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# **CHAPTER ONE**

## ***Matrices, Matrix Algebra, and Elementary Matrix Operations***

# Basic Concepts and Notation

- **Matrix and Vector Notation**

Matrices will be denoted by capital letters (e.g. matrix  $A$ ). Row and column vectors will be denoted by lower case letters (e.g. vector  $x$ ). The  $(i_{\text{th}}, j_{\text{th}})$  element of matrix  $A$  will be denoted by  $a_{ij}$  and the  $i_{\text{th}}$  element of vector  $x$  will be denoted by  $x_i$ .

# Matrix Definition

A rectangular array of numbers of the form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Is called an  $m \times n$  matrix, where  $m$  and  $n$  represent the number of rows and columns, respectively.  $A$  can be written as  $A = [a_{ij}]$ , where  $a_{ij}$  is the  $(i^{\text{th}}, j^{\text{th}})$  element of matrix  $A$ . The elements of a matrix can be real or complex numbers. If the elements are real, it is called a real matrix. On the other hand, if the elements are complex, it is called a complex matrix. Below are some examples of real and complex matrices

$$A = \begin{bmatrix} 2 & 9 & -23 \\ 12 & 4 & 7 \\ -2 & 0 & 8 \\ 15 & 6 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1-j & -2+j & j & 3 \\ 6-j7 & 4-j4 & 3-j & 1+j \\ 2+j & 8-j & 6 & 1+j \\ -6-j7 & 2+j7 & -j & 1+j4 \end{bmatrix}$$

A is a real  $4 \times 3$  matrix and B is a complex  $4 \times 4$  matrix.

An example of the application of matrices in image processing is embodied in the description and manipulation of images. Images are defined as matrices and most image processing techniques are based on matrix operations. A monochromatic digital image is a matrix (an array of numbers) of size  $N \times M$  denoted by  $f(n,m)$ , where the value of  $f$  at spatial coordinates  $(n,m)$  gives the intensity of the image at that location. This is shown in the following equation. Each element of  $f$  is a discrete integer quantity having a value between 0 and 255 for an 8-bit image.

$$f(n, m) = \begin{bmatrix} f(0, 0) & f(0, 1) & \cdots & f(0, M-1) \\ f(1, 0) & f(1, 1) & \cdots & f(1, M-1) \\ \vdots & \vdots & & \vdots \\ f(N-1, 0) & f(N-1, 1) & \cdots & f(N-1, M-1) \end{bmatrix}$$

On the other hand, a digital RGB color image of size  $N \times M$  consists of three channels: red (R), green (G) and blue (B). Each channel is represented by a matrix of size  $N \times M$ . At a given spatial coordinate  $(n,m)$ , the image has three components: each of which is between 0 and 255 for a 24-bit image (8 bits per channel). That is:

$$0 \leq R(n,m) \leq 255$$

$$0 \leq G(n,m) \leq 255$$

$$0 \leq B(n,m) \leq 255$$

## Example

The  $256 \times 256$  digital gray image and an  $8 \times 8$  block of that image are shown below.



139	144	149	153	155	155	155	155
144	151	153	156	159	156	156	156
150	155	160	163	158	156	156	156
159	161	162	160	159	158	159	159
159	160	161	162	162	155	155	155
161	161	161	161	160	157	157	157
162	162	161	163	162	157	157	157
162	162	161	161	163	158	158	158

$8 \times 8$  block



## Example

The  $256 \times 256$  digital color image is shown in below. An  $8 \times 8$  block of this RGB image and the corresponding R, G, and B channel matrices are also shown in the same figure.



$$R = \begin{bmatrix} 229 & 224 & 220 & 222 & 213 & 199 & 194 & 181 \\ 226 & 224 & 222 & 219 & 208 & 204 & 190 & 181 \\ 231 & 228 & 224 & 220 & 209 & 199 & 187 & 184 \\ 233 & 228 & 224 & 220 & 211 & 207 & 189 & 182 \\ 230 & 229 & 224 & 223 & 214 & 203 & 193 & 185 \\ 230 & 231 & 222 & 224 & 215 & 204 & 194 & 190 \\ 230 & 226 & 225 & 216 & 209 & 204 & 192 & 181 \\ 231 & 229 & 221 & 212 & 214 & 209 & 188 & 181 \end{bmatrix}$$

$$G = \begin{bmatrix} 126 & 113 & 113 & 101 & 93 & 82 & 78 & 76 \\ 125 & 117 & 120 & 108 & 94 & 83 & 77 & 69 \\ 124 & 115 & 113 & 109 & 94 & 91 & 80 & 77 \\ 127 & 115 & 115 & 107 & 94 & 87 & 81 & 71 \\ 130 & 124 & 119 & 106 & 100 & 87 & 77 & 73 \\ 120 & 125 & 119 & 110 & 103 & 87 & 75 & 72 \\ 127 & 122 & 113 & 108 & 93 & 85 & 83 & 74 \\ 125 & 117 & 116 & 115 & 98 & 87 & 82 & 74 \end{bmatrix}$$

$$B = \begin{bmatrix} 105 & 92 & 98 & 85 & 87 & 83 & 81 & 78 \\ 106 & 94 & 106 & 95 & 87 & 82 & 77 & 80 \\ 107 & 89 & 97 & 98 & 90 & 89 & 84 & 78 \\ 100 & 93 & 95 & 95 & 83 & 91 & 83 & 75 \\ 108 & 109 & 106 & 89 & 90 & 89 & 84 & 77 \\ 98 & 110 & 105 & 87 & 99 & 88 & 77 & 74 \\ 101 & 106 & 95 & 91 & 88 & 86 & 79 & 82 \\ 106 & 99 & 104 & 105 & 91 & 81 & 86 & 75 \end{bmatrix}$$

An  $8 \times 8$  block

## Elementary Matrices

Matrices with specific structures occur frequently in applications of linear algebra. In this section, we describe some of the most useful elementary matrices.

### 1. Square matrix:

If the number of rows of a matrix is equal to the number of columns of the matrix (i.e.  $m = n$ ), it is called a square matrix.

#### • **Example**

The matrix  $A$  is a  $3 \times 3$  square matrix

$$A = \begin{bmatrix} 2 & 3 & -9 \\ 3 & 5 & 8 \\ -7 & 4 & -3 \end{bmatrix}$$

## 2. Upper triangular matrix:

An  $m \times n$  matrix for which  $a_{ij} = 0$  for  $i > j$  is called an upper triangular matrix.

- **Example**

An upper triangular matrix

$$A = \begin{bmatrix} 2 & 7 & 8 & -3 & 1 \\ 0 & 4 & 6 & 2 & -4 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

### 3. Lower triangular matrix:

An  $m \times n$  matrix for which  $a_{ij} = 0$  for  $j > i$  is called a lower triangular matrix.

- **Example**

A lower triangular matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ -3 & -4 & 2 & 9 \end{bmatrix}$$

## 4. Diagonal matrix:

A square matrix whose nondiagonal elements are equal to zero is called a diagonal matrix.

- **Example**

A  $3 \times 3$  diagonal matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

## 5. Identity matrix:

A diagonal matrix with all diagonal elements equal to one

- **Example**

Identity matrices of different sizes

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The **MATLAB** command to create a  $N \times N$  identity matrix is **eye(N)**.

# Elementary Matrix Operations

In this section, we discuss several elementary matrix operations. These are utilized in a number of matrix manipulation techniques.

## 1. Matrix transpose and Hermitian:

The transpose of matrix  $A$  is obtained by interchanging the rows and columns of  $A$ . It is denoted by  $A^T$ . If matrix  $A$  has dimensions  $n \times m$ , its transpose will be  $m \times n$ .

Therefore:

$$B = A^T \text{ if } b_{ij} = a_{ji}$$

The matrix  $A^H$  is obtained by transposing and then conjugating every element of matrix  $A$ . That is:

$$B = A^H \text{ if } b_{ij} = \bar{a}_{ji}$$

where “bar” stands for complex conjugation. It should be noted that  $A^H = A^T$  if  $A$  is real. Finally, a real square matrix is symmetric if  $A = A^T$ . A complex square matrix is said to be Hermitian if  $A = A^H$ .

The MATLAB® command to transpose matrix  $A$  and store it in matrix  $B$  is

$$\mathbf{B} = \mathbf{A}^T.$$



# Example

$$\text{If } A = \begin{bmatrix} 2 & 7 & 8 & -3 & 1 \\ 1 & 4 & 6 & 2 & -4 \\ -2 & 3 & 2 & 3 & 5 \end{bmatrix} \text{ then } B = A^T = \begin{bmatrix} 2 & 1 & -2 \\ 7 & 4 & 3 \\ 8 & 6 & 2 \\ -3 & 2 & 3 \\ 1 & -4 & 5 \end{bmatrix}$$

$$\text{If } C = \begin{bmatrix} 2-j & 1+j3 \\ j4 & 4 \end{bmatrix} \text{ then } C^H = \begin{bmatrix} 2+j & -j4 \\ 1-j3 & 4 \end{bmatrix}$$

## 2. Properties of matrix transpose (Hermitian)

- The following properties hold for transpose and Hermitian operations.

The proofs are left as an exercise.

a.  $(A^T)^T = A$  or  $(A^H)^H = A$

b.  $(AB)^T = B^T A^T$  or  $(AB)^H = B^H A^H$

### **3. Symmetric matrix:**

An  $n \times n$  real matrix  $A$  is said to be symmetric if

$$A^T = A.$$

### **4. Skew-symmetric matrix:**

An  $n \times n$  real matrix  $A$  is said to be skew-symmetric if

$$A^T = -A.$$

### **5. Trace of a matrix:**

The trace of a square matrix is the sum of the diagonal elements of the matrix, that is:

$$\text{Trace}(A) = \sum_{i=1}^n a_{ii}$$

The MATLAB command to compute the trace of matrix  $A$  is **trace(A)**.

### Example

$$\text{If } A = \begin{bmatrix} 4 & 4 & -5 \\ 3 & 4 & 1 \\ 6 & 7 & -3 \end{bmatrix} \text{ then:}$$

$$\text{Trace}(A) = \sum_{i=1}^n a_{ii} = 4 + 4 - 3 = 5$$

# Properties of the trace of a matrix

The following properties hold for square matrices:

a.  $\text{Trace}(A \pm B) = \text{Trace}(A) \pm \text{Trace}(B)$

b.  $\text{Trace}(A^T) = \text{Trace}(A)$

c.  $\text{Trace}(AB) = \text{Trace}(B^T A^T)$

d.  $\text{Trace}(\alpha A) = \alpha \text{Trace}(A)$

e.  $\text{Trace}(AB) = \text{Trace}(BA)$

# Matrix Algebra

Matrix algebra deals with algebraic operations such as matrix addition, matrix subtraction, matrix multiplication and inversion.

## Matrix Addition and Subtraction

If  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are two matrices of the same size, then:

$$C = A \pm B$$

where

$$c_{ij} = a_{ij} \pm b_{ij}$$

## Example

Find  $C = A + B$  and  $D = A - B$  if:

$$A = \begin{bmatrix} 2 & 3 & -4 \\ -2 & 1 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 6 & -5 \end{bmatrix}$$

**Solution:**

$$C = A + B = \begin{bmatrix} 0 & 4 & -1 \\ 2 & 7 & 1 \end{bmatrix} \text{ and } D = A - B = \begin{bmatrix} 4 & 2 & -7 \\ -6 & -5 & 11 \end{bmatrix}$$

The MATLAB® command to add or subtract two matrices is  $\mathbf{C} = \mathbf{A} \pm \mathbf{B}$ .

- **Properties of Matrix Addition**

The following properties hold for matrix addition. They can be easily proven using the definition of addition.

a. Commutative property:  $A + B = B + A$

b. Associative property:  $A + (B + C) = (A + B) + C$



# Matrix Multiplication

Let  $A$  and  $B$  represent an  $n \times m$  and  $m \times k$  matrix, respectively. Then, the product

$C = AB$  is a matrix of size  $n \times k$  defined as:

$$c_{ij} = \sum_{l=1}^m a_{il}b_{lj} \quad i = 1, 2, \dots, n \quad \text{and} \quad j = 1, 2, \dots, k$$

Note that  $AB$  can be defined only if the number of columns of  $A$  is equal to the number of rows of  $B$ . Also observe that  $AB$  is, in general, different from  $BA$ , that is:

$$AB \neq BA$$

Hence, matrix multiplication is not generally commutative.

- The MATLAB® command to premultiply  $B$  by  $A$  and store it in  $C$  is

$$\mathbf{C} = \mathbf{A} * \mathbf{B}.$$

- The command  $\mathbf{C} = \mathbf{A} .* \mathbf{B}$  is used for point by point array multiplication, where

$$c_{ij} = a_{ij} b_{ij}.$$

In this later case, matrix  $A$  and  $B$  should have the same size.

# Example

Find  $D=AB$ ,  $E=BA$ , and  $F=A \times C$  (array multiplication) if:

$$A = \begin{bmatrix} 2 & 3 & -4 \\ -2 & 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ -2 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

**Solution:**

$$D = AB = \begin{bmatrix} -14 & 3 \\ 12 & 5 \end{bmatrix}, \quad E = BA = \begin{bmatrix} -4 & 10 & 16 \\ -6 & -5 & 14 \\ 2 & 11 & 0 \end{bmatrix}$$

$$\text{and } F = A \times C = \begin{bmatrix} 2 & 6 & -12 \\ -8 & 5 & 36 \end{bmatrix}$$

# ***Properties of Matrix Multiplication***

The following matrix multiplication properties are stated without proof:

- a. Distributive law:  $A(B + C) = AB + AC$
- b. Associative property:  $A(BC) = (AB)C$
- c.  $AI = IA = A$ , where  $I$  is the identity matrix of appropriate size and  $A$  is a square matrix.
- d.  $A^{m+n} = A^m A^n = A^n A^m$  where  $A^n = \overbrace{A \times A \times \cdots \times A}^{n \text{ times}}$