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Half of knowledge is to say "I do not know"

Contents

GROUP THEORY: PART III

Direct product of groups

Recall that the direct product of the groups $(G_1, \star_1), \ldots, (G_n, \star_n)$ is the group $(G, \star)$, where $G = G_1 \times \ldots \times G_n$ and for every $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in G$ we have

$$(a_1, \ldots, a_n) \star (b_1, \ldots, b_n) = (a_1 \star_1 b_1, \ldots, a_n \star_n b_n).$$

THEOREM: If $G = G_1 \times \ldots \times G_n$ and $(a_1, \ldots, a_n) \in G$ such that $o(a_i) = r_i$ for all $i = 1, \ldots, n$, then $o(a) = \text{l.c.m}(r_1, \ldots, r_n)$.

Proof If this true for $n = 2$, then it is true for all positive integer n (by using mathematical induction). So, it is enough to prove that $a = (a_1, a_2) \in G_1 \times G_2 \Longrightarrow o(a) = \text{l.c.m}(r_1, r_2)$.

Let $o(a) = r$. Then $a^r = (a_1^r, a_2^r) = (e_1, e_2)$, where e_1, e_2 are the identities of G_1, G_2 respectively. So, $a_1^r = e_1$ and $a_2^r = e_2$. This implies $r_1|r$ and $r_2|r$ and hence $r = r_1 r_2 s$ for some integer s. Therefore, $\text{l.c.m}(r_1, r_2)|r$.

On the other hand, $a^{\text{l.c.m}(r_1,r_2)} = (a_1^{\text{l.c.m}(r_1,r_2)}, a_2^{\text{l.c.m}(r_1,r_2)}) = (e_1, e_2)$ implies $o(a) = r|\text{l.c.m}(r_1, r_2)$. Thus, $o(a) = \text{l.c.m}(r_1, r_2)$.

EXAMPLE: I. Find the order of the element $(8, 4, 10) \in \mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

Let $a = (8, 4, 10)$. Note that

$$o(8) = \frac{o(1)}{\gcd(8,12)} = \frac{12}{4} = 3 \text{ in } \mathbb{Z}_{12},$$
$$o(4) = \frac{o(1)}{\gcd(4,60)} = \frac{60}{4} = 15 \text{ in } \mathbb{Z}_{60},$$
$$o(10) = \frac{o(1)}{\gcd(10,24)} = \frac{24}{2} = 12 \text{ in } \mathbb{Z}_{24}.$$

Thus, $o(a) = \text{l.c.m}(3, 15, 12) = 60$.

EXAMPLE: II. Find all elements in $\mathbb{Z}_4 \times \mathbb{Z}_3$ of order 12.

We know that $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $\mathbb{Z}_3 = \{0, 1, 2\}$. Let us construct the following table:

$a \in \mathbb{Z}_4$	$r_1 = o(a)$	$b \in \mathbb{Z}_3$	$r_2 = o(b)$	$\text{l.c.m}(r_1, r_2)$
0	1	0	1	1 ✗
1	4	0	1	4 ✗
2	2	0	1	2 ✗
3	4	0	1	4 ✗
0	1	1	3	3 ✗
1	4	1	3	12 ✓
2	2	1	3	6 ✗
3	4	1	3	12 ✓
0	1	2	3	3 ✗
1	4	2	3	12 ✓
2	2	2	3	6 ✗
3	4	2	3	12 ✓

So, the only elements of order 12 in $\mathbb{Z}_4 \times \mathbb{Z}_3$ are $(1, 1), (3, 1), (1, 2)$ and $(3, 2)$.

Remark:

1. Every finite abelian group is a direct product of cyclic groups of orders p^α for some primes p and some positive integers α.

2. If $(G_1, \star_1), \ldots, (G_n, \star_n)$ are groups of order $r_1, \ldots, r_n$, then $G = G_1 \times \ldots \times G_n$ is cyclic group if and only if $\gcd(r_i, r_j) = 1$ for all $i \neq j$.
3. If $m = r_1 \ldots r_n$, then $\mathbb{Z}_m \cong \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_n}$ if $\gcd(r_i, r_j) = 1$ for all $i \neq j$.
4. If $m = p_1^{r_1} \ldots p_n^{r_n}$ is a prime factorization of m, where p_i are all distinct, then $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{r_1}} \times \ldots \times \mathbb{Z}_{p_n^{r_n}}$.

THEOREM: If G_1, G_2, G_3 and G_4 are groups, then

1. $G_1 \cong G_3$ and $G_2 \cong G_4$ implies $G_1 \times G_2 \cong G_3 \times G_4$.
2. $G_1 \times G_2 \cong G_2 \times G_1$.
3. $G_1 \times (G_2 \times G_3) \cong G_1 \times G_2 \times G_3$.

Proof

1. Suppose that $G_1 \cong G_3$ via the isomorphism $\varphi_{1,3} : G_1 \to G_3$ and $G_1 \cong G_3$ via the isomorphism $\varphi_{2,4} : G_2 \to G_4$. Then $G_1 \times G_2 \cong G_3 \times G_4$ via the isomorphism

$$\varphi : G_1 \times G_2 \to G_3 \times G_4, \varphi(a, b) = (\varphi_{1,3}(a), \varphi_{2,4}(b)).$$

2. The map $\varphi : G_1 \times G_2 \to G_2 \times G_1$ defined by $\varphi(a, b) = (b, a)$ is an isomorphism. So, $G_1 \times G_2 \cong G_2 \times G_1$.
3. The map $\varphi : G_1 \times (G_2 \times G_3) \to G_1 \times G_2 \times G_3$ defined by $\varphi(a, (b, c)) = (a, b, c)$ is an isomorphism. Thus $G_1 \times (G_2 \times G_3) \cong G_1 \times G_2 \times G_3$.

EXAMPLE: Find all the abelian non-isomorphic groups of order 720.

First of all, let us find all elementary divisors of $720 = 2^4 3^2 5$:

$2^4, 3^2, 5$	$2^4, 3, 3, 5$
$2^3, 2, 3^2, 5$	$2^3, 2, 3, 3, 5$
$2^2, 2^2, 3^2, 5$	$2^2, 2^2, 3, 3, 5$
$2, 2, 2^2, 3^2, 5$	$2, 2, 2^2, 3, 3, 5$
$2, 2, 2, 2, 3^2, 5$	$2, 2, 2, 2, 3, 3, 5.$

Therefore, the abelian non-isomorphic groups of order 720 are:

$\mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5.$

PROBLEMS:

1. Prove that if G is abelain group of order 15, then G is cyclic.
2. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$. Find the order of $(2, 3) \in G$.
3. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$. Find the order of $(3, 10, 9) \in G$.
4. Find all abelian non-isomorphic groups of oreder 100.

Groups acting on sets

DEFINITION: Let $(G, \star)$ be a group, and let S be a nonempty set. The **action (left action) of** G on S is a map $\curvearrowright: G \times S \to S$ defined by $\curvearrowright (g, s) = gs$ for all $g \in G$ and $s \in S$ such that

1. $es = s$, where e is the identity of G;
2. $(g \star g')s = g(g's)$.

In similar way, we can defined the **right action of** G on S.

EXAMPLE: [**Trivial action**] Let $(G, \star)$ be a group, and let S be a nonempty set. Define the map $\curvearrowright: G \times S \to S$ by $\curvearrowright (g, s) = s$. Then this map represents an action of G on S, called the trivial action. In fact,

1. $es = s$;
2. $(g \star g')s = s = g's = g(g's)$.

EXAMPLE: I. Let $(G, \star)$ be a group. The map $\curvearrowright: G \times G \to G$ defined by $\curvearrowright (g, g') = g \star g'$ is a group action. In fact,

1. $eg = e \star g = g$;
2. $(g \star g')g'' = (g \star g') \star g'' = g \star (g' \star g'') = g \star (g'g'') = g(g'g'')$.

EXAMPLE: II. Let $(G, \star)$ be a group, and let $H \leq G$. The map $\curvearrowright: H \times G \to G$ defined by $\curvearrowright (h, g) = h \star g \star h^{-1}$ is a group action. In fact,

1. $eg = e \star g \star e^{-1} = g \star e = g$;
2. Let $h, h' \in H$ and $g \in G$. Then

$$\begin{aligned}(h \star h')g &= (h \star h') \star g \star (h \star h')^{-1} = (h \star h') \star g \star (h'^{-1} \star h^{-1}) \\ &= h \star (h' \star g \star h'^{-1}) \star h^{-1} = h(h' \star g \star h'^{-1}) = h(h'g).\end{aligned}$$

PROBLEMS: Let $(G, \star)$ be a group, and let $H \leq G$.

1. Prove that $\curvearrowright : H \times G \to G$ defined by $\curvearrowright (h, g) = h \star g$ is a group action.
2. Let $H \trianglelefteq G$. Prove that $\curvearrowright : G \times H \to H$ defined by $\curvearrowright (g, h) = g \star h \star g^{-1}$ is a group action.
3. Let $S = \{H : H \leq G\}$. Prove that $\curvearrowright : G \times S \to S$ defined by $\curvearrowright (g, H) = g \star H \star g^{-1}$ is a group action.

Orbits and isotropic groups

DEFINITION: Let $(G, \star)$ be a group, and let S be a nonempty set. The **orbit** of an element $s \in S$ under the action $\curvearrowright : G \times S \to S$, written $\mathsf{Orb}(s)$, is the set

$$\mathsf{Orb}(s) = \{gs : g \in G\}.$$

The **stabilizer of** an element $s \in S$, written $\mathsf{Stab}(s)$, is the set

$$\mathsf{Stab}(s) = \{g \in G : gs = s\}.$$

In general, the stabilizer of $A \subseteq S$ is define to be the set

$$\mathsf{Stab}(A) = \{g \in G : gA = A\}$$

where $gA = \{gs : s \in A\}$.

THEOREM: Let $(G, \star)$ be a group acting on a set S, and let s be an element in S, $A \subseteq S$. Then

1. $\mathsf{Stab}(s) \leq G$.
2. $\mathsf{Stab}(A) \leq G$.
3. $[G : \mathsf{Stab}(s)] = |\mathsf{Orb}(s)|$.

Proof

1. Recall, $\mathsf{Stab}(s) = \{g \in G : gs = s\} \subseteq G$. Then

   (a). Since $es = s$, where e is the identity of $G \Longrightarrow e \in \mathsf{Stab}(s)$.

   (b). Let $g, g' \in \mathsf{Stab}(s)$. Then $gs = s$ and $g's = s$ and hence $g'^{-1}s = s$. Want to prove that $g * g'^{-1} \in \mathsf{Stab}(s)$. Note that

   $$(g * g'^{-1})s = g(g'^{-1}s) = gs = s.$$

   Thus, $\mathsf{Stab}(s) \leq G$.

2. Similarly, we can prove that $\mathsf{Stab}(A) \leq G$.

3. Suppose $L = \{x \star \mathsf{Stab}(s) : x \in G\}$ be the set of all distinct left cosets of $\mathsf{Stab}(s)$. Define a map $f : L \to \mathsf{Orb}(s)$ by

   $$f(x \star \mathsf{Stab}(s)) =\curvearrowright (x, s) = xs.$$

   Want to show that f is a bijection.

   (a). f is well-defined and one-one:

   $$\begin{aligned} x \star \mathsf{Stab}(s) = y \star \mathsf{Stab}(s) &\Longleftrightarrow y^{-1} \star x \in \mathsf{Stab}(s) \\ &\Longleftrightarrow (y^{-1} \star x)s = y^{-1}(xs) = s \\ &\Longleftrightarrow y(y^{-1}(xs)) = ys \\ &\Longleftrightarrow (y \star y^{-1})(xs)) = ys \\ &\Longleftrightarrow e(xs) = ys \Longleftrightarrow xs = ys \\ &\Longleftrightarrow f(x \star \mathsf{Stab}(s)) = f(y \star \mathsf{Stab}(s)). \end{aligned}$$

   (b). f is onto: Assume that $z \in \mathsf{Orb}(s)$. So, there is $g \in G$ such that $z = gs$. Note that, $g \star \mathsf{Stab}(s) \in L$ and

   $$f(g \star \mathsf{Stab}(s)) = gs = z.$$

   Thus, $[G : \mathsf{Stab}(s)] = |L| = |\mathsf{Orb}(s)|$.

EXAMPLE: Consider the symmetric group $(S_4, \circ)$. Let S_4 acting on the set $S = \{1, 2, 3, 4\}$ by $\curvearrowright (\sigma, i) = \sigma(i)$.

Recall, S_4 has 24 permutations: e, $(1\ 2)$, $(1\ 3)$, $(1\ 4)$, $(2\ 3)$, $(2\ 4)$, $(3\ 4)$, $(1\ 2\ 3)$, $(1\ 3\ 2)$, $(1\ 2\ 4)$, $(1\ 4\ 2)$, $(1\ 3\ 4)$, $(1\ 4\ 3)$, $(2\ 3\ 4)$, $(2\ 4\ 3)$, $(1\ 2\ 3\ 4)$, $(1\ 4\ 3\ 2)$, $(1\ 2\ 4\ 3)$, $(1\ 3\ 4\ 2)$, $(1\ 3\ 2\ 4)$, $(1\ 4\ 2\ 3)$, $(1\ 2) \circ (3\ 4)$, $(1\ 3) \circ (2\ 4)$, $(1\ 4) \circ (2\ 3)$.

1. Let us find $\mathsf{Orb}(3)$ and $\mathsf{Stab}(4)$:
$$\mathsf{Orb}(3) = \{\sigma(3) : \sigma \in S_4\} = \{1, 2, 3, 4\} = S.$$
$$\begin{aligned}\mathsf{Stab}(4) &= \{\sigma \in S_4 : \sigma(4) = 4\}\\ &= \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}.\end{aligned}$$
2. Let us find $\mathsf{Stab}(\{1, 4\})$:
$$\mathsf{Stab}(\{1, 4\}) = \{\sigma \in S_4 : \sigma\{1, 4\} = \{1, 4\}\} = \{e, (2\ 3)\}.$$

Sylow Theorems

Recall, if $(G, \star)$ is a finite group, the by Lagrange theorem, the order of a subgroup of G must be divided the order of G. For the finite abelian groups and finite cyclic group the converse of Lagrange theorem is also true. Now, we consider the **Sylow theorems** for finite group of special order.

THEOREM: [First Sylow theorem] Let $(G, \star)$ be a finite group of order $p^n m$, where p is a prime and $n \in \mathbb{Z}^+$; $\gcd(p, m) = 1$. Then

1. G has subgroup of order p^k for all $1 \leq k \leq n$.
2. If $H \leq G$ and $|H| = p^k; 1 \leq k < n$, then there is a subgroup $K \leq G, |K| = p^{k+1}$ such that $H \trianglelefteq K$.

DEFINITION: Let p be a prime number. A $(G, \star)$ is said to be p**-group** if order of any element in G is p^k for some non-negative integer k. A subgroup $H \leq G$ is called p**-subgroup** if it is p-group. If G a finite a group such that p is a prime divides $|G|$. A subgroup $P \leq G$ is said to be **Sylow** p**-subgroup** if P is a maximal p-subgroup of G. The set of all Sylow p-subgroups of G is denoted by $\text{Syl}_p(G)$.

Note that, the first Sylow theorem emphasizes that $\text{Syl}_p(G) \neq \emptyset$ for any prime p divides $|G|$.

EXAMPLE: Show that $\mathbb{Z}_{10}$ is not 2-group. Find all Sylow 2-subgroups, and Sylow 5-subgroups of $\mathbb{Z}_{10}$.

Answer: We know that $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let us find the order of each element in $\mathbb{Z}_{10}$:

$a \in \mathbb{Z}_{10}$	$o(a)$	$\langle a \rangle$
0	$1 = 3^0$	$\{0\}$
1	$\frac{o(1)}{\gcd(1,10)} = \frac{10}{1} = 10$	$\mathbb{Z}_{10}$
2	$\frac{o(1)}{\gcd(2,10)} = \frac{10}{2} = 5 = 5^1$	$\{2, 4, 6, 8, 0\}$
3	$\frac{o(1)}{\gcd(3,10)} = \frac{10}{1} = 10$	$\mathbb{Z}_{10}$
4	$\frac{o(1)}{\gcd(4,10)} = \frac{10}{2} = 5 = 5^1$	$\{4, 8, 2, 6, 0\}$
5	$\frac{o(1)}{\gcd(5,10)} = \frac{10}{5} = 2 = 2^1$	$\{5, 0\}$
6	$\frac{o(1)}{\gcd(6,10)} = \frac{10}{2} = 5 = 5^1$	$\{6, 2, 8, 4, 0\}$
7	$\frac{o(1)}{\gcd(7,10)} = \frac{10}{1} = 10$	$\mathbb{Z}_{10}$
8	$\frac{o(1)}{\gcd(8,10)} = \frac{10}{2} = 5 = 5^1$	$\{8, 6, 4, 2, 0\}$
9	$\frac{o(1)}{\gcd(9,10)} = \frac{10}{1} = 10$	$\mathbb{Z}_{10}$.

Since $|\mathbb{Z}_{10}| = 10$ which is not positive power of 2, then $\mathbb{Z}_{10}$ is not 2-group.

The only Sylow 2-subgroups of $\mathbb{Z}_{10}$ is $\langle 5\rangle$. The only Sylow 5-subgroups of $\mathbb{Z}_{10}$ is $\langle 2\rangle$.

THEOREM: [Second Sylow theorem] Let $(G, \star)$ be a finite group of order $p^n m$, where p is a prime and $n \in \mathbb{Z}^+$; $\gcd(p, m) = 1$. If $H, K \in \mathsf{Syl}_p(G)$, then H, K are conjugate, i.e., there is $g \in G$ such that $g^{-1} \star K \star g = H$. Moreover, H is unique iff $H \trianglelefteq G$.

THEOREM: [Third Sylow theorem] Let $(G, \star)$ be a finite group of order $p^n m$, where p is a prime and $n \in \mathbb{Z}^+$; $\gcd(p, m) = 1$. If $|\mathsf{Syl}_p(G)| = n_p$, then

1. $n_p = 1(\text{mod} p)$,
2. n_p divides $|G|$.

EXAMPLE: I. Consider the symmetric group $(S_3, \circ)$ which has 6 permutations, i.e., $|S_3| = 6 = 2 \cdot 3$. Let us determine all Sylow subgroup of S_3.

1. $\mathsf{Syl}_2(S_3)$: The divisors of 6 are $1, 2, 3, 6$. According to third Sylow theorem $n_2 = 1(\text{mod} 2)$ and divides $|S_3| = 6$. So, either $n_2 = 1$ or $n_2 = 3$. It is clear that

$$H_1 = \langle (1\ 2)\rangle, H_2 = \langle (1\ 3)\rangle \text{ and } H_3 = \langle (2\ 3)\rangle$$

are subgroups of S_3 of order 2. Thus, $n_2 = 3$.

2. $\mathsf{Syl}_3(S_3)$: Again, by apply the Sylow theorem, we have $n_3 = 1(\text{mod } 3)$ and divides $|S_3| = 6$. So, $n_3 = 1$. It follows that there is only one Sylow 3-subgroup of S_3, namely

$$H_4 = \langle (1\ 2\ 3)\rangle = \{e, (1\ 2\ 3), (1\ 3\ 2)\}.$$

Hence, $H_4 \trianglelefteq S_3$ according to the second Sylow theorem.

EXAMPLE: II. Let $(G, \star)$ be a group of order 231. Prove that there are normal subgroups of G of orders 7 and 11.

Note that, $|G| = 231 = 3 \cdot 7 \cdot 11$. Now

1. $\mathsf{Syl}_7(G)$: The divisors of 231 are $1, 3, 7, 11, 21, 33, 77, 231$. According to third Sylow theorem $n_7 = 1(\text{mod} 7)$ and divides $|G| = 231$. So, $n_7 = 1$. It follows that there is only one Sylow 7-subgroup H of G and hence $H \trianglelefteq G$ according to the second Sylow theorem.
2. $\mathsf{Syl}_{11}(G)$: The divisors of 231 are $1, 3, 7, 11, 21, 33, 77, 231$. According to third Sylow theorem $n_{11} = 1(\text{mod} 11)$ and divides $|G| = 231$. So, $n_{11} = 1$. It follows that there is only one Sylow 11-subgroup K of G and hence $K \trianglelefteq G$ according to the second Sylow theorem.

EXAMPLE: III. Show that there is no simple group G of order 105.

Answer: Note that $105 = 3 \times 5 \times 7$. By using Third Sylow theorem, we have $n_3 = 1$ or 7, $n_5 = 1$ or 21. and $n_7 = 1$ or 15

1. If $n_3 = 1$ or $n_5 = 1$ or $n_7 = 1$, then G is not simple.
2. If G is simple, then $n_3 = 7$, $n_5 = 21$ and $n_7 = 15$. Hence, G has $7 \times 2 = 14$ elements of order 3, G has $21 \times 4 = 84$ elements of order 5, and $15 \times 6 = 90$ elements of order 7. Thus, $|G| \geq 14 + 84 + 90 = 188$ which is impossible. So, G is not simple.

THEOREM: Let G be a finite group of order $p^n m$, where p is a prime and $p > m > 1$. Then, G is not simple.

Proof By using third Sylow theorem, we have

1. $n_p = 1(\text{mod} p)$,

2. n_p divides $|G| = p^n m$.

So, $n_p = pk + 1$ for some integer k and $n_p | p^n m$. Hence, either $n_p = 1$ or $n_p = m$. Since $p > m > 1$, then $n_p \neq m$. Therefore, $n_p = 1$ and hence G has unique normal p-subgroup H. Thus, G is not simple.

Example: Let us show that there is no simple group G of order 6, 10, 14, 15, 18, 20, 21, 22, 26, 28.

According to the above theorem:

1. Since $6 = 3 \times 2$. Take $p = 3$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_3(G)$.
2. Since $10 = 5 \times 2$. Take $p = 5$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_5(G)$.
3. Since $14 = 7 \times 2$. Take $p = 7$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_7(G)$.
4. Since $15 = 5 \times 3$. Take $p = 5$ and $m = 3$. So, there is unique normal $H \in \mathrm{Syl}_5(G)$.
5. Since $18 = 3^2 \times 2$. Take $p = 3$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_3(G)$.
6. Since $20 = 5 \times 4$. Take $p = 5$ and $m = 4$. So, there is unique normal $H \in \mathrm{Syl}_5(G)$.
7. Since $21 = 7 \times 3$. Take $p = 7$ and $m = 3$. So, there is unique normal $H \in \mathrm{Syl}_7(G)$.
8. Since $22 = 11 \times 2$. Take $p = 11$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_{11}(G)$.
9. Since $26 = 13 \times 2$. Take $p = 13$ and $m = 2$. So, there is unique normal $H \in \mathrm{Syl}_{13}(G)$.

10. Since $28 = 7 \times 4$. Take $p = 7$ and $m = 4$. So, there is unique normal $H \in \mathsf{Syl}_7(G)$.

EXERCISES

1. Prove or disprove
   (a). The order of the element $(2,3) \in \mathbb{Z}_6 \times \mathbb{Z}_{15}$ is 5.
   (b). The group $\mathbb{Z}_7 \times \mathbb{Z}_{17} \times \mathbb{Z}_{27} \times \mathbb{Z}_{37}$ is not cyclic.
   (c). There is only one cyclic group of order 2022.
   (d). There is an abelian group isomorphic to a non-abelian group.
   (e). $\mathbb{Z}_3 \times \mathbb{Z}_9 \cong \mathbb{Z}_{27}$.
   (f). If G is an abelian group of order 15 and m divides 15, then G has a subgroup of order m.
   (g). If G is group of order 957, then G is cyclic.
   (h). There is non-abelian group of order 255.
   (i). There is a simple group of order 2021.
   (j). If G is an abelian group of order 72, then G has a subgroup of order 8.
   (k). $\mathbb{Z}_4 \times \mathbb{Z}_{15} \cong \mathbb{Z}_6 \times \mathbb{Z}_{10}$.
   (l). If $g = (2, (3\ 4\ 5)) \in \mathbb{Z}_{10} \times \mathfrak{S}_5$, then $o(g) = 15$.
2. Consider the groups $(\mathbb{R}, +)$ and $(\mathbb{R} \times \mathbb{R}, +)$. Define the map $\curvearrowright$: $\mathbb{R} \times (\mathbb{R} \times \mathbb{R}) \to \mathbb{R} \times \mathbb{R}$ defined by $r \curvearrowright (x,y) = (x + ry, y)$.
   (a). Show that this map is a group action.
   (b). Find $\mathsf{Orb}((1,0))$, $\mathsf{Orb}((1,1))$ and $\mathsf{Stab}((0,0))$.
3. Let $(G, \star)$ be a group of order p, q, where p, q are primes and $p < q$. Prove that
   (a). G has only one normal subgroup of order q.

(b). If $q \neq 1(\mathrm{mod} p)$, then G is cyclic group.

4. Let $(G, \star)$ be a group of order $231 = 3 \times 7 \times 11$ and $H \in \mathsf{Syl}_{11}(G)$, $K \in \mathsf{Syl}_7(G)$. Prove that

(a). $H \trianglelefteq G$ and $K \trianglelefteq G$.

(b). G has a cyclic subgroup of order 77.

5. Let G be a group of order p^2q, where p, q are prime numbers, and $q \not\equiv 1(\mathrm{mod} p)$, $p^2 \not\equiv 1(\mathrm{mod} q)$. Prove that $G \cong \mathbb{Z}_{p^2q}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_{pq}$.

6. Prove that if G is a group of order 231 and $H \in \mathsf{Syl}_{11}(G)$, then $H \subseteq Z(G)$.

7. Prove that if G is a group of order 1045 and $H \in \mathsf{Syl}_{19}(G)$, $K \in \mathsf{Syl}_{11}(G)$, then $K \trianglelefteq G$ and $H \subseteq Z(G)$.

8. Prove that if G is a group of order 60, then either G has 4 elements of order 5, or G has 24 elements of order 5.

9. Prove that if G is a group of order 60 with no non-trivial normal subgroups, then G has no subgroup of order 30.

10. Prove that any group of order 40, 45, 63, 84, 135, 140, 165, 175, 176, 189, 195, 200 is not simple.

11. Show that there is no simple group G of order 33, 34, 35, 38, 39, 42, 44, 46, 50, 51.

12. Show that there is no simple group G of order 132.