, y_m) are the known nodal coordinates of nodes i, j, and m, respectively.
- The nodal displacement matrix is given by

$$\{d\} = \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad (1)$$

- **Step 2: Select Displacement Functions**



- We select a linear displacement function for each element as:

$$u(x,y) = a_1 + a_2\,x + a_3\,y \qquad (2)$$

$$v(x,y) = a_4 + a_5\,x + a_6\,y$$

where *u(x,y)* and *v(x,y)* describe displacements at any interior point (*x*, *y*) of the element.

- The linear function ensures that compatibility will be satisfied.
- A linear function with specified endpoints has only one path through which to pass—that is, through the two points.
- Hence, the linear function ensures that the displacements along the edge and at the nodes shared by adjacent elements, such as edge i-j of the two elements are equal.

- Using Eqs. (2), the general displacement function $\{\psi\}$, which includes the functions u and v, can be expressed as

$$\{\psi\} = \begin{Bmatrix} a_1 + a_2x + a_3y \\ a_4 + a_5x + a_6y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \tag{3}$$

- To obtain the a s in Eqs. (2), we begin by substituting the coordinates of the nodal points into Eqs. (2) to yield

$$\begin{aligned} u_i &= u(x_i, y_i) = a_1 + a_2x_i + a_3y_i \\ u_j &= u(x_j, y_j) = a_1 + a_2x_j + a_3y_j \\ u_m &= u(x_m, y_m) = a_1 + a_2x_m + a_3y_m \\ v_i &= v(x_i, y_i) = a_4 + a_5x_i + a_6y_i \\ v_j &= v(x_j, y_j) = a_4 + a_5x_j + a_6y_j \\ v_m &= v(x_m, y_m) = a_4 + a_5x_m + a_6y_m \end{aligned} \tag{4}$$

- We can solve for the a's beginning with the first three of Eqs. (4) expressed in matrix form as:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (5)$$

And solving for $\{a\}$ to have

$$\{a\} = [x]^{-1}\{u\} \qquad (6)$$

where [x] is the 3× 3 matrix on the right side of Eq. (5).

The method of cofactors is one possible method for finding the inverse of [x].

Thus,

$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \qquad (7)$$

where

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} \tag{8}$$

is the determinant of [x], which equals

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j) \tag{9}$$

Here A is the area of the triangular element.

$$\begin{aligned} \alpha_i &= x_j y_m - y_j x_m & \alpha_j &= y_i x_m - x_i y_m & \alpha_m &= x_i y_j - y_i x_j \\ \beta_i &= y_j - y_m & \beta_j &= y_m - y_i & \beta_m &= y_i - y_j \\ \gamma_i &= x_m - x_j & \gamma_j &= x_i - x_m & \gamma_m &= x_j - x_i \end{aligned} \tag{10}$$

Thus, eq. (6) will be

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \tag{11}$$

- Similarly, using the last three of eqs (4) to obtain

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix} \qquad (12)$$

- The general x displacement function $u(x, y)$ of $\{\psi\}$ will be in terms of the coordinate variables x and y, known coordinate variables α_i, α_j , α_m, β_i …, γ_m and unknown nodal displacements u_i , u_j, and u_m.
- Beginning with eqs. (2) expressed in matrix form, we have

$$\{u\} = [1 \quad x \quad y]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (13)$$

- Substituting Eq. (11) into Eq. (13), we obtain

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \tag{14}$$

- Expanding eq. (14) gives:

$$\{u\} = \frac{1}{2A}[1 \quad x \quad y]\begin{Bmatrix} \alpha_i u_i + \alpha_i u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix} \tag{15}$$

- Multiplying the two matrices in Eq. (15) and rearranging, we obtain

$$u(x,y) = \frac{1}{2A}\{(\alpha_1 + \beta_i x + \gamma_i y)u_i + (\alpha_j + \beta_j x + \gamma_j y)u_j + (\alpha_m + \beta_m x + \gamma_m y)u_m\} \tag{16}$$

- The y displacement function $v(x,y)$, can be obtained by replacing u_i, u_j, and u_m in equation (6) by v_i, v_j, and v_m, respectively

$$v(x,y) = \frac{1}{2A}\{(\alpha_i + \beta_i x + \gamma_i y)v_i + (\alpha_j + \beta_j x + \gamma_j y)v_j + (\alpha_m + \beta_m x + \gamma_m y)v_m\} \quad (17)$$

Define the shape functions as:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i y)$$
$$N_j = \frac{1}{2A}(\alpha_j + \beta_j x + \gamma_j y) \quad (18)$$
$$N_m = \frac{1}{2A}(\alpha_m + \beta_m x + \gamma_m y)$$

Which reduces eqs.(16) and (17) to

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m$$
$$v(x,y) = N_i v_i + N_j v_j + N_m v_m \quad (19)$$

In matrix form

$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \quad (20)$$

Finally, expressing Eq. (20) in abbreviated matrix form, yields

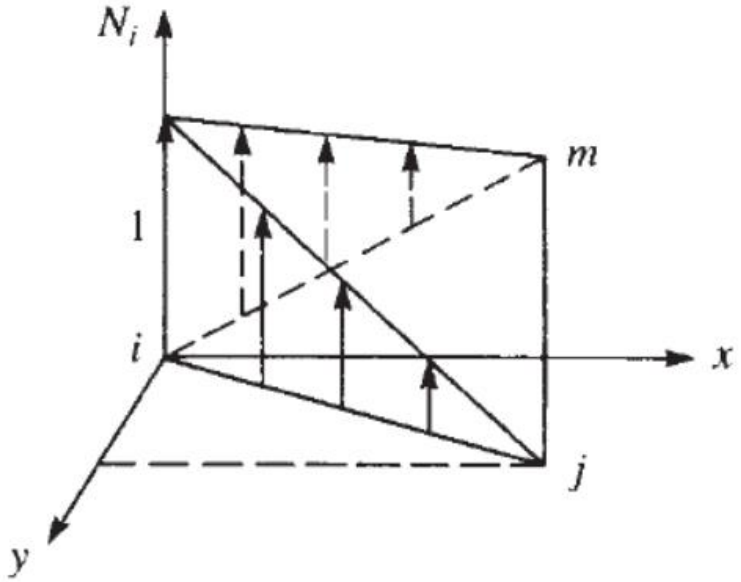
$$\{\psi\} = [N]\{d\} \quad (21)$$

where [N] is given by

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \quad (22)$$

- Now the general displacements have expressed as functions of {d}, in terms of the shape functions N_i , N_j, and N_m.
- The shape functions represent the shape of $\{\psi\}$ when plotted over the surface of a typical element.
- For instance, N_i represents the shape of the variable u when plotted over the surface of the element for u_i =1 and all other degrees of freedom equal to zero, that is, $u_j = u_m = v_i = v_j = v_m = 0$.

- The variation of N_i plotted over the surface of a typical element is shown in figure.
- Note that N_i does not equal zero except along a line connecting and including nodes j and m.
- Finally, $N_i + N_j + N_m = 1$ for all x and y locations on the surface of the element.



EXAMPLE 3.4

The temperatures at the nodes of a triangular element are given by $T_i = 210\ °F$, $T_j = 270\ °F$, and $T_k = 250\ °F$. If the nodal coordinates are $(x_i, y_i) = (50, 30)$ in, $(x_j, y_j) = (70, 50)$ in, and $(x_k, y_k) = (55, 60)$ in, determine (a) the shape functions of the element and (b) temperature at the point $(x, y) = (60, 40)$ in the element.

Solution

1. From the known nodal coordinates, the area of the triangular element and the constants a_i, b_i, c_i, ... involved in the shape functions can be determined as

$$A = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - x_i y_k - x_j y_i - x_k y_j)$$

$$= \frac{1}{2}(50 \times 50 + 70 \times 60 + 55 \times 30 - 50 \times 60 - 70 \times 30 - 55 \times 50) = 250\ \text{in}^2$$

$$a_i = x_j y_k - x_k y_j = 70 \times 60 - 55 \times 50 = 1450$$

$$a_j = x_k y_i - x_i y_k = 55 \times 30 - 50 \times 60 = -1350$$

$$a_k = x_i y_j - x_j y_i = 50 \times 50 - 70 \times 30 = 400$$

$$b_i = y_j - y_k = 50 - 60 = -10$$

$$b_j = y_k - y_i = 60 - 30 = 30$$

$$b_k = y_i - y_j = 30 - 50 = -20$$

$$c_i = x_k - x_j = 55 - 70 = -15$$

$$c_j = x_i - x_k = 50 - 55 = -5$$

$$c_k = x_j - x_i = 70 - 50 = 20$$

Notes:

- Formulation of problems of heat transfer is similar to plain stress/strain problems.
- In this example the symbols a, b, and c are used instead of α, β, and γ

The shape functions can be found as

$$N_i(x,y) = \frac{1}{2A}(a_i + b_i x + c_i y) = \frac{1}{500}(1450 - 10x - 15y) = 2.9 - 0.02x - 0.03y$$

$$N_j(x,y) = \frac{1}{2A}(a_j + b_j x + c_j y) = \frac{1}{500}(-1350 + 30x - 5y) = -2.7 + 0.06x - 0.01y$$

$$N_k(x,y) = \frac{1}{2A}(a_k + b_k x + c_k y) = \frac{1}{500}(400 - 20x + 20y) = 0.8 - 0.04x + 0.04y$$

2. The temperature distribution in the element can be expressed as

$$T(x,y) = N_i(x,y)T_i + N_j(x,y)T_j + N_k(x,y)T_k$$

$$= 210(2.9 - 0.02x - 0.03y) + 270(-2.7 + 0.06x - 0.01y) + 250(0.8 - 0.04x + 0.04y)$$

The temperature at the point $(x, y) = (60, 40)$ in can be found as

$$T(60,40) = 210(2.9 - 1.2 - 1.2) + 270(-2.7 + 3.6 - 0.4) + 250(0.8 - 2.4 + 1.6) = 240^\circ\text{F}$$

EXAMPLE 3.9

The nodal coordinates of a triangular plate element subjected to in-plane loads follow:

Node i: $(x_i, y_i) = (10, 10)$ mm

Node j: $(x_j, y_j) = (20, 50)$ mm

Node k: $(x_k, y_k) = (-10, 30)$ mm

The in-plane displacement components of the nodes are given by

$$(u_i v_i) = (1, -1) \text{ mm}, (u_j, v_j) = (-2, 1) \text{ mm}, (u_k, v_k) = (0.5, -0.5) \text{ mm}$$

Express the variations of $u(x, y)$ and $v(x, y)$ in the element in terms of the shape functions.

Approach: Express the linear variations of $u(x, y)$ and $v(x, y)$ in the element using Eqs. (3.51) and (3.52).

Solution

The shape functions of the element can be derived as (see Problem 3.56)

$$N_i(x, y) = 1.1 + 0.02x - 0.03y$$
$$N_j(x, y) = -0.4 + 0.02x + 0.02y$$
$$N_k(x, y) = 0.3 - 0.04x + 0.01y$$

The variation of the u-displacement inside the element can be expressed, using Eq. (3.51), as:

$$u(x, y) = (1.1 + 0.02x - 0.03y)u_i + (-0.4 + 0.02x + 0.02y)u_j + (0.3 - 0.04x + 0.01y)u_k$$
$$= (1.1 + 0.02x - 0.03y)(1) + (-0.4 + 0.02x + 0.2y)(-2) + (0.3 - 0.04x + 0.01y)(0.5)$$
$$= 2.05 - 0.04x - 0.065y$$

Similarly, the variation of the v-displacement inside the element can be expressed, using Eq. (3.52), as:

$$v(x, y) = (1.1 + 0.02x - 0.03y)v_i + (-0.4 + 0.02x + 0.02y)v_j + (0.3 - 0.04x + 0.01y)v_k$$
$$= (1.1 + 0.02x - 0.03y)(-1) + (-0.4 + 0.02x + 0.02y)(1) + (0.3 - 0.04x + 0.01y)(-0.5)$$
$$= -1.65 + 0.02x + 0.045y$$

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

- The element strains and stresses are expressed in terms of the unknown nodal displacements.

Element Strains:

- The strains associated with the two-dimensional element are given by:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad (23\text{-a})$$

Substituting displacement functions for *u* and *v* from Eqs. (2) into Eq. (23a), yields

$$\varepsilon_x = a_2 \quad , \quad \varepsilon_y = a_6 \, , \quad \gamma_{xy} = a_3 + a_5 \qquad (23\text{-b})$$

- It can be observed from Eq. (23-b) that the strains in the element are constant.
- The element is then called a constant-strain triangle (CST).
- It should be also noted that based on the assumption of choosing displacement functions that are linear in x and y, all lines in the triangle element remain straight as the element deforms.
- Using eqs. (19) for the displacements, we have

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}(N_i u_i + N_j u_j + N_m u_m) \tag{24}$$

or

$$u_{,x} = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m \tag{25}$$

where the comma followed by a variable indicates differentiation with respect to that variable.

- We have used $u_{i,x} = 0$ because $u_{i,x} = u(x_i , y_i)$ is a constant value (a nodal value); similarly, $u_{j,x} = 0$ and $u_{m,x} = 0$.

- Using Eqs. (18), we can evaluate the expressions for the derivatives of the shape functions in Eq. (25) as follows:

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}(\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A} \tag{26}$$

similarly

$$N_{j,x} = \frac{\beta_j}{2A} \qquad \text{and} \qquad N_{m,x} = \frac{\beta_m}{2A} \tag{27}$$

- Therefore, using eqs. (26) and (27) in Eq. (25), we have

$$\frac{\partial u}{\partial x} = \frac{1}{2A}(\beta_i u_i + \beta_j u_j + \beta_m u_m) \tag{28}$$

similarly

$$\frac{\partial v}{\partial y} = \frac{1}{2A}(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m) \tag{29}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}(\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m)$$

- Using Eqs. (28) and (29) in Eq. (23-a), we obtain

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \tag{30}$$

- Or in abbreviated matrices

$$\{\varepsilon\} = \begin{bmatrix} [B_i] & [B_j] & [B_m] \end{bmatrix} \begin{Bmatrix} \{d_i\} \\ \{d_j\} \\ \{d_m\} \end{Bmatrix} \tag{31}$$

where

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix} \tag{32}$$

- Finally, in simplified matrix form, Eq. (31) can be written as

$$\{\varepsilon\} = [B]\{d\} \tag{33}$$

where

$$[B] = \begin{bmatrix} [B_i] & [B_j] & [B_m] \end{bmatrix} \tag{34}$$

- The [B] matrix (sometimes called a gradient matrix) is independent of the x and y coordinates.
- It depends solely on the element nodal coordinates, as seen from Eqs. (32) and (10).
- The strains in Eq. (33) will be constant (consistent with the simple expressions previously given by Eq. (23-b).

Stress/Strain Relationship:

- In general, the in-plane stress/strain relationship is given by

where

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad (35\text{-a})$$

where [D] is given by:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \qquad (35\text{-b})$$

for plane stress problems.

and

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{35-c}$$

for plain strain problems.

- Using Eq. (33) in Eq. (35-a), we obtain the in-plane stresses in terms of the unknown nodal degrees of freedom as

$$\{\sigma\} = [D]\,[B]\,\{d\} \tag{36}$$

where the stresses $\{\sigma\}$ are also constant everywhere within the element.

Step 4: Derive the Element Stiffness Matrix and Equations

- Using the principle of minimum potential energy, we can generate the equations for a typical constant-strain triangular element.
- Keep in mind that for the basic plane stress element, the total potential energy is now a function of the nodal displacements u_i , v_i , u_j, …, v_m (that is, $\{d\}$) such that

$$\pi_p = \pi_p(u_i, v_i, u_j, \ldots, v_m) \tag{37}$$

- Here the total potential energy is given by

$$\pi_p = U + \Omega_b + \Omega_p + \Omega_s \tag{38}$$

where the strain energy is given by

$$U = \frac{1}{2} \iiint_V \{\varepsilon\}^T \{\sigma\} \, dV \tag{39}$$

substituting eq. (35) in eq.(39), gives

$$U = \frac{1}{2}\iiint_V \{\varepsilon\}^T [D]\{\varepsilon\}\, dV \qquad (40)$$

The potential energy of the body forces is given by

$$\Omega_b = -\iiint_V \{\psi\}^T \{X\}\, dV \qquad (41)$$

where $\{\psi\}$ is again the general displacement function, and {X} is the body weight/unit volume or weight density matrix (typically, in units of pounds per cubic inch or kilonewtons per cubic meter).

The potential energy of concentrated loads is given by

$$\Omega_p = -\{d\}^T \{P\} \qquad (42)$$

where {d} represents the usual nodal displacements, and {P} now represents the concentrated external loads.

The potential energy of distributed loads (or surface tractions) moving through respective surface displacements is given by

$$\Omega_s = -\iint_S \{\psi_S\}^T \{T_S\}\, dS \tag{43}$$

where $\{T_S\}$ represents the surface tractions (typically in units of pounds per square inch or kilonewtons per square meter),

$\{\psi_S\}$ represents the field of surface displacements through which the surface tractions act, and S represents the surfaces over which the tractions $\{T_S\}$ act.

Similar to Eq. (21), we express $\{\psi_S\}$ as

$\{\psi_S\} = [N_S]\{d\}$,

where $[N_S]$ represents the shape function matrix evaluated along the surface where the surface traction acts.

- Using Eq. (21) for $\{\psi_S\}$ and Eq. (33) for the strains in Eqs. (40) through (43), we have

$$\pi_p = \frac{1}{2}\iiint_V \{d\}^T[B]^T[D][B]\{d\}\,dV - \iiint_V \{d\}^T[N]^T\{X\}\,dV - \{d\}^T\{P\} - \iint_S \{d\}^T[N_S]^T\{T_S\}\,dS \tag{44}$$

- The nodal displacements $\{d\}$ are independent of the general x – y coordinates, so $\{d\}$ can be taken out of the integrals of Eq. (44). Therefore,

$$\pi_p = \frac{1}{2}\{d\}^T\iiint_V [B]^T[D][B]\,dV\{d\} - \{\mathrm{d}\}^T\iiint_V [N]^T\{X\}\,dV - \{d\}^T\{P\} - \{\mathrm{d}\}^T\iint_S [N_S]^T\{T_S\}\,dS \tag{45}$$

From Eqs. (41) through (43), we can see that the last three terms of Eq. (45) represent the total load system $\{f\}$ on an element; that is,

$$\{f\} = \iiint_V [N]^T \{X\}\, dV + \{P\} + \iint_S [N_S]^T \{T_S\}\, dS \tag{46}$$

where the first, second, and third terms on the right side of Eq. (46) represent the body forces, the concentrated nodal forces, and the surface tractions, respectively.

Using Eq. (46) in Eq. (45), we obtain

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T [D][B]\, dV \{d\} - \{d\}^T \{f\} \tag{47}$$

- Taking the first variation, or equivalently, the partial derivative of π_p with respect to the nodal displacements since $\pi_p = \pi_p(\{d\})$ (as was previously done for the bar element),we obtain

$$\frac{\partial \pi_p}{\partial \{d\}} = \left[\iiint_V [B]^T [D][B]\, dV \right] \{d\} - \{f\} = 0 \tag{48}$$

Rewriting Eq. (48), we have

$$\iiint_V [B]^T [D][B]\, dV \{d\} = \{f\} \tag{49}$$

From Eq. (49) we can see that

$$[k] = \iiint_V [B]^T [D][B]\, dV \tag{50}$$

- For an element with constant thickness, t, Eq. (50) becomes

$$[k] = t \iint_A [B]^T [D][B]\, dx\, dy \tag{51}$$

- where the integrand is not a function of x or y for the constant-strain triangular element and thus can be taken out of the integral to yield

$$[k] = t\, A\, [B]^T [D][B] \tag{52}$$

where A is given by Eq. (9), [B] is given by Eq. (34), and [D] is given by Eq. (35-b) or Eq. (35-c).

We will assume elements of constant thickness. (This assumption is convergent to the actual situation as the element size is decreased.)

- From Eq. (52) we see that [k] is a function of the nodal coordinates (because [B] and A are defined in terms of them) and of the mechanical properties E and v (of which [D] is a function).
- The expansion of Eq. (52) for an element is

$$[k] = \begin{bmatrix} [k_{ii}] & [k_{ij}] & [k_{im}] \\ [k_{ji}] & [k_{jj}] & [k_{jm}] \\ [k_{mi}] & [k_{mj}] & [k_{mm}] \end{bmatrix} \tag{53}$$

- where the 2×2 submatrices are given by

$$\begin{aligned} [k_{ii}] &= [B_i]^T [D][B_i] tA \\ [k_{ij}] &= [B_i]^T [D][B_j] tA \\ [k_{im}] &= [B_i]^T [D][B_m] tA \end{aligned} \tag{54}$$

and so forth.

- In Eqs. (54), $[B_i]$, $[B_j]$, and $[B_m]$ are defined by Eqs. (32).
- The [k] matrix is seen to be a 6×6 matrix (equal in order to the number of degrees of freedom per node, two, times the total number of nodes per element, three).
- In general, Eq. (46) must be used to evaluate the surface and body forces.
- When Eq. (46) is used to evaluate the surface and body forces, these forces are called consistent loads because they are derived from the consistent (energy) approach.
- For higher-order elements, typically with quadratic or cubic displacement functions, Eq. (46) should be used.
- However, for the CST element, the body and surface forces can be lumped at the nodes with equivalent results and added to any concentrated nodal forces to obtain the element force matrix.

- The element equations are then given by

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{16} \\ k_{21} & k_{22} & \dots & k_{26} \\ \vdots & \vdots & & \vdots \\ k_{61} & k_{62} & \dots & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{55}$$

- Finally, realizing that the strain energy U is the first term on the right side of Eq. (47) and using the expression for the stiffness matrix given by Eq. (50), we can again express the strain energy in the quadratic form

$$U = \frac{1}{2} \{d\}^T [k] \{d\}$$

Step 5: Assemble the Element Equations to Obtain the Global Equations and Introduce Boundary Conditions

- The global structure stiffness matrix and equations can be obtain by using the direct stiffness method as

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \tag{56}$$

and

$$\{F\} = [K]\{d\} \tag{57}$$

where, in Eq. (56), all element stiffness matrices are defined in terms of the global x – y coordinate system,

{d} is now the total structure displacement matrix, and

$$\{F\} = \sum_{e=1}^{N} \{f^{(e)}\} \tag{58}$$

is the column of equivalent global nodal loads obtained by lumping body forces and distributed loads at the proper nodes (as well as including concentrated nodal loads) or by consistently using Eq. (46).

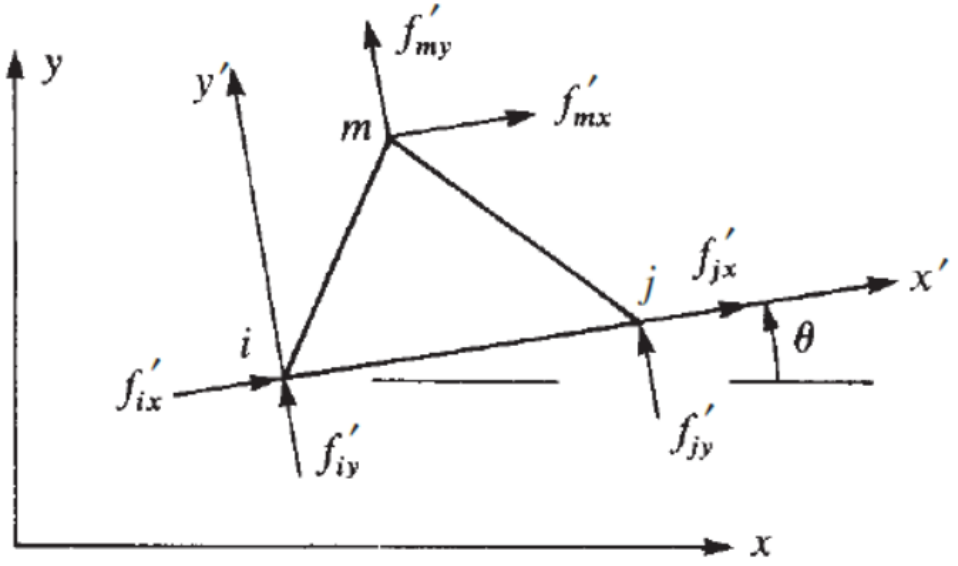
Note:

- In the formulation of the element stiffness matrix Eq. (52), the matrix has been derived for a general orientation in global coordinates.
- Equation (52) then applies for all elements.
- All element matrices are expressed in the global-coordinate orientation.
- Therefore, no transformation from local to global equations is necessary.

Transformation of Element Matrices from Local to Global Coordinates:

- If the local axes for the constant-strain triangular element are not parallel to the global axes for the whole structure, we must apply rotation-of-axes transformations similar to those used for truss element.
- This transformation should be applied to the element stiffness matrix, as well as to the element nodal force and displacement matrices.

- The transformation of axes for the triangular element shown in the figure is illustrated by considering the element to have local axes $x' - y'$ not parallel to global axes x – y.



- Local nodal forces are shown in the figure.
- The transformation from local to global equations follows the procedure of truss elements.
- The general expressions to relate local to global displacements, forces, and stiffness matrices, are the same; that is,

$$\{d'\} = [T]\{d\} \qquad \{f'\} = [T]\{f\} \qquad [k] = [T]^T[k'][T] \tag{59}$$

- where for the transformation matrix [T] used in truss element must be expanded because two additional degrees of freedom are present in the constant-strain triangular element.
- Thus,

$$[T] = \left[\begin{array}{cc:cc:cc} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{array}\right] \begin{array}{c} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{array}$$

where C = cos θ , S = sin θ

Step 6: Solve for the Nodal Displacements

- The unknown global structure nodal displacements are determined by solving the system of algebraic equations given by Eq. (57).

Step 7: Solve for the Element Forces (Stresses)

- Having solved for the nodal displacements, we obtain the strains and stresses in the global x and y directions in the elements by using Eqs. (33) and (36).
- Finally, we determine the maximum and minimum in-plane principal stresses σ_1 and σ_2 by using the equations:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sigma_{\min} \qquad (60)$$

- These stresses are usually assumed to act at the centroid of the element.
- The angle that one of the principal stresses makes with the x-axis is given by

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{61}$$

Example:

Evaluate the stiffness matrix for the element shown in Figure 6–11. The coordinates are shown in units of mm. Assume plane stress conditions. Let $E = 210\,\text{GPa}$, $\nu = 0.25$, and thickness $t = 20\,\text{mm}$. Assume the element nodal displacements have been determined to be $u_1 = 0.0$, $v_1 = 0.05\,\text{mm}$, $u_2 = 0.025\,\text{mm}$, $v_2 = 0.0$, $u_3 = 0.0$, and $v_3 = 0.05\,\text{mm}$. Determine the element stresses.

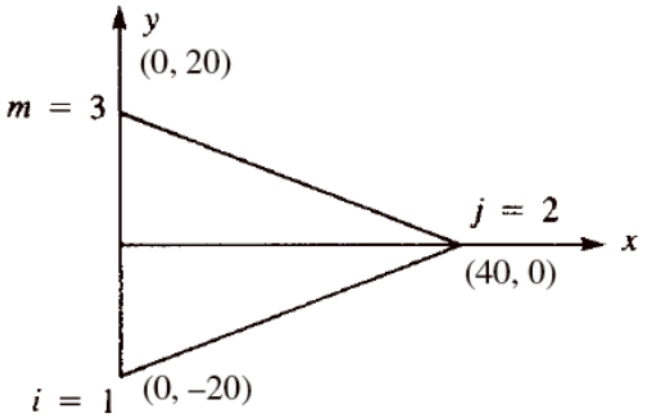


Figure 6–11 Plane stress element for stiffness matrix evaluation

See pages 355-357 of Daryl L. Logan
" A First Course in the Finite Element Method"
Sixth edition

Treatment of Body and Surface Forces:

1. Body Forces:

Using the first term on the right side of Eq. (46), we can evaluate the body forces at the nodes as

$$\{f_b\} = \iiint_V [N]^T \{X\}\, dV \qquad (6.3.1)$$

where

$$\{X\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \qquad (6.3.2)$$

and X_b and Y_b are the weight densities in the x and y directions in units of force/unit volume, respectively.

- These forces may arise, for instance, because of actual body weight (gravitational forces), angular velocity (called centrifugal body forces), or inertial forces in dynamics.

- In Eq. (6.3.1), [N] is a linear function of x and y; therefore, the integration must be carried out.
- Without lack of generality, the integration is simplified if the origin of the coordinates is chosen at the centroid of the element.
- For example, consider the element with coordinates shown in figure.
- With the origin of the coordinate placed at the centroid of the element, we have, from the definition of the centroid,

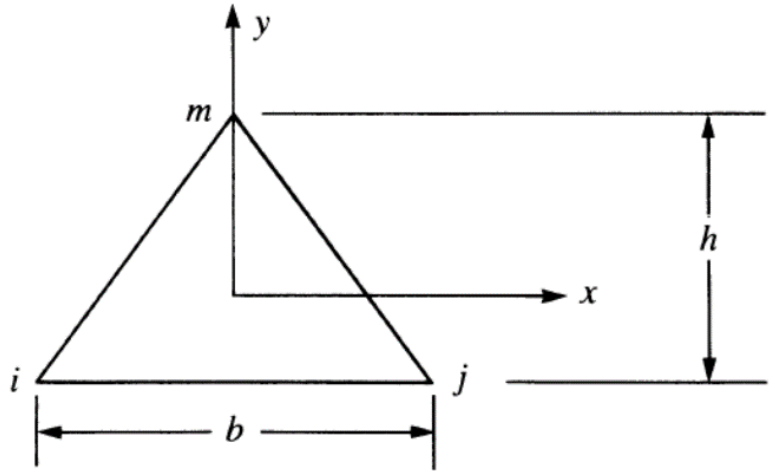
$$\iint x\, dA = \iint y\, dA = 0$$

- and therefore,

$$\iint \beta_i x\, dA = \iint \gamma_i y\, dA = 0 \qquad (6.3.3)$$

and

$$\alpha_i = \alpha_j = \alpha_m = \frac{2A}{3} \qquad (6.3.4)$$



- Using Eqs. (6.3.2) through (6.3.4) in Eq. (6.3.1), the body force at node *i* is then represented by

$$\{f_{bi}\} = \begin{Bmatrix} X_b \\ Y_b \end{Bmatrix} \frac{tA}{3} \qquad (6.3.5)$$

- Similarly, considering the j and m node body forces, we obtain the same results as in Eq. (6.3.5). In matrix form, the element body forces are

$$\{f_b\} = \begin{Bmatrix} f_{bix} \\ f_{biy} \\ f_{bjx} \\ f_{bjy} \\ f_{bmx} \\ f_{bmy} \end{Bmatrix} = \begin{Bmatrix} X_b \\ Y_b \\ X_b \\ Y_b \\ X_b \\ Y_b \end{Bmatrix} \frac{At}{3} \qquad (6.3.6)$$

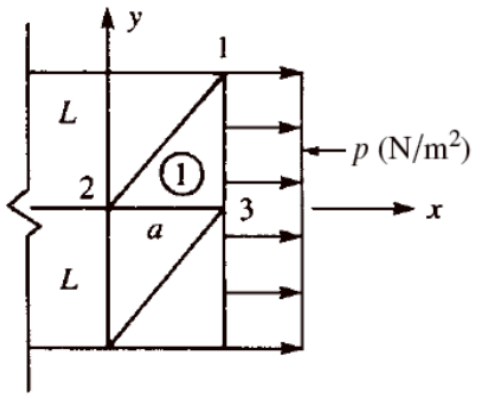
- From the results of Eq. (6.3.6), we can conclude that the body forces are distributed to the nodes in three equal parts.
- The signs depend on the directions of X_b and Y_b with respect to the positive x and y global coordinates.
- For the case of body weight only, because of the gravitational force associated with the y direction, we have only Y_b ($X_b = 0$).

2. Surface Forces:

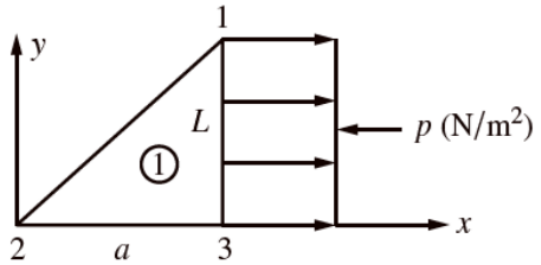
Using the third term on the right side of Eq. (46), we can evaluate the surface forces at the nodes as

$$\{f_s\} = \iint_S [N_S]^T \{T_S\}\, dS \qquad (6.3.7)$$

- The subscript S in $[N_S]$ in Eq. (6.3.7) means the shape functions evaluated along the surface where the surface traction is applied.
- We will now illustrate the use of Eq. (6.3.7) by considering the example of a uniform stress p (say, in newtons per square meter) acting between nodes 1 and 3 on the edge of element 1 in the figure.



(a)



(b)

- In Eq. (6.3.7), the surface traction now becomes

$$\{T_S\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix} = \begin{Bmatrix} p \\ 0 \end{Bmatrix} \tag{6.3.8}$$

and

$$[N_S]^T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \text{ evaluated at } x = a,\ y = y \tag{6.3.9}$$

- As the surface traction p acts along the edge at $x = a$ and $y = y$ from $y = 0$ to $y = L$, we evaluate the shape functions at $x = a$ and $y = y$ and integrate over the surface from 0 to L in the y direction and from 0 to t in the z- direction, as shown by Eq. (6.3.10).

- Using Eqs. (6.3.8) and (6.3.9), we express Eq. (6.3.7) as

$$\{f_s\} = \int_0^t \int_0^L \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{Bmatrix} p \\ 0 \end{Bmatrix} dz\, dy \quad \text{evaluated at } x = a,\ y = y \tag{6.3.10}$$

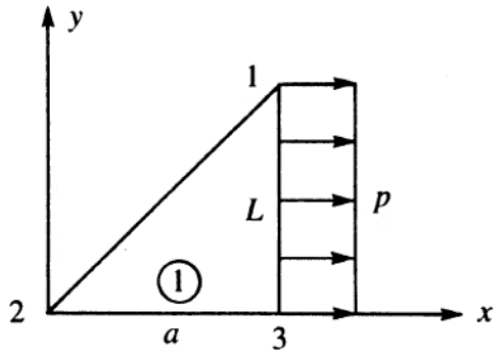
Simplifying Eq. (6.3.10), we obtain

$$\{f_s\} = t \int_0^L \begin{bmatrix} N_1 p \\ 0 \\ N_2 p \\ 0 \\ N_3 p \\ 0 \end{bmatrix} dy \quad \text{evaluated at } x = a, y = y \tag{6.3.11}$$

- Now, by Eqs. (6.2.18) (with $i = 1$), we have

$$N_1 = \frac{1}{2A}(\alpha_1 + \beta_1 x + \gamma_1 y) \qquad (6.3.12)$$

- For convenience, we choose the coordinate system for the element as shown in the figure.
- Using the definition Eqs. (10), we obtain



$$\alpha_i = x_j y_m - y_j x_m$$

or, with $i = 1$, $j = 2$, and $m = 3$,

$$\alpha_1 = x_2 y_3 - y_2 x_3 \qquad (6.3.13)$$

- Substituting the coordinates into Eq. (6.3.13), we obtain

$$\alpha_1 = 0 \qquad (6.3.14)$$

- Similarly, again using Eqs. (10), we obtain

$$\beta_1 = 0 \qquad \gamma_1 = a \qquad (6.3.15)$$

- Therefore, substituting Eqs. (6.3.14) and (6.3.15) into Eq. (6.3.12), we obtain

$$N_1 = \frac{ay}{2A} \qquad (6.3.16)$$

- Similarly, using Eqs. (18), we can show that

$$N_2 = \frac{L(a-x)}{2A} \quad \text{and} \quad N_3 = \frac{Lx - ay}{2A} \qquad (6.3.17)$$

- On substituting Eqs. (6.3.16) and (6.3.17) for N_1, N_2, and N_3 into Eq. (6.3.11),
- Evaluating N_1, N_2, and N_3 at $x = a$ and $y = y$ (the coordinates corresponding to the location of the surface load p),
- and then integrating with respect to y, we obtain

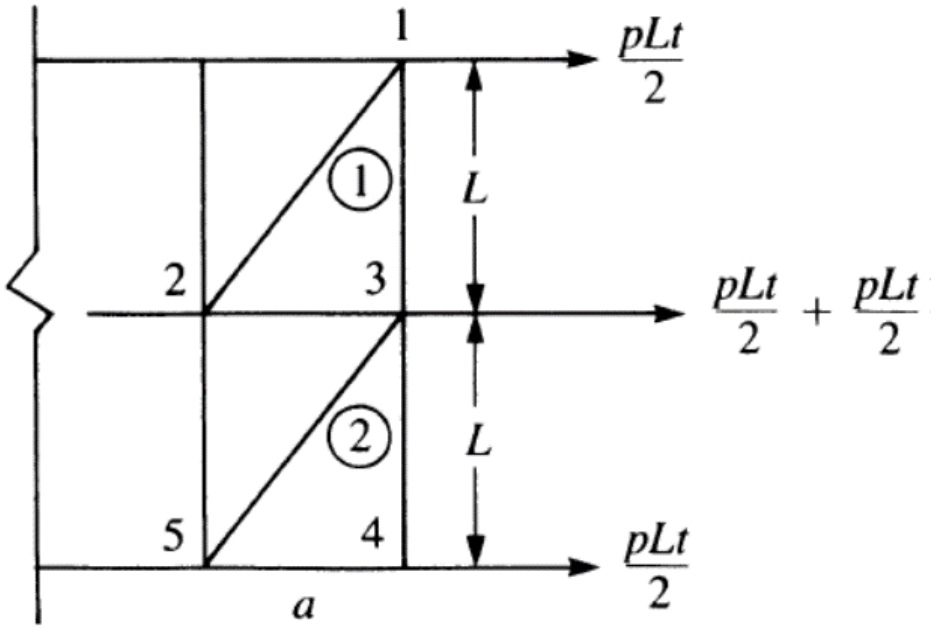
$$\{f_s\} = \frac{t}{2(aL/2)} \begin{Bmatrix} a\left(\frac{L^2}{2}\right)p \\ 0 \\ 0 \\ 0 \\ \left(L^2 - \frac{L^2}{2}\right)ap \\ 0 \end{Bmatrix} \qquad (6.3.18)$$

- where the shape function $N_2 = 0$ between nodes 1 and 3, as should be the case according to the definitions of the shape functions.

- Simplifying Eq. (6.3.18), we finally obtain

$$\{f_s\} = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \begin{Bmatrix} pLt/2 \\ 0 \\ 0 \\ 0 \\ pLt/2 \\ 0 \end{Bmatrix} \qquad (6.3.19)$$

- The figure shown illustrates the results for the surface load equivalent nodal forces for both elements 1 and 2.



- We can conclude that for a constant-strain triangle, a distributed load on an element edge can be treated as concentrated loads acting at the nodes associated with the loaded edge by making the two kinds of load statically equivalent.
- However, for higher-order elements such as the linear-strain triangle, the load replacement should be made by using Eq. (6.3.7), which was derived by the principle of minimum potential energy.
- For higher-order elements, this load replacement by use of Eq. (6.3.7) is generally not equal to the apparent statically equivalent one; however, it is consistent in that this replacement results directly from the energy approach.

Explicit Expression for the Constant-Strain Triangle Stiffness Matrix:

- Recall that the stiffness matrix is given by

$$[k] = tA[B]^T[D][B] \qquad (6.4.1)$$

- Consider the plain strain case, this leads to

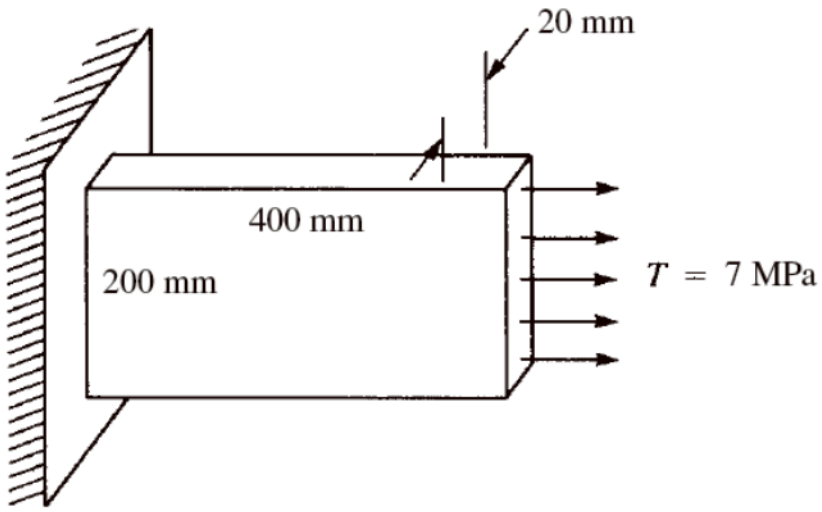
$$[k] = \frac{tE}{4A(1+\nu)(1-2\nu)} \begin{bmatrix} \beta_i & 0 & \gamma_i \\ 0 & \gamma_i & \beta_i \\ \beta_j & 0 & \gamma_j \\ 0 & \gamma_j & \beta_j \\ \beta_m & 0 & \gamma_m \\ 0 & \gamma_m & \beta_m \end{bmatrix} \times \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \qquad (6.4.2)$$

- On multiplying, gives

$$[k] = \frac{tE}{4A(1+\nu)(1-2\nu)} \times \begin{bmatrix} \beta_i^2(1-\nu)+\gamma_i^2\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_i\nu+\beta_i\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_j(1-\nu)+\gamma_i\gamma_j\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_j\nu+\beta_j\gamma_i\left(\frac{1-2\nu}{2}\right) & \beta_i\beta_m(1-\nu)+\gamma_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_i\gamma_m\nu+\beta_m\gamma_i\left(\frac{1-2\nu}{2}\right) \\ & \gamma_i^2(1-\nu)+\beta_i^2\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_i\nu+\beta_i\gamma_j\left(\frac{1-2\nu}{2}\right) & \gamma_i\gamma_j(1-\nu)+\beta_i\beta_j\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_i\nu+\beta_i\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_i\gamma_m(1-\nu)+\beta_i\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & \beta_j^2(1-\nu)+\gamma_j^2\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_j\nu+\beta_j\gamma_j\left(\frac{1-2\nu}{2}\right) & \beta_j\beta_m(1-\nu)+\gamma_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \beta_j\gamma_m\nu+\gamma_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & & \gamma_j^2(1-\nu)+\beta_j^2\left(\frac{1-2\nu}{2}\right) & \beta_m\gamma_j\nu+\beta_j\gamma_m\left(\frac{1-2\nu}{2}\right) & \gamma_j\gamma_m(1-\nu)+\beta_j\beta_m\left(\frac{1-2\nu}{2}\right) \\ & & & & \beta_m^2(1-\nu)+\gamma_m^2\left(\frac{1-2\nu}{2}\right) & \gamma_m\beta_m\nu+\beta_m\gamma_m\left(\frac{1-2\nu}{2}\right) \\ \text{Symmetry} & & & & & \gamma_m^2(1-\nu)+\beta_m^2\left(\frac{1-2\nu}{2}\right) \end{bmatrix} \tag{6.4.3}$$

For the plane stress case, we need only replace $1-\nu$ by 1, $(1-2\nu)/2$ by $(1-\nu)/2$, and $(1+\nu)(1-2\nu)$ outside the brackets by $1-\nu^2$ in Eq. (6.4.3).

Example: For a thin plate subjected to the surface traction shown in Figure 6–16, determine the nodal displacements and the element stresses. The plate thickness $t = 20$ mm, $E = 210$ GPa, and $\nu = 0.30$.



■ **Figure 6–16** Thin plate subjected to tensile stress

Rectangular Plane Element (Bilinear Rectangle, Q4):

- The four-noded rectangular plane element stiffness matrix will be developed.
- This element is also called the bilinear rectangle because of the linear terms in *x* and *y* for the x and y displacement functions.
- The "*Q4*" symbol represents the element as a quadrilateral with four corner nodes.
- Two advantages of the rectangular element over the triangular element are:

a) ease of data input, and

b) simpler interpretation of output stresses.

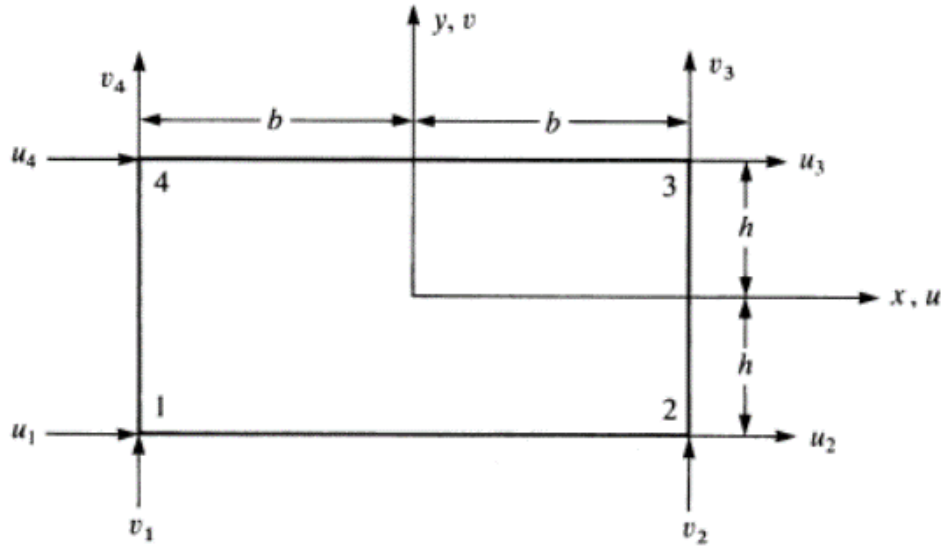
- A disadvantage of the rectangular element is that the simple linear-displacement rectangle with its associated straight sides poorly approximates the real boundary condition edges.

Step 1 Select Element Type:

Consider the rectangular element shown in Figure 6–20 (all interior angles are 90°) with corner nodes 1–4 (again labeled counterclockwise) and base and height dimensions $2b$ and $2h$, respectively.

The unknown nodal displacements are now given by

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (6.6.1)$$



Step 2 Select Displacement Functions

For a compatible displacement field, the element displacement functions u and v must be linear along each edge because only two points (the corner nodes) exist along each edge. We then select the linear displacement functions as

$$\begin{aligned} u(x,y) &= a_1 + a_2x + a_3y + a_4xy \\ v(x,y) &= a_5 + a_6x + a_7y + a_8xy \end{aligned} \tag{6.6.2}$$

There are a total of eight generalized degrees of freedom (a's) in Eq. (6.6.2) and a total of eight specific degrees of freedom (u_1, v_1 at node 1 through u_4, v_4 at node 4) for the element.

We can proceed in the usual manner to eliminate the a_i's from Eqs. (6.6.2) to obtain

$$\begin{aligned} u(x,y) = \frac{1}{4bh}[&(b-x)(h-y)u_1 + (b+x)(h-y)u_2 \\ &+ (b+x)(h+y)u_3 + (b-x)(h+y)u_4] \\ v(x,y) = \frac{1}{4bh}[&(b-x)(h-y)v_1 + (b+x)(h-y)v_2 \\ &+ (b+x)(h+y)v_3 + (b-x)(h+y)v_4] \end{aligned} \tag{6.6.3}$$

These displacement expressions, Eqs. (6.6.3), can be expressed equivalently in terms of the shape functions and unknown nodal displacements as

$$\{\psi\} = [N]\{d\} \tag{6.6.4}$$

where the shape functions are given by

$$\begin{aligned} N_1 &= \frac{(b-x)(h-y)}{4bh} \qquad N_2 = \frac{(b+x)(h-y)}{4bh} \\ N_3 &= \frac{(b+x)(h+y)}{4bh} \qquad N_4 = \frac{(b-x)(h+y)}{4bh} \end{aligned} \tag{6.6.5}$$

and the N_i s are again such that $N_1 = 1$ at node 1 and $N_1 = 0$ at all the other nodes, with similar requirements for the other shape functions. In expanded form, Eq. (6.6.4) becomes

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{6.6.6}$$

Step 3 Define the Strain-Displacement and Stress/Strain Relationships

Again the element strains for the two-dimensional stress state are given by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{6.6.7a}$$

Using Eq. (6.6.2) in Eq. (6.6.7a), we express the strains in terms of the a s as

$$\begin{aligned} \varepsilon_x &= a_2 + a_4 y \\ \varepsilon_y &= a_7 + a_8 x \\ \gamma_{xy} &= (a_3 + a_6) + a_4 x + a_8 y \end{aligned} \tag{6.6.7b}$$

Using Eq. (6.6.6) in Eq. (6.6.7a) and taking the derivatives of u and v as indicated, we can express the strains in terms of the unknown nodal displacements as

$$\{\varepsilon\} = [B]\{d\} \tag{6.6.8}$$

where

$$[B] = \frac{1}{4bh}\begin{bmatrix} -(h-y) & 0 & (h-y) & 0 & (h+y) & 0 & -(h+y) & 0 \\ 0 & -(b-x) & 0 & -(b+x) & 0 & (b+x) & 0 & (b-x) \\ -(b-x) & -(h-y) & -(b+x) & (h-y) & (b+x) & (h+y) & (b-x) & -(h+y) \end{bmatrix} \tag{6.6.9}$$

From Eqs. (6.6.7b), (6.6.8), and (6.6.9), we observe that ε_x is a function of y, ε_y is a function of x, and γ_{xy} is a function of both x and y. The stresses are again given by the formulas in Eq. (6.2.36), where $[B]$ is now that of Eq. (6.6.9) and $\{d\}$ is that of Eq. (6.6.1).

$$\{\sigma\} = [D][B]\{d\} \tag{6.2.36}$$

Step 4 Derive the Element Stiffness Matrix and Equations:

- The stiffness matrix is determined by

$$[k] = \int_{-h}^{h}\int_{-b}^{b}[B]^T[D][B]t\,dx\,dy \qquad (6.6.10)$$

- with [D] again given by the usual plane stress or plane strain conditions, Eq. (6.1.8) or (6.1.10).

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \qquad (6.1.8)$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \qquad (6.1.10)$$

- Because the [B] matrix is a function of x and y, integration of Eq. (6.6.10) must be performed.
- The [k] matrix for the rectangular element is now of order 8×8.

- A numerical evaluation of Eq. (6.6.10) for [k] is shown in Eq. (6.6.11) using b= 4 in. , h = 2 in. , t =1 in. , E = 30 $\times 10^6$ psi, and v = 0.3.

$$[K] = \begin{bmatrix} 1.35e10 & 5.486e9 & -1.688e9 & -4.22e8 & -6.752e9 & -5.486e9 & -5.064e9 & 4.22e8 \\ 5.486e9 & 2.447e10 & 4.22e8 & 9.284e9 & -5.486e9 & -1.224e10 & -4.22e8 & -2.152e10 \\ -1.688e9 & 4.22e8 & 1.35e10 & -5.486e9 & -5.064e9 & -4.22e8 & -6.752e9 & 5.486e9 \\ -4.22e8 & 9.284e9 & -5.486e9 & 2.447e10 & 4.22e8 & -2.152e10 & 5.486e9 & -1.224e10 \\ -6.752e9 & -5.486e9 & -5.064e9 & 4.22e8 & 1.35e10 & 5.486e9 & -1.688e9 & -4.22e8 \\ -5.486e9 & -1.224e10 & -4.22e8 & -2.152e10 & 5.486e9 & 2.447e10 & 4.22e8 & 9.284e9 \\ -5.064e9 & -4.22e8 & -6.752e9 & 5.486e9 & -1.688e9 & 4.22e8 & 1.35e10 & -5.486e9 \\ 4.22e8 & -2.152e10 & 5.486e9 & -1.224e10 & -4.22e8 & 9.284e9 & -5.486e9 & 2.447e10 \end{bmatrix} \quad (6.6.11)$$

The element force matrix is determined by Eq. (46) as

$$\{f\} = \iiint_V [N]^T \{X\} dV + \{P\} + \iint_S [N_S]^T \{T\} dS \quad (6.6.12)$$

where [N] is the rectangular matrix in Eq. (6.6.6), and N_1 through N_4 are given by Eqs. (6.6.5). The element equations are then given by

$$\{f\} = [k]\{d\} \quad (6.6.13)$$

Step 5 through 7:

- Steps 5 through 7, which involve assembling the global stiffness matrix and equations, determining the unknown nodal displacements, and calculating the stress, are identical to those for the CST.
- However, the stresses within each element now vary in both the *x* and *y* directions.

Numerical Comparison of CST to Q4 Element Models and Element Defects:

- Table 6–1 compares the free end deflection and maximum principal stress for a cantilevered beam modeled with 2, 4, and 8 rows of either all triangular CST elements or all rectangular Q4 elements.

Typical Q4 and CST models:

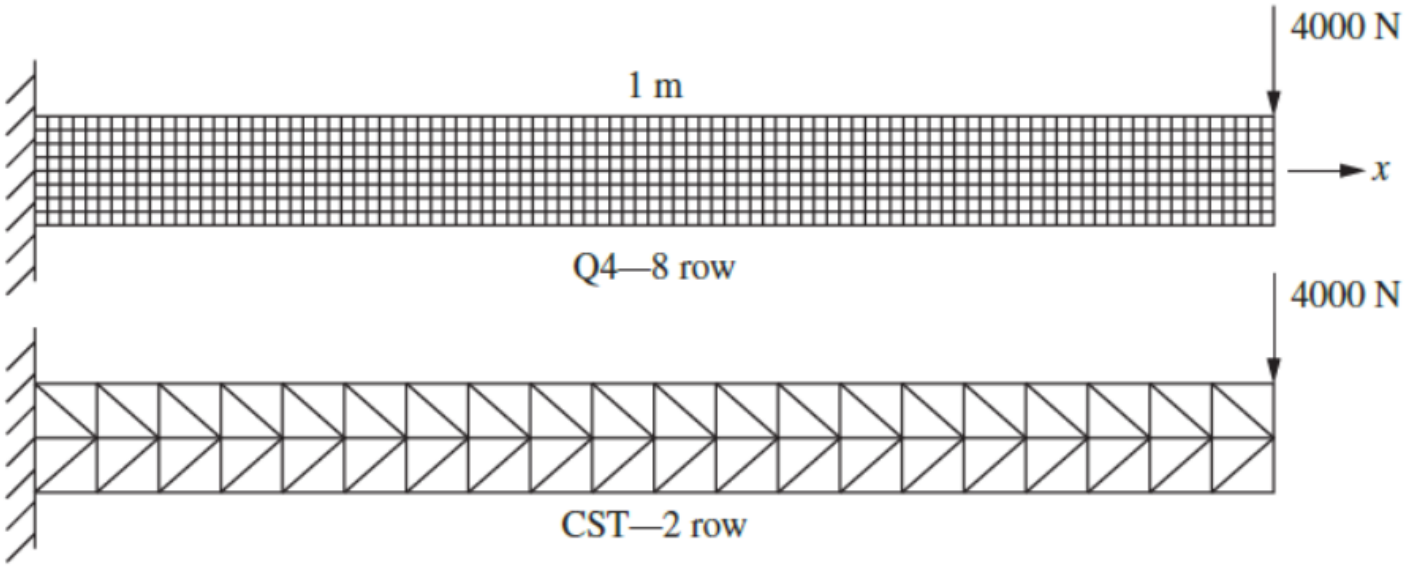


Table 6-1 Table comparing free-end deflections and largest principal stresses for CST and Q4 compatible element models (end force $P = 4000\,\text{N}$, length $L = 1\,\text{m}$, $I = 1 \times 10^{-5}\,\text{m}^4$, thickness $= 0.12\,\text{m}$, $E = 200\,\text{GPa}$, $G = 77.5\,\text{GPa}$)

Plane Element Used/Rows	Number of Nodes	Number of Degrees of Freedom	Free End Displ.,m	Tensor Stress, (0.05m from wall) σ_x, MPa
Q4/2	60	120	5.944×10^{-4}	17.34
Q4/4	200	400	6.509×10^{-4}	18.71
Q4/8	720	1440	6.661×10^{-4}	18.94
CST/2	60	120	3.630×10^{-4}	7.10
CST/4	200	400	5.537×10^{-4}	13.20
CST/8	720	1440	6.385×10^{-4}	16.91
Exact Values			6.672×10^{-4}	19.00

SIMPLEX, COMPLEX, AND MULTIPLEX ELEMENTS:

- Finite elements can be classified into three categories simplex, complex, and multiplex elements depending on the geometry of the element and the order of the polynomial used in the interpolation function.
- The simplex elements are those for which the approximating polynomial consists of constant and linear terms.
- Thus, the following polynomials represent the simplex functions for one-, two-, and three-dimensional elements.
- Noting that a simplex is defined as a geometric figure obtained by joining n + 1 joints (nodes) in an n-dimensional space, we can consider the corners of the elements as nodes in simplex elements.
- For example, the simplex element in two dimensions is a triangle with three nodes (corners).

For $n = 1$ (linear model)

One-dimensional case:

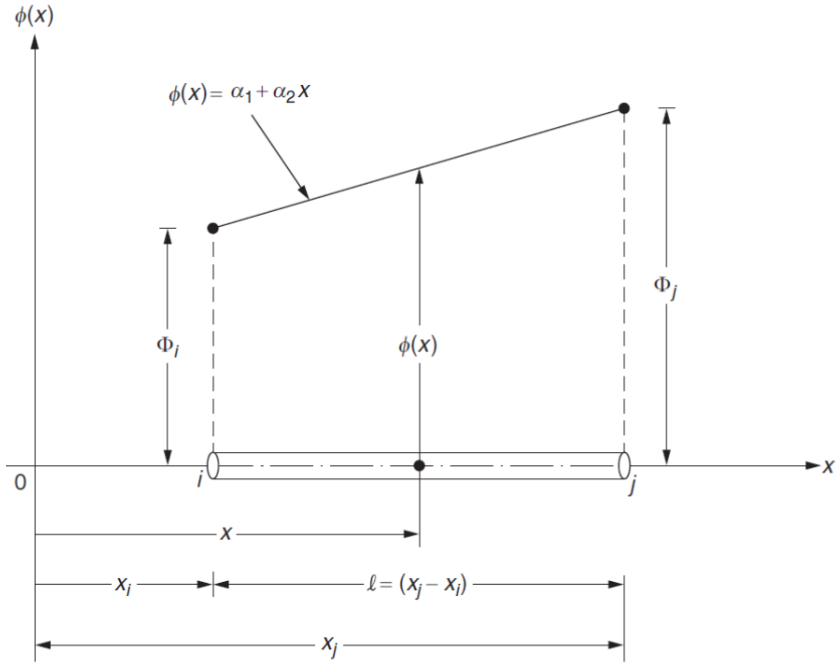
$$\phi(x) = \alpha_1 + \alpha_2 x$$

Two-dimensional case:

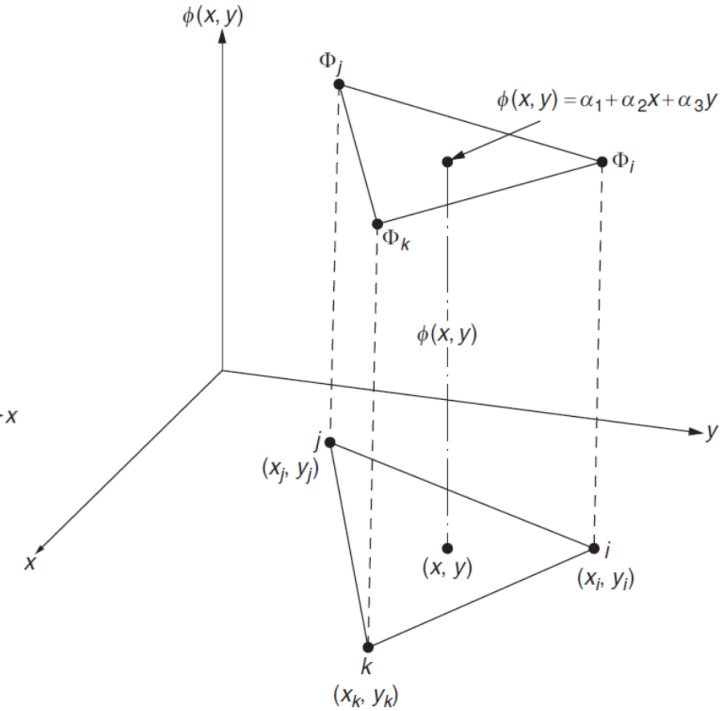
$$\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

Three-dimensional case:

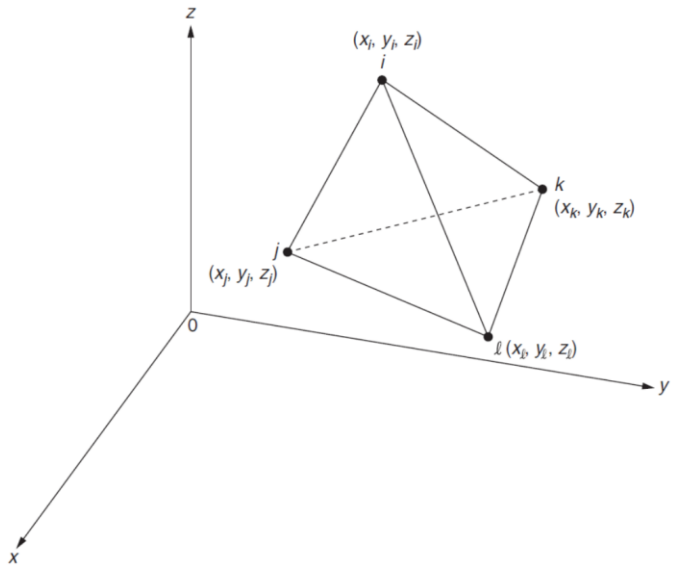
$$\phi(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$



One-dimensional simplex element.



Two-dimensional simplex element.



A three-dimensional simplex element.

- The complex elements are those for which the approximating polynomial consists of quadratic, cubic, and higher order terms, according to the need, in addition to the constant and linear terms.
- Thus, the following polynomials denote complex functions.

For $n = 2$ (quadratic model)
One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

Two-dimensional case:

$$\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy$$

Three-dimensional case:

$$\phi(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz$$

For $n = 3$ (cubic model)
One-dimensional case:

$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

Two-dimensional case:

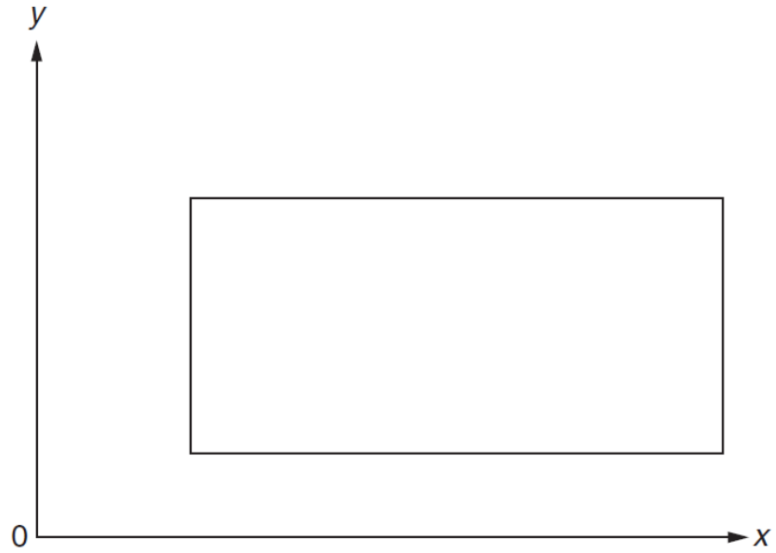
$$\phi(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^3 + \alpha_8 y^3 + \alpha_9 x^2 y + \alpha_{10} xy^2$$

Three-dimensional case:

$$\begin{aligned}\phi(x, y, z) = {} & \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 y^2 + \alpha_7 z^2 + \alpha_8 xy + \alpha_9 yz + \alpha_{10} xz + \alpha_{11} x^3 + \alpha_{12} y^3 + \alpha_{13} z^3 \\ & + \alpha_{14} x^2 y + \alpha_{15} x^2 z + \alpha_{16} y^2 z + \alpha_{17} xy^2 + \alpha_{18} xz^2 + \alpha_{19} yz^2 + \alpha_{20} xyz\end{aligned}$$

- The complex elements may have the same shapes as the simplex elements but will have additional boundary nodes and, sometimes, internal nodes.
- For example, the interpolating polynomial for a two-dimensional complex element (including terms up to quadratic terms) is given by the second of the preceding equations.
- Since this equation has six unknown coefficients α_i, the corresponding complex element must have six nodes.
- Thus, a triangular element with three corner nodes and three mid-side nodes satisfies this requirement.

- The multiplex elements are those whose boundaries are parallel to the coordinate axes to achieve interelement continuity, and whose approximating polynomials contain higher order terms.
- The rectangular element shown is an example of a multiplex element in two dimensions.
- Note that the boundaries of the simplex and complex elements need not be parallel to the coordinate axes.

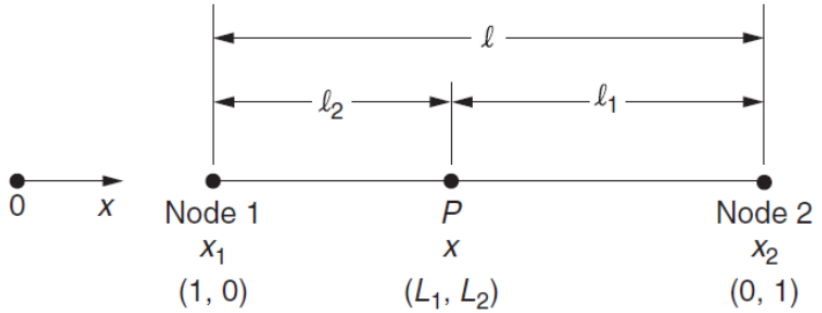


LINEAR INTERPOLATION POLYNOMIALS IN TERMS OF NATURAL COORDINATES:

- The derivation of element characteristic matrices and vectors involves the integration of the shape functions or their derivatives or both over the element.
- These integrals can be evaluated easily if the interpolation functions are written in terms of a local coordinate system that is defined separately for each element.
- In this section, we derive the interpolation functions of simplex elements in terms of a particular type of local coordinate systems, known as natural coordinate systems.
- A natural coordinate system is a local coordinate system that permits the specification of any point inside the element by a set of nondimensional numbers whose magnitude lies between 0 and 1.
- Usually, natural coordinate systems are chosen such that some of the natural coordinates will have unit magnitude at primary or corner nodes of the element.

One-Dimensional Element:

- The natural coordinates for a one-dimensional (line) element are as shown.



- Any point P inside the element is identified by two natural coordinates, L_1 and L_2, which are defined as

$$L_1 = \frac{l_1}{l} = \frac{x_2 - x}{x_2 - x_1} \qquad \text{and} \qquad L_2 = \frac{l_2}{l} = \frac{x - x_1}{x_2 - x_1} \tag{1}$$

- where l_1 and l_2 are the distances from a point in the element to nodes 2 and 1, respectively, and l is the length of the element.
- Since it is a one-dimensional element, there should be only one independent coordinate to define location of any point P.

- This is true even with natural coordinates because the two natural coordinates L_1 and L_2 are not independent but are related as

$$L_1 + L_2 = \frac{l_1}{l} + \frac{l_2}{l} = 1 \qquad (2)$$

- The natural coordinates L_1 and L_2 are also the shape functions for the line element, i.e.

$$N_i = L_1, \quad N_j = L_2 \qquad (3)$$

- Any point x within the element can be expressed as a linear combination of the nodal coordinates of nodes 1 and 2 as

$$x = x_1 L_1 + x_2 L_2 \qquad (4)$$

- Combining equations (2) and (2) gives

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} \qquad (5)$$

- Solving for L_1 and L_2 yields,

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{l} \begin{Bmatrix} x_2 & -1 \\ -x_1 & 1 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \qquad (6)$$

- In determining the stiffness matrix, it was found that the strains and stresses are derivatives of displacement, and hence the derivatives of shape functions.
- In natural coordinates, the shape functions for one dimensional elements are L_1 and L_2.
- If f is a function of L_1 and L_2, differentiation of f with respect to x can be performed, using the chain rule, as

$$\frac{df}{dx} = \frac{\partial f}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial f}{\partial L_2}\frac{\partial L_2}{\partial x} \tag{7}$$

$$\frac{\partial L_1}{\partial x} = -\frac{1}{x_2 - x_1} \quad \text{and} \quad \frac{\partial L_2}{\partial x} = \frac{1}{x_2 - x_1} \tag{8}$$

- Shape functions are also need to be integrated.
- Integration of polynomial terms in natural coordinates can be performed by using the simple formula

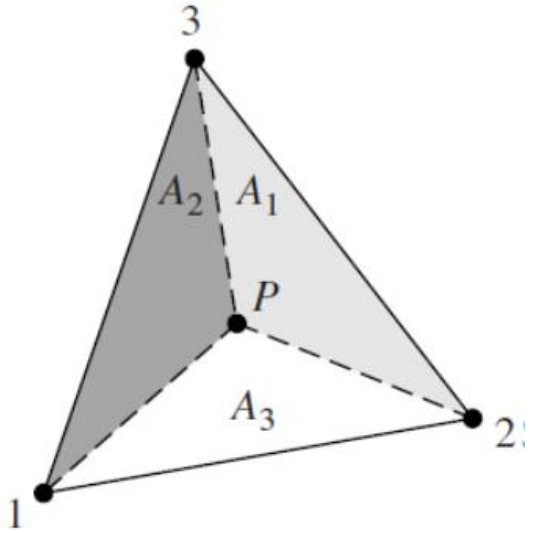
$$\int_{x_1}^{x_2} L_1^{\alpha} L_2^{\beta}\, \mathrm{d}x = \frac{\alpha!\beta!}{(\alpha+\beta+1)!} l \tag{9}$$

where $\alpha!$ is the factorial of α given by $\alpha! = \alpha\ (\alpha - 1)\ (\alpha - 2)\ \ldots$

Value of		Value of the Integral in Eq. (3.67)/*I*
α	β	
0	0	1
1	0	1/2
1	1	1/6
2	0	1/3
1	2	1/12
3	0	1/4
4	0	1/5
2	2	1/30
3	1	1/20
1	4	1/30
3	2	1/60
5	0	1/6

Two-Dimensional (Triangular) Element:

- When expressed in Cartesian coordinates, the interpolation functions for the triangular element are algebraically complex.
- Further, the integrations required to obtain element characteristic matrices are cumbersome.
- Considerable simplification of the interpolation functions as well as the subsequently required integration is obtained via the use of *area coordinates*.
- The figure below shows a three-node triangular element divided into three areas defined by the nodes and an arbitrary interior point $P(x, y)$.



- **Note: *P* is *not* a node.**
- **P is the point (x,y) at which the value of the field variable (displacement) is to be used.**

- The area coordinates (location) of *P* are defined as

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}$$

where

A_1 is the area of the triangle formed by the points P, 2 and 3;

A_2 is the area of the triangle formed by the points P, 1 and 3;

A_3 is the area of the triangle formed by the points P, 1 and 2; and

A is the area of the triangle 123.

- Because *Li* are defined in terms of areas, they are also known as **area coordinates**.
- Since $A_1 + A_2 + A_3 = A$
- then $\frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = L_1 + L_2 + L_3 = 1$

- A study of the properties of L_1, L_2, and L_3 shows that they are also the shape functions for the two-dimensional simplex (triangular) element:

$$N_i = L_1, \quad N_j = L_2, \quad N_K = L_3$$

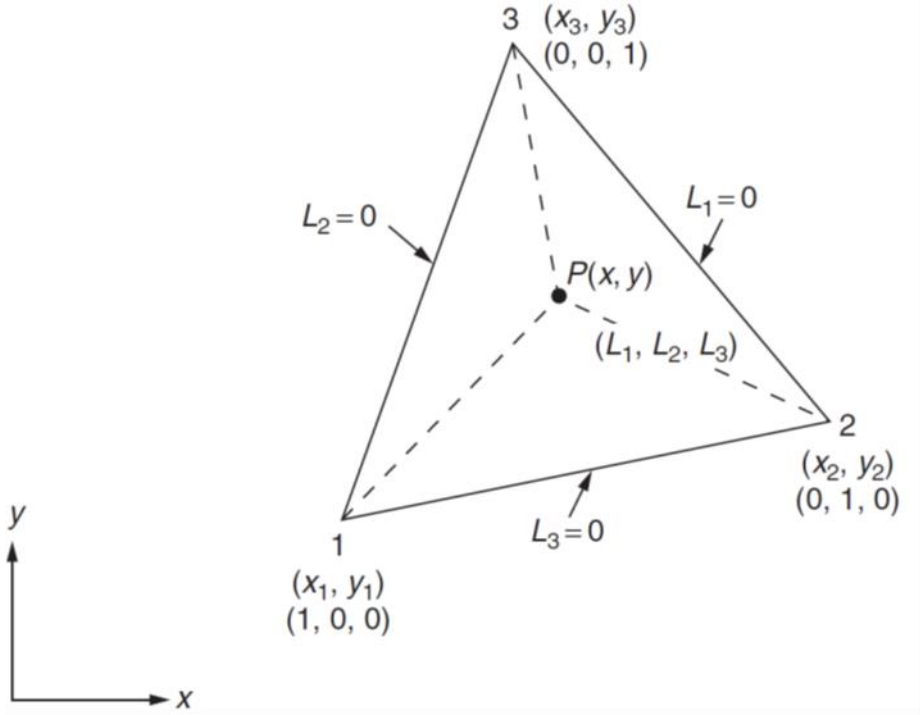
- The relation between the natural and Cartesian coordinates is given by:

$$\left.\begin{aligned} x &= x_1L_1 + x_2L_2 + x_3L_3 \\ y &= y_1L_1 + y_2L_2 + y_3L_3 \end{aligned}\right\}$$

- To every set of natural coordinates (L_1, L_2, L_3) which are not independent but are related by:

$L_1 + L_2 + L_3 = 1.0$, there corresponds a unique set of Cartesian coordinates (x, y).

- At node 1, $L_1 = 1$ and $L_2 = L_3 = 0$, and so on.

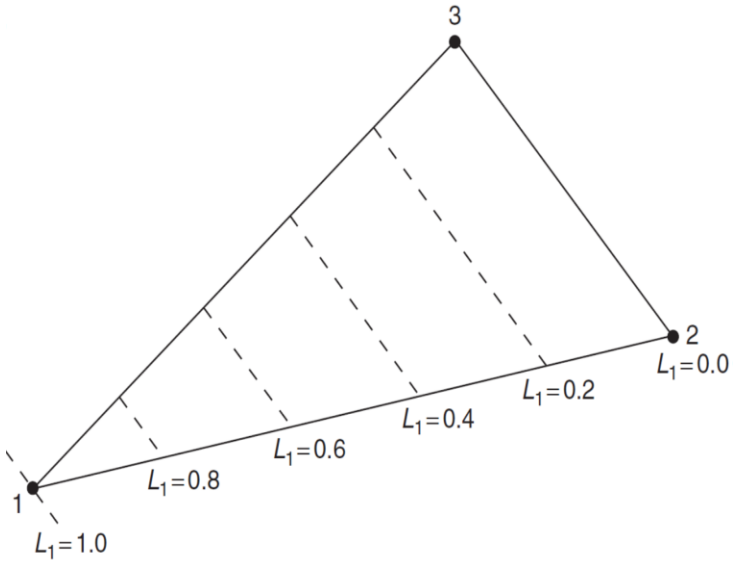


- The linear relationship between L_i ($i = 1, 2, 3$) and (x, y) implies that the contours of L_1 are straight lines parallel to the side 2, 3 of the triangle (on which $L_1 = 0$), and so on.
- The relation between area coordinates and their

relations with Cartesian coordinates can be expressed as

in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}$$



Variation of the area coordinate L_1

can be inverted to obtain

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3y_1 - x_1y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1y_2 - x_2y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

- where A is the area of the triangle 1, 2, 3 given by

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

If f is a function of L_1, L_2, and L_3, the differentiation with respect to x and y can be performed as

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial x} \\ \frac{\partial f}{\partial y} &= \sum_{i=1}^{3} \frac{\partial f}{\partial L_i}\frac{\partial L_i}{\partial y} \end{aligned}\right\}$$

where

$$\left.\begin{aligned} \frac{\partial L_1}{\partial x} &= \frac{y_2 - y_3}{2A}, & \frac{\partial L_1}{\partial y} &= \frac{x_3 - x_2}{2A} \\ \frac{\partial L_2}{\partial x} &= \frac{y_3 - y_1}{2A}, & \frac{\partial L_2}{\partial y} &= \frac{x_1 - x_3}{2A} \\ \frac{\partial L_3}{\partial x} &= \frac{y_1 - y_2}{2A}, & \frac{\partial L_3}{\partial y} &= \frac{x_2 - x_1}{2A} \end{aligned}\right\}$$

For integrating polynomial terms in natural coordinates, we can use the relations

$$\int_L L_1^{\alpha} L_2^{\beta} \cdot d\mathscr{L} = \frac{\alpha!\beta!}{(\alpha+\beta+1)!}\mathscr{L}$$

and

$$\iint_A L_1^{\alpha} L_2^{\beta} L_3^{\gamma} \cdot \mathrm{d}A = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}2A$$

- The first equation is used to evaluate an integral that is a function of the length along an edge of the element.
- Thus, the quantity $\mathscr{L}$ denotes the distance between the two nodes that define the edge under consideration.
- Second equation is used to evaluate area integrals

Value of			Value of the Integral in Eq. (3.77)/$\mathscr{L}$	Value of the Integral in Eq. (3.78)/A
α	β	γ		
0	0	0	1	1
1	0	0	1/2	1/3
2	0	0	1/3	1/6
1	1	0	1/6	1/12
3	0	0	1/4	1/10
2	1	0	1/12	1/30
1	1	1	—	1/60
4	0	0	1/5	1/15
3	1	0	1/20	1/60
2	2	0	1/30	1/90
2	1	1	—	1/180
5	0	0	1/6	1/21
4	1	0	1/30	1/105
3	2	0	1/60	1/210
3	1	1	—	1/420
2	2	1	—	1/630

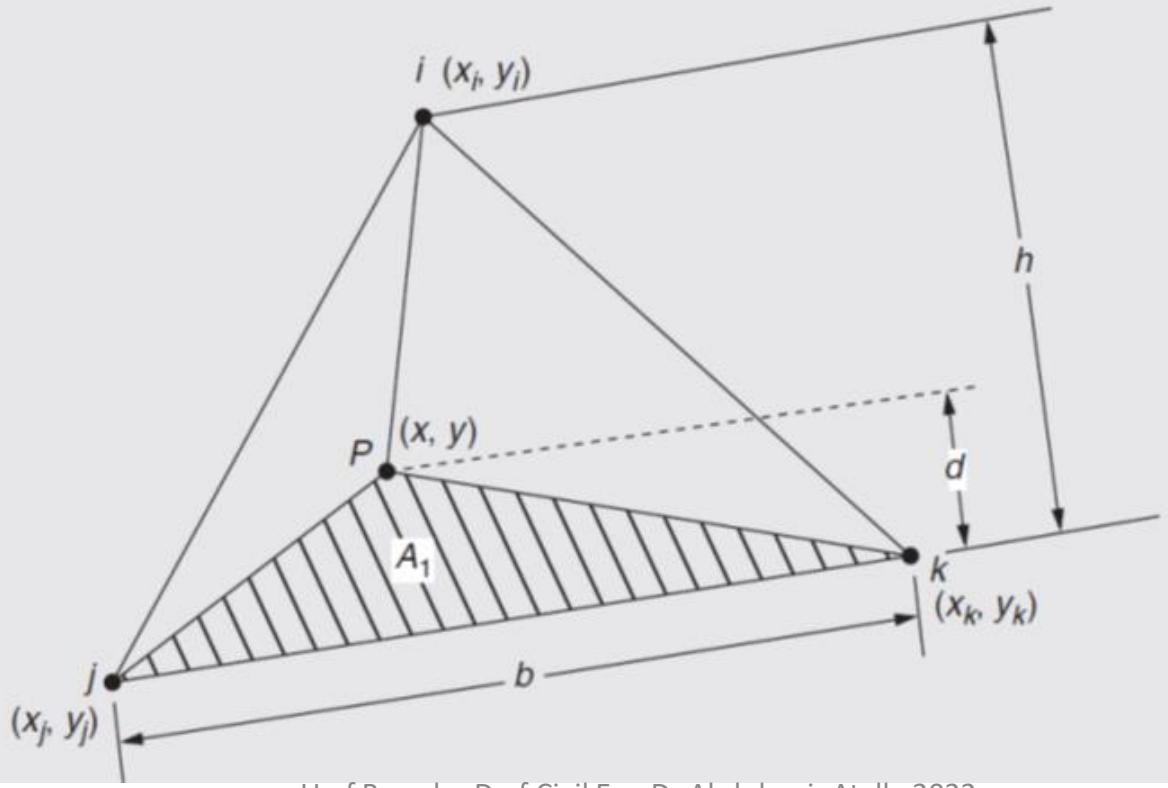
EXAMPLE 3.11

Show that the natural (area) coordinate L_i $(i = 1, 2, 3)$ is the same as the shape function N_i given by Eq. (3.35).

Solution

The area coordinate L_1, defined as the ratio of the area of the shaded triangle to the total area of the triangle ijk shown in Fig. 3.14, can be expressed as

$$L_1 = \frac{A_1}{A} = \frac{\frac{1}{2} b\, d}{\frac{1}{2} b\, h} = \frac{d}{h} \tag{E.1}$$



where d and h denote the distances of the perpendiculars from the points P and i to the base jk of the triangle. The area A_1 of the triangle Pjk can be determined in terms of the coordinates of P, j, and k as

$$2A_1 = \begin{vmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = x_j y_k - x_k y_j + x(y_j - y_k) + y(x_k - x_j) \tag{E.2}$$

Eqs. (E.1) and (E.2) lead to

$$L_1 = \frac{2A_1}{2A} = \frac{1}{2A}\{x_j y_k - x_k y_j + x(y_j - y_k) + y(x_k - x_j)\} \tag{E.3}$$

which can be seen to be identical to the shape function N_i given by Eq. (3.35). This shows that the area coordinates of a linear triangular element are identical to the shape functions.

2.2 NUMERICAL INTEGRATION

In finite element method, numerical integration has acquired immense importance, particularly in the context of isoparametric elements. For evaluating element stiffness matrix and element load matrix, one comes across integrations of the type

$$I = \int_{-1}^{+1} f(\xi)\, d\xi \qquad \ldots(2.43a)$$

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi, \eta)\, d\xi\, d\eta \qquad \ldots(b)$$

and

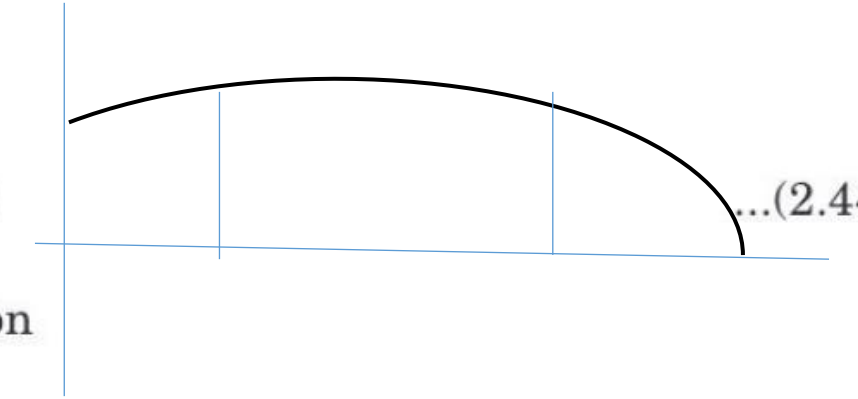
$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta \qquad \ldots(c)$$

for one, two and three dimensional problems respectively. The reader must be familiar with the numerical integration process using trapezoidal rule and Simpson's rule. In these methods, the sampling points are equispaced and weightages are decided by considering a local piecewise curve passing through two and three consecutive points respectively. Trapezoidal rule and Simpson's rule are particular cases of a general Newton-Cotes quadrature (integration) method.

In finite element method, another method which is more popular and accurate is used. This is Gauss or Gauss-Legendre method of integration. In this, the locations of the sampling points and corresponding weigthages are decided so as to obtain exact integral for a polynomial of given degree. Each Gauss point involves two unknowns namely, location of the point ξ_i and weightage W_i. For n Gauss points, there are $2n$ unknowns involved. With $2n$ unknowns, it is possible to represent a polynomial of $2n - 1$ degree exactly. Hence, n Gauss points can exactly integrate a polynomial of $2n - 1$ degree or less.

2.2.1 One Dimensional Integration

The integral in Eq. 2.43a is written as



$$\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{i=1}^{n} W_i\, f(\xi_i) \qquad \ldots(2.44)$$

It is easy to verify that the integral of function

$$f(\xi) = \int_{-1}^{+1} \xi^k\, d\xi \qquad \ldots(2.45)$$

is zero when k is odd. Thus, for example,

$$\int_{-1}^{+1} \xi d\xi = \int_{-1}^{+1} \xi^3\, d\xi = \int_{-1}^{+1} \xi^5\, d\xi = 0 \qquad \ldots(2.46)$$

Further, these functions have zero values at $\xi = 0$. Hence, to obtain zero value of such odd functions [$f(\xi) = -f(-\xi)$], following rules are decided.

(i) For even number of Gauss points (i.e. $n = 2, 4, 6\ldots$), the sampling points must be in pairs and the points in pairs must have equal weightages. Thus,

$$\left.\begin{aligned} \xi = -a \text{ and } \xi = +a;\ W(a) = W(-a) \\ \xi = -b \text{ and } \xi = +b;\ W(b) = W(-b) \end{aligned}\right\} \qquad \ldots(2.47)$$

(ii) For odd number of points (*i.e.*, $n = 1, 3, 5\ldots$), one point must be at $\xi = 0$ and remaining points must be in pairs as mentioned in rule (i) mentioned above.

The integrals of the function (Eq. 2.45) when k is even are written as

$$\left.\begin{aligned} \int_{-1}^{+1} 1 d\xi &= 2 \\ \int_{-1}^{+1} \xi^2 \, d\xi &= \frac{2}{3} \\ \int_{-1}^{+1} \xi^4 \, d\xi &= \frac{2}{5} \end{aligned}\right\} \quad \ldots(2.48)$$

The locations and weightages for $n = 1, 2, 3\ldots$ are derived below.

Consider $n = 1$

$$\int_{-1}^{+1} f(\xi) d\xi = \int_{-1}^{+1} (1 + \xi) d\xi = 2 + 0 \quad \ldots(2.49)$$

Only one Gauss point is needed. Hence, it will be located at $\xi = 0$. Using Eq. 2.44

$$\left.\begin{aligned} W(0) f(0) &= 2 \\ W(0)(1) &= 2 \\ W(0) &= 2 \end{aligned}\right\} \quad \ldots(2.50)$$

i.e.,

Hence,

Consider $n = 2$

$$\int_{-1}^{+1} f(\xi) \, d\xi = \int_{-1}^{+1} (1 + \xi + \xi^2 + \xi^3) \, d\xi = 2 + 0 + \frac{2}{3} + 0 \quad \ldots(2.51)$$

The terms ξ and ξ^3 will be taken care of by the two rules mentioned above. To get exact integral of remaining terms (separately for each term), using Eq. 2.44

For $$f(\xi) = 1 \; ; \; W(a)(1) + W(-a)(1) = 2 \qquad \ldots(2.52a)$$

For $$f(\xi) = \xi^2 \; ; \; W(a)a^2 + W(-a)a^2 = \frac{2}{3} \qquad \ldots(2.52b)$$

Noting that $$W(a) = W(-a)$$

$$W(a) + W(a) = 2 \qquad \ldots(2.52c)$$

and $$W(a)a^2 + W(a)a^2 = \frac{2}{3} \qquad \ldots(2.52d)$$

solution gives $$W(a) = 1 \qquad \ldots(2.52e)$$

and $$a = \sqrt{\frac{1}{3}} = 0.577\,350\,269 \qquad \ldots(2.52f)$$

Consider $n = 3$

$$\int_{-1}^{+1} (1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5)\, d\xi = 2 + 0 + \frac{2}{3} + 0 + \frac{2}{5} + 0 \qquad \ldots(2.53a)$$

Three Gauss points are needed. Obviously, one will be at $\xi = 0$ and remaining two will be in pair at $\xi = -a$ and $\xi = +a$ with $W(a) = W(-a)$. Using Eq. 2.44

For $$f(\xi) = 1 \; ; \; W(-a)(1) + W(0)(1) + W(a)(1) = 2 \qquad \ldots(2.53b)$$

For $$f(\xi) = \xi^2 \; ; \; W(-a)a^2 + 0 + W(a)a^2 = \frac{2}{3} \qquad \ldots(2.53c)$$

For $$f(\xi) = \xi^4 \; ; \; W(-a)a^4 + 0 + W(-a)a^4 = \frac{2}{5} \qquad ...(2.53d)$$

Solution is obtained noting that $W(a) = W(-a)$.

$$\left.\begin{aligned} W(0) &= \frac{8}{9} = 0.888\,888\,889 \\ W(a) &= \frac{5}{9} = 0.555\,555\,556 \\ a &= \sqrt{\frac{3}{5}} = 0.774\,596\,699 \end{aligned}\right\} \qquad ...(2.54)$$

It is difficult to derive weightages and locations for $n > 3$. Interested reader can see Ref. 1 for information upto $n = 10$. The locations and weightages upto $n = 4$ are given in Table 2.2.

Table 2.2: Locations and Weightages for Gauss Integration $\int_{-1}^{+1} f(\xi)d\xi = \sum_{i=1}^{n} W_i f(\xi_i)$

n	*Location*	*Weightage*
1	0	2.000
2	− 0.577 350 269	1.000
	+ 0.577 350 269	1.000
3	− 0.774 596 669	0.555 555 556
	0	0.888 888 889
	+ 0.774 596 669	0.555 555 556
4	− 0.861 136 312	0.347 854 845
	− 0.339 981 044	0.652 145 155
	+ 0.339 981 044	0.652 145 155
	+ 0.861 136 312	0.347 854 845

Information of Table 2.2 is useful to integrate a function in the range $(-1 \le \xi \le +1)$. It is, however, possible to integrate a function $f(\xi)$ in the range $(a \le x \le b)$ by suitably correlating coordinates in actual and parent regions.

Example 2.1. Find I using Gauss integration method.

$$I = \int_z^b f(x)\,dx = \int_2^6 (3 + 4x + 6x^2 + 4x^3)\,dx \qquad ...(2.55)$$

The relation between x and ξ can be readily expressed as

$$x = \frac{a+b}{2} + \left(\frac{b-a}{2}\right)\xi = 4 + 2\xi$$

$$dx = 2\,d\xi$$

$$\frac{dx}{d\xi} = S = \text{length scale factor} = 2$$

Two Gauss points are needed for exact integration. Gauss points are located at

$$x_1 = 4 - \frac{2 \times 1}{\sqrt{3}} = 2.8453$$

$$x_2 = 4 + \frac{2 \times 1}{\sqrt{3}} = 5.1547$$

$$f(x_1) = 155.0947\ ;\ W(1) = 1.0$$

$$f(x_2) = 730.9051\ ;\ W(2) = 1.0$$

Hence,
$$I = 2\,(155.0947 \times 1 + 730.9051 \times 1)$$

$$I = 1771.9996$$

Exact value of integral is 1772. Only four digits after decimal are used in the calculations. Hence, there is small difference.