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Half of knowledge is to say "I do not know"

Contents

Hilbert's Nullstellensatz

We will be working in $k[X_1, \ldots, X_n]$, the ring of polynomials in n variables over the field k. (Any application of the Nullstellensatz requires that k be algebraically closed, but we will not make this assumption until it becomes necessary.) The set $\mathbb{A}^n = \mathbb{A}_k^n$ of all n-tuples with components in k is called affine n-space. If S is a set of polynomials in $k[X_1, \ldots, X_n]$, then the zero-set of S, that is, the set $V = \mathbb{V}(S)$ of all $x \in \mathbb{A}^n$ such that $f(x) = 0$ for every $f \in S$, is called a variety. (The term "affine variety" is more precise, but we will use the short form because we will not be discussing projective varieties in this section.) Thus a variety is the solution set of simultaneous polynomial equations.

If I is the ideal generated by S, then I consists of all finite linear combinations $\sum g_i f_i$ with $g_i \in k[X_1, \ldots, X_n]$ and $f_i \in S$. It follows that $\mathbb{V}(S) = \mathbb{V}(I)$, so every variety is the variety of some ideal. Also, we proved that we can make $\mathbb{A}^n$ into a topological space by taking varieties as the closed sets.

On the other hand, if X is an arbitrary subset of $\mathbb{A}^n$, we defined the ideal of X as

$$\mathbb{I}(X) = \{f \in k[X_1, \ldots, X_n] : f \text{ vanishes on } X\}.$$

By definition we have:

1. If $X \subseteq Y$ then $\mathbb{I}(X) \supseteq \mathbb{I}(Y)$; if $S \subseteq T$ then $\mathbb{V}(S) \supseteq \mathbb{V}(T)$.
2. $\mathbb{I}(\mathbb{V}(S)) \supseteq S$ and $\mathbb{V}(\mathbb{I}(X)) \supseteq X$.
3. $\mathbb{V}(\mathbb{I}(\mathbb{V}(S))) = \mathbb{V}(S)$ and $\mathbb{I}(\mathbb{V}(\mathbb{I}(X))) = \mathbb{I}(X)$.
4. $\mathbb{I}(\emptyset) = k[X_1, \ldots, X_n]$.
5. If k is an infinite field, then $\mathbb{I}(\mathbb{A}^n) = \{0\}$.
6. If $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$, then $I(\{a\}) = \langle X_1 - a_1, ..., X_n - a_n \rangle$.

Let us prove 5: Property (5) holds for $n = 1$ since a nonconstant polynomial in one variable has only finitely many zeros. Thus $f \neq 0$ implies that $f \notin \mathbb{I}(\mathbb{A}^1)$. If $n > 1$, let $f = a_r X_1^r + \ldots + a_1 X_1 + a_0$ where the a_i are polynomials in $X_2, \ldots, X_n$ and $a_r \neq 0$. By the induction hypothesis, there is a point $(x_2, \ldots, x_n)$ at which a_r does not vanish. Fixing this point, we can regard f as a polynomial in X_1, which cannot possibly vanish at all $x_1 \in k$. Thus $f \notin \mathbb{I}(\mathbb{A}^n)$.

To prove (6), note that the right side is contained in the left side because $X_i - a_i$ is 0 when $X_i = a_i$. Also, the result holds for $n = 1$ by the remainder theorem. Thus assume

$n > 1$ and let $f = b_r X_1^r + \ldots + b_1 X_1 + b_0 \in \mathbb{I}(\{a\})$, where the b_i are polynomials in $X_2, \ldots, X_n$ and $b_r \neq 0$. By the division algorithm, we have

$$f = (X_1 - a_1)g(X_1, \ldots, X_n) + h(X_2, \ldots, X_n)$$

and h must vanish at $(a_2, \ldots, a_n)$. By the induction hypothesis, $h \in \langle X_2 - a_2, \ldots, X_n - a_n\rangle$, hence $f \in \langle X_1 - a_1, \ldots, X_n - a_n\rangle$.

Irreducible algebraic sets

A variety is said to be **reducible** if it can be expressed as the union of two proper subvarieties; otherwise the variety is **irreducible**.

Problem 1: Assume that $\mathbb{I}(V)$ is not prime, and let $f_1 f_2 \in \mathbb{I}(V)$ with $f_1, f_2 \notin \mathbb{I}(V)$. If $x \in V$, then $x \notin \mathbb{V}(f_1)$ implies $x \in \mathbb{V}(f_2)$. Similarly, $x \notin \mathbb{V}(f_1)$ implies $x \in \mathbb{V}(f_2)$.

Proof If $x \in V$ and $f_1(x) \neq 0$, then $f_2(x)$ must be 0 since $f_1 f_2 \in \mathbb{I}(V)$; the result follows.

Problem 2: Show that V is reducible if $\mathbb{I}(V)$ is not prime.

Proof By Problem 1, $V \subseteq \mathbb{V}(f_1) \cup \mathbb{V}(f_2)$. Thus

$$V = (V \cap \mathbb{V}(f_1)) \cup (V \cap \mathbb{V}(f_2)) = V_1 \cup V_2.$$

By Problem 1, since $f_1 \notin \mathbb{I}(V)$, there exists $x \in V$ such that $f_1(x) \neq 0$. Thus $x \notin V_1$, so $V_1 \subsetneq V$; similarly, $V_2 \subsetneq V$.

Problem 3: If V and W are varieties with $V \subset W$, then $\mathbb{I}(V) \supsetneq \mathbb{I}(W)$.

Proof $V \subsetneq W$ implies $\mathbb{I}(V) \supseteq \mathbb{I}(W)$. If $\mathbb{I}(V) = \mathbb{I}(W)$, let $V = \mathbb{V}(S)$, $W = \mathbb{V}(T)$. Then $\mathbb{I}(\mathbb{V}(S)) = \mathbb{I}(\mathbb{V}(T))$, and by applying $\mathbb{V}$ to both sides, we have $V = W$.

Problem 4: Assume that $V = V_1 \cup V_2$, with $V_1, V_2 \subsetneq V$. By Problem 3, we can choose $f_i \in \mathbb{I}(V_i)$ with $f_i \notin \mathbb{I}(V)$. Show that $f_1 f_2 \in \mathbb{I}(V)$, so $\mathbb{I}(V)$ is not a prime ideal.

Proof Let $x \in V$; if $f_1(x) \neq 0$, then since $f_1 \in \mathbb{I}(V_1)$, we have $x \notin V_1$. But then $x \in V_2$, and therefore $f_2(x) = 0$ (since $f_2 \in \mathbb{I}(V_2)$. Thus $f_1 f_2 = 0$ on V, so $f_1 f_2 \in \mathbb{I}(V)$.

Proposition: Any variety in k^n is the union of finitely many irreducible subvarieties. Moreover, this decomposition is unique “assuming that we discard any subvariety that is contained in another one”.

Proof First, let V be reducible variety. Then V is the union of proper subvarieties V_1 and V_2. If V_1 is reducible, then it too is the union of proper subvarieties. This decomposition process must terminate in a finite number of steps, for otherwise by Problems 1-4, there would be a strictly increasing infinite sequence of ideals, contradicting the fact that $k[X_1, \ldots, X_n]$ is Noetherian. Now, if $V = \bigcup_i V_i = \bigcup_j W_j$, then

$$V_i = V_i \bigcap \Big(\bigcup_j W_j \Big) = \bigcup_j (V_i \cap W_j).$$

So by irreducibility, $V_i = V_i \cap W_j$ for some j. Thus $V_i \subseteq W_j$, and similarly $W_j \subseteq V_k$ for some k. But then $V_i \subseteq V_k$, hence $i = k$ (otherwise we would have discarded V_i). Thus each V_i can be paired with a corresponding W_j, and vice versa.

Warmup: Assume that k is algebraically closed. Suppose that $\mathbb{A}^n$ is covered by open sets $\mathbb{A}^n \backslash \mathbb{V}(I_i)$ in the Zariski topology. Let I is the ideal generated by the I_i, so that $I = \sum I_i$, the set of all finite sums $x_{i_1} + \ldots + x_{i_r}$ with $x_{i_j} \in I_{i_j}$. Show that $1 \in I$. Furthermore, Show that $\mathbb{A}^n$ is **compact** in the Zariski topology.

Recall that the Hilbert Basis Theorem states: If R is a Noetherian ring, then $R[X_1, \ldots, X_n]$ is also Noetherian. It follows that every variety is the intersection of finitely many **hypersurfaces** (zero-sets of single polynomials). In fact, if $V = \mathbb{V}(I)$ is a variety, then I has finitely many generators, namely $f_1, \ldots, f_r$. But then $V = \bigcap_{i=1}^r \mathbb{V}(f_i)$.

The Nullstellensatz

We have observed that every variety V defines an ideal $\mathbb{I}(V)$ and every ideal I defines a variety $\mathbb{V}(I)$. Moreover, if $\mathbb{I}(V_1) = \mathbb{I}(V_2)$, then $V_1 = V_2$. But it is entirely possible for many ideals to define the same variety. For example, the ideals $\langle f \rangle$ and $\langle f^m \rangle$ need not coincide, but their zero-sets are identical. A variety V is, by definition, always expressible as $\mathbb{V}(S)$ for some collection S of polynomials, but an ideal I need not be of the special form $\mathbb{I}(X)$. Hilbert's Nullstellensatz says that if two ideals define the same variety, then, informally, the ideals are the same "up to powers". More precisely, if g belongs to one of the ideals, then g^r belongs to the other ideal for some positive integer r. Thus the only factor preventing a one-to-one correspondence between ideals and varieties is that a polynomial can be raised to a power without affecting its zero-set. In this section we collect some results needed for the proof of the Nullstellensatz. We begin by showing that each point of $\mathbb{A}^n$ determines a maximal ideal.

Proposition: If $a = (a_1, \dots, a_n) \in \mathbb{A}^n$, then $I = I(\{a\}) = \langle X_1 - a_1, \dots, X_n - a_n \rangle$ is a maximal ideal.

Proof Suppose that I is properly contained in the ideal J, with $f \in J \backslash I$. Apply the division algorithm n times to get

$$f = f_1(X_1 - a_1) + f_2(X_2 - a_2) + \dots + f_n(X_n - a_n) + r$$

where $f_1, f_2, \dots, f_n \in k[X_1, \dots, X_n], r \in k$. Note that r cannot be 0 since $f \notin I$. But $f \in J$, so by solving the above equation for r we have $r \in J$, hence $1 = (1/r)r \in J$. Consequently, $J = k[X_1, \dots, X_n]$.

Recall that the radical of an ideal I (in any commutative ring R) is the set of all elements $f \in R$ such that $f^r \in I$ for some positive integer r. It is clear, if f^r and g^s belong to I, then by the binomial theorem, $(f+g)^{r+s-1} \in I$, and it follows that $\sqrt{I}$ is an ideal. Let us introduce the following proposition:

Proposition: If I is any ideal of $k[X_1, \dots, X_n]$, then $\sqrt{I} \subseteq \mathbb{I}(\mathbb{V}(I))$.

Proof If $f \in \sqrt{I}$, then $f^r \in I$ for some positive integer r. But then f^r vanishes on $\mathbb{V}(I)$, hence so does f. Therefore $f \in \mathbb{I}(\mathbb{V}(I))$.

Remark: The Nullstellensatz states that $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$, and the hard part is to prove that $\mathbb{I}(\mathbb{V}(I)) \subseteq \sqrt{I}$. The technique is known as the "Rabinowitsch trick", and it is indeed very clever.

The Nullstellensatz: Equivalent Versions

We are now in position to establish the equivalence of several versions of the Nullstellensatz.

Theorem: For any field k and any positive integer n, the following statements are equivalent.

1. **Maximal Ideal Theorem:** The maximal ideals of $k[X_1, \dots, X_n]$ are the ideals of the form $\langle X_1 - a_1, \dots, X_n - a_n \rangle$, $a_1, \dots, a_n \in k$. Thus maximal ideals correspond to points.
2. **Weak Nullstellensatz:** If I is an ideal of $k[X_1, \dots, X_n]$ and $\mathbb{V}(I) = \emptyset$, then $I = k[X_1, \dots, X_n]$. Equivalently, if I is a proper ideal, then $\mathbb{V}(I)$ is not empty.
3. **Nullstellensatz:** If I is an ideal of $k[X_1, \dots, X_n]$, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$

4. k is algebraically closed.

Proof (1) implies (2): Let I be a proper ideal, and let J be a maximal ideal containing I. It follows that $\mathbb{V}(J) \subseteq \mathbb{V}(I)$, so it suffices to show that $\mathbb{V}(J)$ is not empty. By (1), J has the form $\langle X_1 - a_1, \ldots, X_n - a_n \rangle$. But then $a = (a_1, \ldots, a_n) \in \mathbb{V}(J)$. [In fact $\mathbb{V}(J) = \{a\}$.]

(2) implies (3): This was done in previous section.

(3) implies (2): We use the fact that the radical of an ideal I is the intersection of all prime ideals containing I. Let I be a proper ideal of $k[X_1, \ldots, X_n]$. Then I is contained in a maximal, hence prime ideal P. Consequently, $\sqrt{I}$ is also contained in P, hence $\sqrt{I}$ is a proper ideal. By (3), $\mathbb{I}(\mathbb{V}(I))$ is a proper ideal. But if $\mathbb{V}(I) = \emptyset$, then $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = k[X_1, \ldots, X_n]$, a contradiction.

(2) implies (1): If I is a maximal ideal, then by (2) there is a point $a = (a_1, \ldots, a_n) \in \mathbb{V}(I)$. Thus every $f \in I$ vanishes at a, in other words, $I \subseteq \mathbb{I}(\{a\})$. But $\langle X_1 - a_1, \ldots, X_n - a_n \rangle = \mathbb{I}(\{a\})$; to see this, decompose $f \in \mathbb{I}(\{a\})$. Therefore the maximal ideal I is contained in the maximal ideal $\langle X_1 - a_1, \ldots, X_n - a_n \rangle$, and it follows that $I = \langle X_1 - a_1, \ldots, X_n - a_n \rangle$.

(4) implies (1): Let I be a maximal ideal of $k[X_1, \ldots, X_n]$, and let $K = k[X_1, \ldots, X_n]/I$, a field containing an isomorphic copy of k via $c \to c + I, c \in k$. Consequently, K is a finite extension of k, so by (4), $K = k$. But then $X_i + I = a_i + I$ for some $a_i \in k, i = 1, \ldots, n$. Therefore $X_i - a_i$ is zero in $k[X_1, \ldots, X_n]/I$, in other words, $X_i - a_i \in I$. Thus, $I \supseteq \langle X_1 - a_1, \ldots, X_n - a_n \rangle$, and we must have equality.

(1) implies (4): Let f be a nonconstant polynomial in $k[X_1]$ with no root in k. We can regard f is a polynomial in n variables with no root in $\mathbb{A}^n$. Let I be a maximal ideal containing the proper ideal $\langle f \rangle$. By (1), I is of the form $\langle X_1 - a_1, \ldots, X_n - a_n \rangle = \mathbb{I}(\{a\})$ for some $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$. Therefore f vanishes at a, a contradiction.

Corollary 1: If the ideals I and J define the same variety and a polynomial g belongs to one of the ideals, then some power of g belongs to the other ideal.

Proof If $\mathbb{V}(I) = \mathbb{V}(J)$, then by the Nullstellensatz, $\sqrt{I} = \sqrt{J}$. If $g \in I \subseteq \sqrt{I}$, then $g^r \in J$ for some positive integer r.

Corollary 2: The maps $V \to \mathbb{I}(V)$ and $I \to \mathbb{V}(I)$ set up a one-to-one correspondence between varieties and radical ideals.

Proof We know that $\mathbb{V}(\mathbb{I}(V)) = V$. By the Nullstellensatz, $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = I$ for radical ideals. It remains to prove that for any variety V, $\mathbb{I}(V)$ is a radical ideal. If $f^r \in \mathbb{I}(V)$, then f^r, hence f, vanishes on V, so $f \in \mathbb{I}(V)$.

Corollary 3: Let $f_1, \ldots, f_r, g \in k[X_1, \ldots, X_n]$, and assume that g vanishes wherever the f_i all vanish. Then there are polynomials $h_1, \ldots, h_r \in k[X_1, \ldots, X_n]$ and a positive integer s such that $g^s = h_1 f_1 + \ldots + h_r f_r$.

Proof Let I be the ideal generated by $f_1, \ldots, f_r$. Then $\mathbb{V}(I)$ is the set of points at which all f_i vanish, so that $\mathbb{I}(\mathbb{V}(I))$ is the set of polynomials that vanish wherever all f_i vanish. Thus g belongs to $\mathbb{I}(\mathbb{V}(I))$, which is $\sqrt{I}$ by the Nullstellensatz. Consequently, for some positive integer s, we have $g^s \in I$, and the result follows.

Aside on commutative algebra: Let B be a subring of A.

- A is finitely generated over (or finitely generated as a B- algebra) if there are finitely many elements $a_1, \ldots, a_n$ such that $A = B[a_1, \ldots, a_n]$.
- A is a finite B-algebra if there are finitely many elements $a_1, \ldots, a_n$ with $A = Ba_1 + \ldots + Ba_n$.

Example: The polynomial ring $k[X_1, \ldots, X_n]$ finitely generated k-algebra, but not a finite k-algebra.

The Coordinate Ring of a Variety

Throughout this section V denotes an affine variety in $\mathbb{A}_k^n$. Recall, a **polynomial function** on V is a map $f : V \to k$ such that there is a polynomial $F \in k[X_1, \ldots, X_n]$ with $f(P) = F(P)$ for all $P \in V$.

Remark: The polynomial F is not uniquely determined by the values it takes on V. In particular, for F and G are elements in $k[X_1, \ldots, X_n]$ we have

$$F \downharpoonright_V = G \downharpoonright_V \iff (F - G) \downharpoonright_V = 0 \iff F - G \in \mathbb{I}(V).$$

The **coordinate ring of** V is defined by

$$k[V] := k[X_1, \ldots, X_n]/\mathbb{I}(V).$$

From the above remarks we can make the following identification:

$$k[V] = \{f : f : V \to k \text{ is a polynomial function}\}.$$

It follows that

$$V \text{ is irreducible} \iff k[V] \text{ is an integral domain}.$$

Note that the coordinate functions $X_1, \ldots, X_n$ generate $k[V]$, which explains the terminology "coordinate ring". In previous section we studied the relationship between the subsets of $\mathbb{A}_k^n$ and the ideals in the coordinate ring $k[\mathbb{A}_k^n] = k[X_1, \ldots, X_n]$. The ring $k[V]$ plays the same role for V that $k[X_1, \ldots, X_n]$ plays for $\mathbb{A}_k^n$. In particular, there is a correspondence between the closed sets W contained in V and the ideals of $k[V]$. To describe this relationship, first note that the projection $\pi : k[X_1, \ldots, X_n] \to k[V] = k[X_1, \ldots, X_n]/\mathbb{I}(V)$ induces a bijection

$$\{\text{ideals } J \subseteq k[X_1, \ldots, X_n] : J \supseteq \mathbb{I}(V)\} \overset{1:1}{\longleftrightarrow} \{\text{ideals } J' \subseteq k[V]\},$$

defined by $J \mapsto J/\mathbb{I}(V)$, with inverse map $J' \mapsto \pi^{-1}(J')$. This mapping preserves radical ideals, prime ideals and maximal ideals. Hence, we have the following correspondences:

$$\begin{array}{ccc}
\{\text{radical ideals } J' \subseteq k[V]\} & \overset{1:1}{\longleftrightarrow} & \{\text{closed sets } W \subseteq V\} \\
\cup\!\mid & & \cup\!\mid \\
\{\text{prime ideals } J' \subseteq k[V]\} & \overset{1:1}{\longleftrightarrow} & \{\text{irreducible sets } W \subseteq V\} \\
\cup\!\mid & & \cup\!\mid \\
\{\text{maximal ideals } J' \subseteq k[V]\} & \overset{1:1}{\longleftrightarrow} & \{\text{points of } V\}
\end{array}$$

Here we have been talking about closed sets of V in the sense of the topology induced by the Zariski topology on $\mathbb{A}_k^n$. This result shows that this is the same as the topology defined by taking the closed sets of V to be sets of the form $\mathbb{V}(J)$, where J is a radical ideal in $k[V]$.

We will now discuss a further important characteristic property of coordinate rings.

An algebra A is **reduced** if A contains no non-zero nilpotent elements, i.e., for $x \in A$, if $x^n = 0$ for some $n > 1$, then $x = 0$.

The algebra $k[X_1, \ldots, X_n]/I$ is reduced if and only if I is a radical ideal, and so, since $\mathbb{I}(V)$ is a radical ideal, the coordinate ring is a reduced algebra. By construction, the coordinate ring $k[V]$ of an affine variety V is a finitely generated k-algebra. These properties characterize coordinate rings of varieties, in the sense that, given any finitely generated reduced k-algebra A, we can construct a corresponding algebraic variety as follows. By choosing generators $a_1, \ldots, a_n$ we can write $A = k[a_1, \ldots, a_n]$, and we have a surjective homomorphism

$$\pi : k[X_1, \ldots, X_n] \to A = k[a_1, \ldots, a_n], X_i \mapsto a_i.$$

Let $I = \ker(\pi)$. Then $V = \mathbb{V}(I)$ is a variety which is irreducible if and only if A is an integral domain. Since A is reduced, I is a radical ideal, so $\mathbb{I}(V) = I$, and so by construction $A = k[V]$.

Example For the usual parabola

$$C_0 = \{(X, Y) \in \mathbb{A}_k^2 : Y - X^2 = 0\}$$

we have

$$k[C_0] = k[X, Y]/\langle Y - X^2\rangle \cong k[X] \cong k[\mathbb{A}_k^1].$$

For the semi-cubical parabola, given by

$$C_1 = \{(X, Y) \in \mathbb{A}_k^2 : Y^2 - X^3 = 0\}$$

we have

$$k[C_1] = k[X, Y]/\langle Y^2 - X^3\rangle.$$

Notice that $k[C_1]$ is not a UFD. As sets, there are bijections between each of C_0 and C_1 and $\mathbb{A}_k^1$, since each curve has a rational parametrization, given by $t \mapsto (t, t^2)$ and $t \mapsto (t^2, t^3)$ respectively However, as algebraic varieties C_0 and C_1 behave differently.

Polynomial Maps

We now consider maps between algebraic sets. Throughout this section $V \subseteq \mathbb{A}_k^n$ and $W \subseteq \mathbb{A}_k^m$ are closed sets, and X_i, for $1 \leq i \leq n$, and Y_i, for $1 \leq j \leq m$, are the coordinate functions on $\mathbb{A}_k^n$ and $\mathbb{A}_k^m$ respectively.

A map $f : V \to W$ is called a **polynomial map** if there are polynomials $F_1, \ldots, F_m \in k[X_1, \ldots, X_n]$ such that

$$f(P) = (F_1(P), \ldots, F_m(P)) \in \mathbb{A}_k^m$$

for all points $P \in V$.

Proposition: Let $Y_1, \ldots, Y_m$ be the coordinate functions on $\mathbb{A}_k^m$. A map $f : V \to W$

is a polynomial map if and only if $f_j := Y_j \circ f \in k[V]$ for $j = 1, \dots, m$.

Proof Composing f with Y_j gives the projection onto the j-ih coordinate:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "f_j"'] & W \subseteq \mathbb{A}_k^m \arrow[d, "Y_j"] \\ & \mathbb{A}_k^1 \end{tikzcd}$$

Let $f_j = Y_j \circ f$. Then if f is a polynomial map we have $f_j(P) = F_j(P)$ for some $F_j \in k[X_1, \dots, X_n]$. Thus f_j is a polynomial map, and hence $f_j \in k[V]$.

On the other hand, if $f_j = Y_j \circ f$ is a polynomial map for every j, then by definition there are polynomials $F_1, \dots, F_m \in k[X_1, \dots, X_n]$ with

$$f(P) = (F_1(P), \dots, F_m(P)) \in \mathbb{A}_k^m$$

for all $P \in V$.

Remark: The above proposition shows that any polynomial map $f : V \to W$ can be written in the form $f = (f_1, \dots, f_m)$ with $f_1, \dots, f_m \in k[V]$.

Proposition: A polynomial map $f : V \to W$ is continuous in the Zariski topology.

Proof We must show that if $Z \subseteq W$ is closed then $f^{-1}(Z)$ is also closed. But this is clear, since if $= \{h_1 = \dots = h_r = 0\}$ then $f^{-1}(Z) = \{h_1 \circ f = \dots = h_r \circ f = 0\}$, and so is also closed.

Example 1: Consider the two curves

$$C_0 = \{(X, Y) \in \mathbb{A}_k^2 : Y - X^2 = 0\},$$
$$C_1 = \{(X, Y) \in \mathbb{A}_k^2 : Y^2 - X^3 = 0\}.$$

The map from $\mathbb{A}_k^1$ to the parabola

$$f : \mathbb{A}_k^1 \to C_0, t \mapsto (t, t^2),$$

and the map

$$f : \mathbb{A}_k^1 \to C_1, t \mapsto (t^2, t^3),$$

are both bijective polynomial maps.

Example 2: If $V \subseteq \mathbb{A}_k^n$, $W \subseteq \mathbb{A}_k^n$ and $X \subseteq \mathbb{A}_k^l$ are algebraic sets, and $f : V \to W$

and $g : W \to X$ are polynomial maps, then $g \circ f : V \to X$ is also a polynomial map. This follows immediately from the fact that the composition of a polynomial with a polynomial is again a polynomial.

Now let $f : V \to W$ be a polynomial map. For $g \in k[W]$ we define $f^*(g) := g \circ f$:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "f^*(g)"'] & W \arrow[d, "g"] \\ & \mathbb{A}^1_k \end{tikzcd}$$

Since g is a polynomial function, $g \circ f$ is also a polynomial function. Thus we have a map

$$f^* : k[W] \longrightarrow k[V],$$
$$g \longmapsto f^*(g) := g \circ f.$$

If $f : V \to W$, $g : W \to X$ are polynomial maps, then

$$(g \circ f)^* = f^* \circ g^* : k[X] \longrightarrow k[V].$$

This follows immediately from the fact that for $h \in k[X]$ we have

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = g^*(h) \circ f = f^*(g^*(h)).$$

The map f^* is a ring homomorphism, since we have

$$f^*(g_1 + g_2) = (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f = f^*(g_1) + f^*(g_2),$$
$$f^*(g_1 \cdot g_2) = (g_1 \cdot g_2) \circ f = g_1 \circ f \cdot g_2 \circ f = f^*(g_1) \cdot f^*(g_2).$$

For any constant $c \in k$ we have $f^*(c) = c$, so f^* is also a k-algebra homomorphism. Thus every polynomial map $f : V \to W$ gives rise to a k-algebra homomorphism $f^* : k[W] \to k[V]$. The next theorem says that this procedure has an inverse.

Proposition: If $\varphi : k[W] \to k[V]$ is a k-algebra homomorphism, then there exists a unique polynomial map $f : V \to W$ such that $\varphi = f^*$.

Proof Exercise.

In fact, There is a bijection:

$$\left\{\begin{array}{c}\text{polynomial maps}\\ f:V\to W\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}k-\text{algebra homomorphisms}\\ \varphi:k[W]\to k[V]\end{array}\right\}$$
$$f \longmapsto f^*$$

A polynomial map $f : V \to W$ is an **isomorphism** if there is a polynomial map $g : W \to V$ such that $f \circ g = id_W$ and $g \circ f = id_V$.

Proposition: A polynomial map $f : V \to W$ is an isomorphism of varieties if and only if $f^* : k[W] \to k[V]$ is an isomorphism of k-algebras.

Proof This follows from the fact that $(f \circ g)^* = g^* \circ f^*$.

Example 1: Let $A = [\alpha_{ij}]$ be an invertible $(n \times n)$ matrix. Then the linear forms $y_i = \sum_{j=1}^{n} \alpha_{ij}x_j$ define a bijective polynomial map

$$f = (y_1, \ldots, y_n) : \mathbb{A}^n_k \to \mathbb{A}^n_k.$$

Example 2: Consider the parabola $C_0 = \{y - x^2 = 0\}$ in $\mathbb{A}^2_k$ and the parametrization

$$f : \mathbb{A}^1_k \longrightarrow C_0,$$
$$t \longmapsto (t, t^2).$$

The projection $p : \mathbb{A}^2_k \longrightarrow \mathbb{A}^1_k$ to the first coordinate, restricted to C_0, gives an inverse map

$$p\downharpoonleft_{C_0} = g : C_0 \longrightarrow \mathbb{A}^1_k,$$
$$(x, y) \longmapsto x.$$

Thus f is an isomorphism. We can also see this by considering the map $f^* : k[C_0] \to k[\mathbb{A}^1_k]$, since

$$f^* : k[C_0] \cong k[x] \longrightarrow k[\mathbb{A}^1_k] = k[t],$$
$$x \longmapsto t,$$

is an isomorphism.

Example 3: A different kind of behavior can be observed in the case of the semi-cubical parabola, $C_1 = \{(x, y) : y^2 = x^3\}$. Specifically, the map

$$f : \mathbb{A}^1_k \longrightarrow C_1,$$
$$t \longmapsto (t^2, t^3),$$

is a bijection, but the image $f^*(k[C_1]) \subseteq k[\mathbb{A}^1_k] = k[t]$ is generated by $f^*(x) = t^2$ and $f^*(y) = t^3$, and so $f^*(k[C_1]) \neq k[t]$, and thus f is not an isomorphism. Though f is a bijection, the inverse map

$$g : C_1 \longrightarrow \mathbb{A}^1_k,$$

$$g(x,y) = \begin{cases} y/x, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

is not a polynomial map.

Projective Varieties

In this chapter we introduce projective varieties and investigate morphisms between them.

Projective Space

Let V be a finite dimensional vector space over k. We consider the following equivalence relation on $V\backslash\{\mathbf{0}\}$:

$$u \sim v \iff \text{there exists } \lambda \in k^* \text{ with } u = \lambda v.$$

The **projective space** associated to V is defined by

$$\mathbb{P}(V) := V\backslash\{\mathbf{0}\}/\sim.$$

The **dimension** of $\mathbb{P}(V)$ is defined by $\dim \mathbb{P}(V) := \dim V - 1$.

Two vectors are equivalent if and only if they span the same line in V, so geometrically, the projective space space associated to V is the set of all lines through the origin in V. In particular, taking $V = k^{n+1}$, we define

$$\mathbb{P}^n := \mathbb{P}^n_k := \mathbb{P}(k^{n+1}).$$

Example 1: The space $\mathbb{P}^1_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^2)$ is homeomorphic to S^1, as shown in Figure 1.

Example 2: The real projective plane has a decomposition

$$\mathbb{P}^2_{\mathbb{R}} := \mathbb{P}(\mathbb{R}^3) = \mathbb{R}^2 \cup \mathbb{P}^1(\mathbb{R}).$$

Under this decomposition $\mathbb{R}^2$ corresponds to the set of lines that do not lie in the (x,y)-plane, and $\mathbb{P}^1(\mathbb{R})$ corresponds to the set of lines in the (x,y)-plane, as in Figure 2.

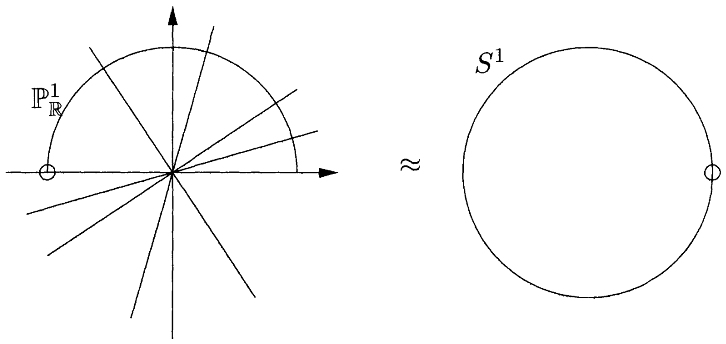

Figure 1: The real projective line $\mathbb{P}^1_{\mathbb{R}}$

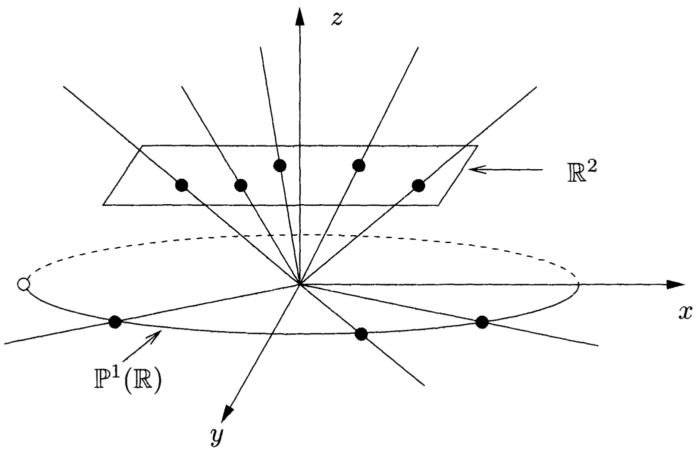

Figure 2: Decomposition of real projective space $\mathbb{P}^2_{\mathbb{R}}$

In this section we will denote the residue class map as follows:

$$\pi : V\backslash\{\mathbf{0}\} \longrightarrow \mathbb{P}(V).$$

For the special case $\mathbb{P}(V) = \mathbb{P}^n_k$, we use the notation

$$(x_0 : \ldots : x_n) := \pi((x_0, \ldots, x_n)),$$

and we call $(x_0 : \ldots : x_n)$ the homogeneous coordinates of the point $= \pi((x_0, \ldots, x_n)) \in \mathbb{P}^n_k$. These are well defined only up to multiplication by a common scalar. Nevertheless, we will see that we can "compute" with them.

In the above example we saw how $\mathbb{P}^2_{\mathbb{R}}$ decomposes into a union of an affine and a projective part. In fact, any projective space can be decomposed in a similar way into an affine subspace and a projective subspace of smaller dimension. For $\mathbb{P}^n_k$ such a decomposition is given by setting

$$U_l := \{(x_0 : \ldots : x_n) \in \mathbb{P}^n_k : x_l \neq 0\},$$

$$H_l := \{(x_0 : \ldots : x_n) \in \mathbb{P}^n_k : x_l = 0\}.$$

The space H_l can be identified with $\mathbb{P}^{n-1}_k$, and U_l can be identified with $\mathbb{A}^n_k$, for example, by the mutually inverse maps

$$i_l : \mathbb{A}_k^n \longrightarrow U_l,$$
$$(x_1, \ldots, x_n) \longmapsto (x_1 : \ldots : x_{l-1} : 1 : x_{l+1} : \ldots : x_n),$$

and

$$j_l : U_l \longrightarrow \mathbb{A}_k^n,$$
$$(x_0 : \ldots : x_{l-1} : x_l : x_{l+1} : \ldots : x_n) \longmapsto (\frac{x_0}{x_l}, \ldots, \frac{x_{l-1}}{x_l}, \frac{x_{l+1}}{x_l}, \ldots, \frac{x_n}{x_l}).$$

This gives us a decomposition

$$\mathbb{P}_k^n = U_l \cup H_l = \mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}.$$

Remark: Generally we fix the value of l (usually $l = 0$ or n), and refer to U_l as the affine part of $\mathbb{P}_k^n$ and to H_l as the hyperplane at infinity. Points in H_l are called "points at infinity". Whilst this particular decomposition into an affine and a projective piece is conventional, in fact any projective hyperplane can be taken in $\mathbb{P}_k^n$, and the complement will always be an affine space.

A **projective subspace** of $\mathbb{P}(V)$ is a subset of the form $\pi(W\backslash\{\mathbf{0}\})$, where $W \subseteq V$ is a linear subspace and π is the residue class map. We write $\mathbb{P}(W) \subseteq \mathbb{P}(V)$.

A projective subspace is itself naturally a projective space. If $\dim W = \dim V - 1$, then we call $\mathbb{P}(W)$ a hyperplane in $\mathbb{P}(V)$. Projective spaces of dimensions 1 and 2 are called **projective lines** and **projective planes** respectively.

Proposition: Let $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ be projective subspaces of an n-dimensional projective space $\mathbb{P}(V)$. If $\dim \mathbb{P}(W_1) + \dim \mathbb{P}(W_2) \geq n$, then $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ intersect, i.e. $\mathbb{P}(W_1) \cap \mathbb{P}(W_2) \neq \emptyset$.

Proof We have $\dim W_1 + \dim W_2 \geq 2(n+1) \geq n+2 = \dim V + 1$. So W_1 and W_2 intersect at least in a line.

In particular two lines in the projective plane always intersect. This is in contrast to the situation in the affine plane, where two lines may be parallel. In projective space the distinction between the cases of parallel and nonparallel lines no longer exists.

Remark: Any projective space has a covering by affine spaces

$$\mathbb{P}_k^n = U_0 \cup U_1 \cup \ldots \cup U_n,$$

where

$$U_i := \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_i \neq 0\}.$$

Projective Varieties

We want to consider the zero sets in projective space of polynomial equations defined on $\mathbb{P}_k^n$. Since the homogeneous coordinates of a point $= (x_0 : \ldots : x_n) \in \mathbb{P}_k^n$ are only determined up to multiplication by a common scalar, we must make a restriction to homogeneous polynomials. A polynomial

$$f(x_0, \ldots, x_n) = \sum a_{\nu_0 \ldots \nu_n} x_0^{\nu_0} \ldots x_n^{\nu_n}$$

is called **homogeneous** of degree d if all the monomials have the same degree $d = \nu_0 + \ldots + \nu_n$. We will also use the word **form** to refer to homogeneous polynomials, e.g., **linear form**, **quadratic form**, **cubic form**, etc. If f is homogeneous of degree d, then we have

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n).$$

In particular, the zero set of f,

$$\mathbb{V}(f) := \{(x_0, \ldots, x_n) \in \mathbb{P}_k^n : f(x_0, \ldots, x_n) = 0\} \subseteq \mathbb{P}_k^n,$$

is well defined.

A **projective variety** is a subset $V \subseteq \mathbb{P}_k^n$ such that there is a set of homogeneous polynomials $T \subseteq k[x_0, \ldots, x_n]$ with

$$V = \{P \in \mathbb{P}_k^n : f(P) = 0 \text{ for all } f \in T\}.$$

As in the affine case, we may assume that has only finitely many elements. We now give a few examples of projective varieties.

Example 1: We have already seen the projective subvariety of $\mathbb{P}_k^n$ given by the hyperplane at infinity,

$$H_n = \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_n = 0\}.$$

Example 2: Consider the map

$$\varphi : \mathbb{P}_k^1 \longrightarrow \mathbb{P}_k^3,$$

$$\varphi(t_0 : t_1) \longmapsto (t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3),$$

The image $:= \varphi(\mathbb{P}^1_k)$ is a projective variety, given by

$$C = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{P}^3_k : \mathsf{rank} \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1 \right\}.$$

This means that is the intersection of three quadrics $C = Q_1 \cap Q_2 \cap Q_3$, where

$$Q_1 = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_k : x_0x_2 - x_1^2 = 0\},$$

$$Q_2 = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_k : x_0x_3 - x_1x_2 = 0\},$$

$$Q_3 = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_k : x_1x_3 - x_2^2 = 0\}.$$

The curve cannot be defined by only two quadratic equations. On the other hand, we have $C = Q_1 \cap F$, where

$$F := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_k : x_0x_3^2 - 2x_1x_2x_3 + x_2^3 = 0\},$$

that is, the quadric Q_1 and the cubic F meet along the curve C. The curve is called the (projective) **rational normal curve** of degree 3.

More about the projective algebraic sets

Recall, a projective algebraic subset of $\mathbb{P}^n_k$ is the common zero set of a collection of homogeneous polynomials in $k[x_0, \ldots, x_n]$.

Let $\{F_\lambda\}_{\lambda\in\Lambda} \subseteq k[x_0, \ldots, x_n]$ be a collection of homogeneous polynomials. The affine algebraic set $V = \mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) \subseteq \mathbb{A}^{n+1}_k$ is cone-shaped, i.e., $\forall p \in V$, the line through p and the origin is in V.

Example 1: (Linear subspaces). Say $W \subseteq k^{n+1}$ is a sub-vector space. Then

$$\mathbb{P}(W) = \text{one-dimensional subspaces of } W = \mathbb{P}(k^{n+1}) = \mathbb{P}^n_k.$$

Note that $\mathbb{P}(W) = \mathbb{V}(L_1, \ldots, L_t) \subseteq \mathbb{P}^n_k$, where $L_i = \sum_{j=0}^n a_{ij}x_j$ are the set of linear functionals in V^* which define W.

Example 2: (Some special cases). W is one-dimensional$\Longrightarrow \mathbb{P}(W)$ is a point.
W is 2-dimensional$\Longrightarrow \mathbb{P}(W)$ is a line in $\mathbb{P}^n_k$.
In general, if W is $(d+1)$-dimensional, then $\mathbb{P}(W)$ is a d-hyperplane in $\mathbb{P}^n_k$. If W has codimension 1 in V, then $\mathbb{V}(W) = \mathbb{P}(W) \subseteq \mathbb{P}(V) = \mathbb{P}^n_k$ is called a hyperplane in $\mathbb{P}^n_k$.

In fact, every projective algebraic set in $\mathbb{P}^n_k$ is defined by finitely many homogeneous equations. Note, as in the affine case,

$$\begin{aligned}\mathbb{V}(\{F_\lambda\}_{\lambda\in\Lambda}) = \mathbb{V}(\langle F_\lambda\rangle_{\lambda\in\Lambda}) &= \mathbb{V}\Big(\text{any set of (homogeneous) generators for } \langle F_\lambda\rangle_{\lambda\in\Lambda}\Big)\\ &= \mathbb{V}\Big(\sqrt{\langle F_\lambda\rangle_{\lambda\in\Lambda}}\Big).\end{aligned}$$

Recall, an ideal $I \subseteq k[x_0, \dots, x_n]$ is homogeneous if it admits a set of generators consisting of homogeneous polynomials.

Example: $I = \langle x^3 - y^2, y^2 - z, z\rangle$ is homogeneous because $I = \langle x^3, y^2, z\rangle$.

In fact, the projective algebraic sets form the closed sets of a topology on $\mathbb{P}^n_k$, the Zariski topology.

The projective Nullstellensatz

The **homogeneous ideal of a projective algebraic set** $V \subseteq \mathbb{P}^n_k$ is the ideal $\mathbb{I}(V) \subseteq k[x_0, \dots, x_n]$ generated by all homogeneous polynomials which vanish at every point of V.

Note that, if I is a homogeneous ideal in $k[x_0, \dots, x_n]$, we can define both an affine algebraic set $\mathbb{V}(I) \subseteq k^{n+1}$ and a projective algebraic set $\mathbb{V}(I) \subseteq \mathbb{P}^n_k$. These have the same radical ideal in $k[x_0, \dots, x_n]$.

In fact, for any projective algebraic set $V \subseteq \mathbb{P}^n_k$, $\mathbb{V}(\mathbb{I}(V)) = V$. Moreover, $k = \bar{k}$ implies the following theorem:

Theorem (Projective Nullstellensatz).

$$\{\text{projective algebraic sets in } \mathbb{P}^n_k\} \longrightarrow \left\{\begin{array}{c}\text{radical homogeneous ideals in}\\ k[x_0,\dots,x_n] \text{ except for}\\ \langle x_0,\dots,x_n\rangle\end{array}\right\}.$$

We call $\langle x_0, \dots, x_n\rangle$ the **irrelevant ideal**. In general, the Zariski topology in $\mathbb{P}^n_k$ restricts to the Zariski topology in each affine chart:

$$\begin{aligned}\mathbb{P}^n_k \supseteq V &= \mathbb{V}\Big(F_1(x_0,\dots,x_n),\dots,F_t(x_0,\dots,x_n)\Big)\\ \supseteq V\cap U_i &= \mathbb{V}\Big(F_1(y_0,\dots,1,\dots,y_n),\dots,F_t(y_0,\dots,1,\dots,y_n)\Big),\end{aligned}$$

where the coordinates are given by

$$U_i \longrightarrow k^n,$$
$$(x_0 : \ldots : x_n) \longmapsto (\frac{x_0}{x_i}, \ldots, \hat{i}, \ldots, \frac{x_n}{x_i}).$$

Projective closure

The projective closure of an affine algebraic set $V \subseteq \mathbb{A}^n_k$ is the closure of V in $\mathbb{P}^n_k$, under the standard chart embedding $\mathbb{A}^n_k = U_0 \hookrightarrow \mathbb{P}^n_k$.

Example 1:. Consider $V = \mathbb{V}(xy-1) \subseteq \mathbb{A}^2_k$:

$$\overline{V} = \overline{\mathbb{V}(xy-1)} = \mathbb{V}(xy - z^2) \subseteq \mathbb{P}^2_k.$$

Look at $\overline{V} \cap U_z = V$.

Look at $\overline{V} \cap \{$"line at infinity"$\}$:

$$\overline{V} \cap \mathbb{V}(z) = \mathbb{V}(xy - z^2, z) = \mathbb{V}(xy, z) = \{(1:0:0), (0:1:0)\} \subseteq \mathbb{P}^2_k.$$

Given a polynomial $f \in k[x_1, \ldots, x_n]$, its **homogenization** is the polynomial $F \in k[X_0, \ldots, X_n]$ obtained as follows: If f has degree d, write

$$f = \sum a_I x_1^{i_1} \ldots x_n^{i_n} = f_d + f_{d-1} + f_{d-2} + \ldots + f_0,$$

where f_i is the homogeneous component of degree i. Then

$$F = f_d + X_0 f_{d-1} + X_0^2 f_{d-2} + \ldots + X_0^d f_0.$$

Caution: Given $V = \mathbb{V}(f_1, \ldots, f_t) \subseteq k^n$, the projective closure $\overline{V}$ in $\mathbb{P}^n_k$ is not necessarily defined by the homogenization of the f_i. For example:

$$\{(t, t^2, t^3) : t \in k\} \subseteq k^3 \hookrightarrow \mathbb{P}^3_k$$
$$(t, t^2, t^3) \longmapsto (1 : t : t^2 : t^3) = (\frac{1}{t^3} : \frac{1}{t^2} : \frac{1}{t} : 1),$$

so it has exactly one point at infinity, $(0:0:0:1)$.

Problems

1. Let f be a polynomial in $k[X_1, \ldots, X_n]$, and assume that the factorization of f into irreducibles is $f = f_1^{n_1} \ldots f_r^{n_r}$. Show that the decomposition of the variety

$\mathbb{V}(f)$ into irreducible subvarieties is given by $\mathbb{V}(f) = \bigcup_{i=1}^{r} \mathbb{I}(\mathbb{V}(f_i))$. Moreover, $\mathbb{I}(\mathbb{V}(f)) = \langle f_1 \dots f_r \rangle)$.

2. Show that there is a one-to-one correspondence between irreducible polynomials in $k[X_1, \dots, X_n]$ and irreducible hypersurfaces in $\mathbb{A}_k^n$, if polynomials that differ by a nonzero multiplicative constant are identified.
3. For any collection of subsets X_i of $\mathbb{A}^n$, show that $\mathbb{I}(\bigcup_i X_i) = \bigcap_i \mathbb{I}(X_i)$.
4. Show that every radical ideal I of $k[X_1, \dots, X_n]$ is the intersection of finitely many prime ideals. Moreover, show that the decomposition is unique, subject to the condition that the prime ideals P are minimal, that is, there is no prime ideal Q with $I \subseteq Q \subsetneq P$.
5. Suppose that X is a variety in $\mathbb{A}^2$, defined by equations $f_1(x, y) = \dots = f_m(x, y) = 0, m \geq 2$. Let g be the greatest common divisor of the f_i. If g is constant, show that X is a finite set (possibly empty).
6. Show that every variety in $\mathbb{A}^2$ except for $\mathbb{A}^2$ itself is the union of a finite set and an algebraic curve.
7. Give an example of two distinct irreducible polynomials in $k[X, Y]$ with the same zero set, and explain why this cannot happen if k is algebraically closed.
8. Give an explicit example of the failure of a version of the Nullstellensatz in a non-algebraically closed field.
9. Let k be algebraically closed field. Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}_k^3 : t \in k\}$ is closed in $\mathbb{A}_k^3$ and find $\mathbb{I}(X)$.
10. Consider the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}_k^3 : t \in k\}$. Show that $Y = \mathbb{V}(I)$, where $I = \langle y^2 - xz, yz - x^3, z^2 - x^2 y \rangle \subseteq k[x, y, z]$.
11. Find $\sqrt{\langle y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3 \rangle}$ in $k[x, y]$. [**Hint**. $\sqrt{I} = \mathbb{I}(\mathbb{V}(I))$]
12. Let $\varphi : \mathbb{A}_k^1 \to \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}_k^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.
13. Prove that a single point in $\mathbb{R}^n$ is an affine variety. Moreover, the union of any finite number of points in $\mathbb{R}^n$ is an affine variety. Give an example of an infinite set of points in $\mathbb{R}^2$ whose union is an affine variety. Justify your answer.
14. Let $X = \{(m, m^3 + 1) \in \mathbb{R}^2 : m \in \mathbb{Z}\}$. Show that X is not an affine variety.
15. Consider the infinite family of polynomials $f_1, f_2, f_3, \dots$ with

$$f_i = 5x^i + 2021y^{i+7}(i^2 + 3)x^{i+2022}y \in \mathbb{R}[x, y] \text{ (where } i = 1, 2, 3, \dots).$$

Prove that there is some integer n so that every f_j with $j > n$ can be written as a linear combination of $f_1, f_2, f_3, \ldots, f_n$. [**Hint**. the form of the f_i is a red herring. Also, I do not want to know specifically what n is.]