University of Basrah

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Advanced Vibration Lecture-6

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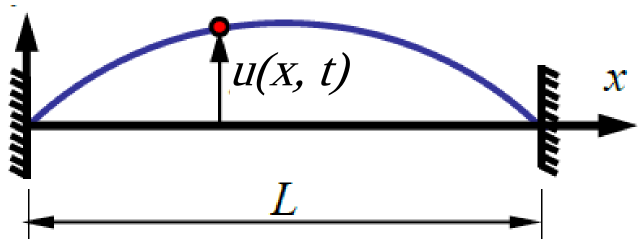
Continuous Systems

Broadly speaking, models of vibratory systems can be divided into two classes, *lumped* and *continuous*, depending on the nature of the parameters. In the case of lumped systems, the components are discrete, with the mass assumed to be rigid and concentrated at specific points, and with the stiffness taking the form of massless springs connecting the concentrated masses. The masses and springs represent the system parameters, and such models are referred as discrete or lumped-parameter models. The motion of discrete systems is governed by a set of ordinary differential equations.

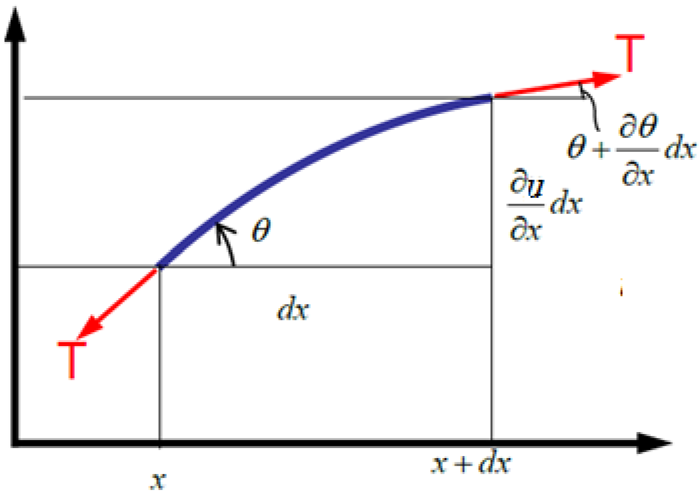
Continuous systems, on the other hand, differ from discrete systems in that the mass and elasticity are continuously distributed. Such systems are also known as distributed-parameter systems, and examples include strings, rods, beams, plates and shells. While discrete systems possess a finite number of degrees of freedom, continuous systems have an infinite number of degrees of freedom because we need an infinite number of coordinates to specify the displacement of every point in an elastic body. The displacement is governed by a function of spatial and temporal variables. As a result, the motion of continuous systems is governed by partial differential equations to be satisfied over the entire domain of the system, subject to boundary conditions and initial conditions.

The discrete and continuous systems are indeed closely connected, and in fact, both systems possess natural frequencies and normal modes of vibration.

Vibration of Strings



It is assumed that both displacement and slope are small. It is also assumed that the tension remain constant along the string during vibration.



$$T\ \sin(\theta + \frac{\partial\theta}{\partial x}dx) - T\ \sin\theta = \rho dx\ \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$T\ \frac{\partial\theta}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{2}$$

But $\theta = \dfrac{\partial u}{\partial x}$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

Where $c = \sqrt{\dfrac{T}{\rho}}$ is the velocity of wave propagation through in the string. Equation (3) is a 1-dim wave equation. Separation of variables method can be used to solve the this equation:

$$u(x,t) = U(x).G(t) \tag{4}$$

Substituting in the differential equation follows that:

$$\frac{1}{U}\frac{d^2U}{dx^2} = \frac{1}{c^2}\frac{1}{G}\frac{d^2G}{dt^2} \tag{5}$$

Which can be re-written as:

$$c^2\frac{1}{U}\frac{d^2U}{dx^2} = \frac{1}{G}\frac{d^2G}{dt^2} \tag{6}$$

Equation (6) cannot be satisfied unless it equals a given constant other than zero:

$$c^2\frac{1}{U}\frac{d^2U}{dx^2} = \frac{1}{G}\frac{d^2G}{dt^2} = -\omega^2 \tag{7}$$

Taking the left hand side:

$$\frac{d^2U}{dx^2} = -\frac{\omega^2}{c^2}U = -\left(\frac{\omega}{c}\right)^2 U \tag{8}$$

Which can be arranged as:

$$\frac{d^2U}{dx^2} + \left(\frac{\omega}{c}\right)^2 U = 0 \tag{9}$$

The solution of equation (9) is given by:

$$U(x) = A\sin\left(\frac{\omega}{c}\right)x + B\cos\left(\frac{\omega}{c}\right)x \tag{10}$$

Similarly, we can show that:

$$G(\mathrm{t}) = C \sin \omega t + D \cos \omega t \tag{11}$$

Hence:

$$u(x,t) = \left(A \sin\left(\frac{\omega}{c}\right) x + B \cos\left(\frac{\omega}{c}\right) x \right) \left(C \sin \omega t + D \cos \omega t \right) \tag{12}$$

<u>**Example-1**</u>**:** For the fixed-fixed boundary conditions:

$u(0,t) = 0$

$u(L,t) = 0$

$u(x,0) = h(x)$

$\dot{u}(x,0) = v(x)$

Applying boundary conditions;

$$u(0,t) = 0 = B\left(G(\mathrm{t}) = C \sin \omega t + B \cos \omega t\right) \Rightarrow B = 0$$

$$u(L,t) = 0 = A \sin\left(\frac{\omega}{c}\right) L \left(C \sin \omega t + D \cos \omega t\right) \Rightarrow A \sin\left(\frac{\omega}{c}\right) L = 0$$

Constant A cannot be zero, hence: $\sin\left(\frac{\omega}{c}\right) L = 0$

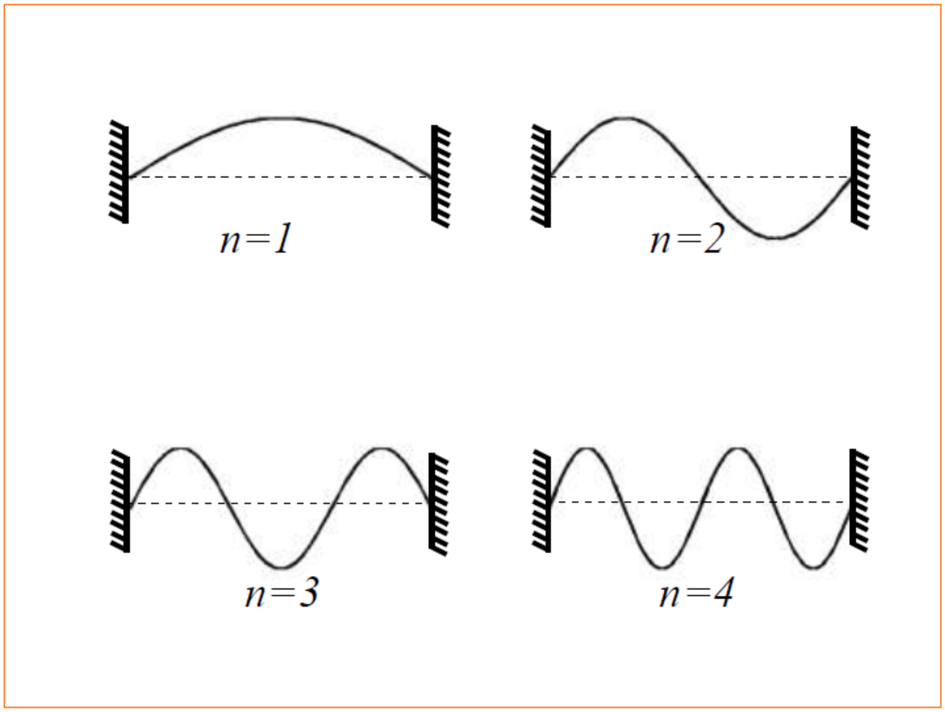
Hence:

$\omega_n = \dfrac{n\pi c}{L}$, $n = 1, 2, 3, \ldots$ These are *eigenvalues*

Therefore:

$$u_n(x,t) = \left(A_n \sin \frac{n\pi}{L} x \right) \left(C_n \sin \omega_n t + D_n \cos \omega_n t \right)$$
$$= U_n(x).G_n(t)$$

Note that *eigenvectors* are given by $U_n(x) = A_n \sin \dfrac{n\pi}{L} x$



The complete solution can be written as:

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin\frac{n\pi}{L}x$$

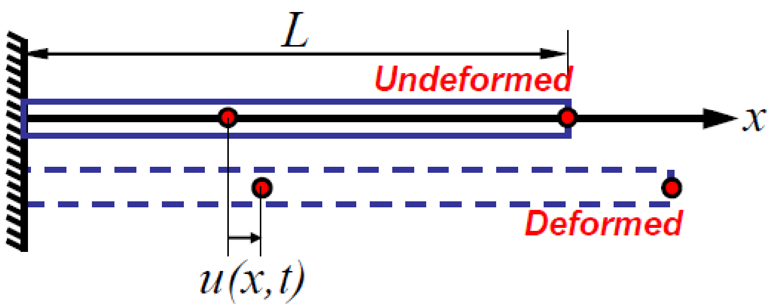
The constants C_n and D_n can be evaluated from the initial conditions:

$$D_n = \frac{2}{L}\int_0^L h(x)\sin\frac{n\pi}{L}x\,dx$$

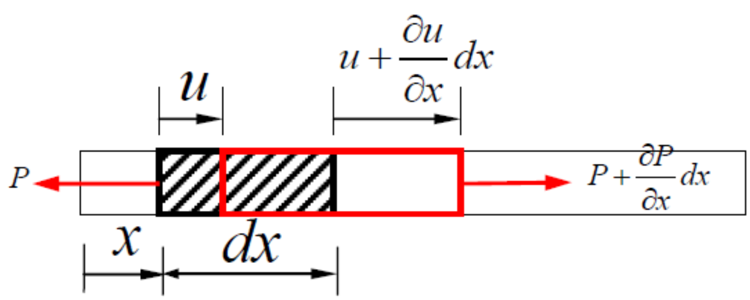
$$C_n = \frac{2}{\omega_n L}\int_0^L v(x)\sin\frac{n\pi}{L}x\,dx$$

Longitudinal vibration of Rods

Longitudinal vibration in rods can also be characterized as 1-dim wave equation.



Considering an infinitesimal strip of length dx:



$$\sigma = \frac{P}{A} \tag{13}$$

Since $\sigma = \varepsilon E$ and $\varepsilon = \partial u / \partial x$,equation (13) becomes:

$$\frac{P}{A} = E\,\frac{\partial u}{\partial x} \tag{14}$$

$$P = EA\,\frac{\partial u}{\partial x} \tag{15}$$

Differentiating with respect to x:

$$\frac{\partial P}{\partial x} = EA\,\frac{\partial^2 u}{\partial x^2} \tag{16}$$

Applying Newton's second low of motion on the strip:

$$\frac{\partial P}{\partial x}dx = (\rho A dx)\frac{\partial^2 u}{\partial t^2} \tag{17}$$

Substituting eq. (16) in (17):

$$EA\,\frac{\partial^2 u}{\partial x^2} = \rho A\,\frac{\partial^2 u}{\partial t^2} \tag{18}$$

Which can be re-arranged as:

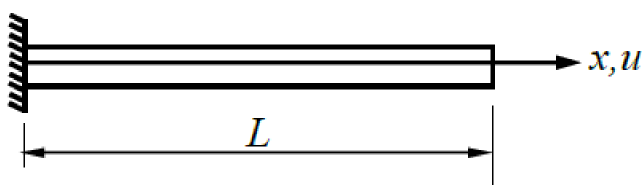
$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{19}$$

Where $c = \sqrt{\frac{E}{\rho}}$ is the velocity of wave propagation through rod material.

The solution is given by:

$$u(x,t) = \left(A\sin\left(\frac{\omega}{c}\right)x + B\cos\left(\frac{\omega}{c}\right)x\right)(C\sin\omega t + D\cos\omega t) \tag{20}$$

<u>Example-2</u>: cantilever beam:



Boundary conditions:
u(0, t) = 0 which leads to: B = 0

At the free end, the axial force must vanishes $P = 0$

$$P = EA\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0 = \frac{\omega}{c}A\cos\left(\frac{\omega}{c}\right)L.G(t) \Rightarrow A\cos\left(\frac{\omega}{c}\right)L = 0$$

$$\therefore \cos\left(\frac{\omega}{c}\right)L = 0$$

Which is the frequency or **characteristics** equation of the system. Hence:

$$\frac{\omega_n L}{c} = \frac{(2n-1)}{2}\pi, \quad n = 1,\ 2,\ 3,\ldots$$

Or $\omega_n = \frac{(2n-1)}{2}\frac{\pi c}{L}, \quad n = 1,\ 2,\ 3,\ldots$

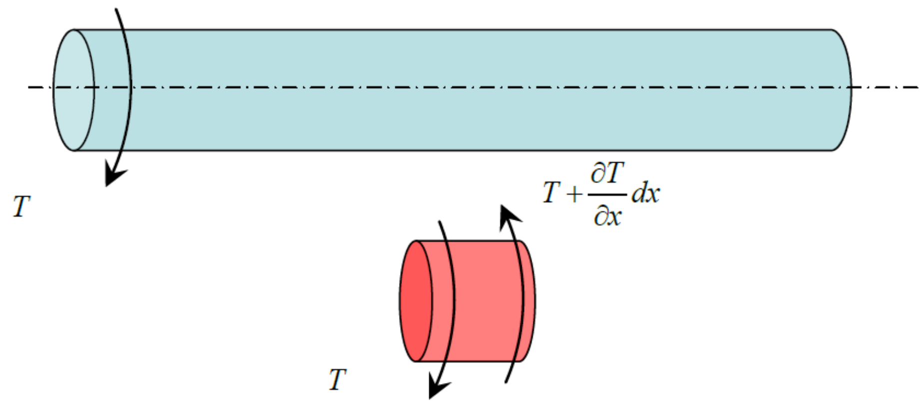
$$u_n(x,t) = \left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin\frac{\omega_n}{c}x$$

The complete solution is:

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin\omega_n t + D_n \cos\omega_n t\right)\sin\frac{(2n-1)\pi}{2L}x$$

The constants $C_n,\ D_n$ are to be determined from initial conditions.

Torsional Vibration of Rods and Shafts



The angle of twist can be expressed as: $d\theta = \frac{Tdx}{GJ}$ (21)

We can write:

$$\frac{\partial T}{\partial x}dx = GJ\frac{\partial^2\theta}{\partial x^2}dx \qquad (22)$$

Applying inertial law of motion:

$$GJ\frac{\partial^2\theta}{\partial x^2}dx = \rho J dx\frac{\partial^2\theta}{\partial t^2} \qquad (23)$$

Hence:

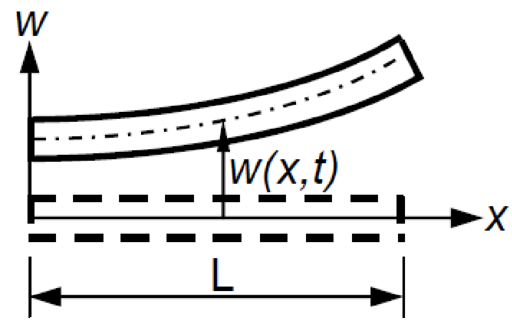
$$\frac{G}{\rho}\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial t^2} \qquad (24)$$

Which can be re-arranged as:

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} \qquad (25)$$

Where $c = \sqrt{\frac{G}{\rho}}$ is the velocity of wave propagation through rod material.

Transverse Vibration of Beams



From strength of materials, the bending moment

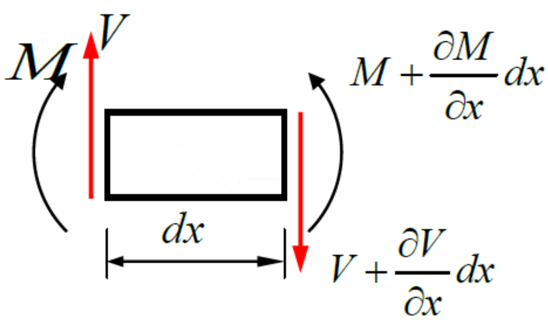
$$M = -EI \frac{\partial^2 w}{\partial x^2} \tag{26}$$

And also, the shear force:

$$V = \frac{\partial M}{\partial x} \tag{27}$$

Where *w* is the transverse displacement, *I* is the second moment of cross sectional area and E is the Young's modulus.

Consider the infinitesimal element:



Applying inertial law of motion in the vertical direction:

$$\frac{\partial V}{\partial x}dx = \rho A dx \frac{\partial^2 w}{\partial t^2} \tag{28}$$

Hence:

$$V = \frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left(-EI\frac{\partial^2 w}{\partial x^2}\right) \tag{29}$$

So, we can write:

$$V = -EI\frac{\partial^3 w}{\partial x^3} \tag{30}$$

differentiating eq. (30) with respect to *x*:

$$\frac{\partial V}{\partial x} = -EI\frac{\partial^4 w}{\partial x^4} \tag{31}$$

From eq. (28):

$$\rho A\frac{\partial^2 w}{\partial t^2} = -EI\frac{\partial^4 w}{\partial x^4} \tag{32}$$

Which can be re-arranged as:

$$-\frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} = \frac{\partial^2 w}{\partial t^2} \tag{33}$$

Separation of variables method can be used to solve this equation:

$$w(x,t) = W(x).G(t) \tag{34}$$

Substituting in the differential equation follows that:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4 W}{dx^4} = \frac{1}{G}\frac{d^2 G}{dt^2} \tag{35}$$

Which can be cannot be satisfied unless it equals a given constant other than zero:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4W}{dx^4}=\frac{1}{G}\frac{d^2G}{dt^2}=-\omega^2 \tag{36}$$

Taking the left hand side:

$$-\frac{EI}{\rho A}\frac{1}{W}\frac{d^4W}{dx^4}=-\omega^2 \tag{37}$$

Which can be arranged as:

$$\frac{d^4W}{dx^4}-\frac{\rho A}{EI}\omega^2 W=0 \tag{38}$$

Denoting $\beta^4=\frac{\rho A}{EI}\omega^2$

$$\frac{d^4W}{dx^4}-\beta^4 W=0 \tag{39}$$

The solution of equation (38) is given by:

$$W(x)=A\sin\beta x+B\cos\beta x+C\sinh\beta x+D\cosh\beta x \tag{40}$$

Similarly, taking right side of eq. (35), we can show that:

$$G(\mathrm{t})=E\sin\omega t+F\cos\omega t \tag{41}$$

Hence:

$$w(x,t)=\left(A\sin\beta x+B\cos\beta x+C\sinh\beta x+D\cosh\beta x\right)\left(E\sin\omega t+F\cos\omega t\right) \tag{42}$$

Example-1: Cantilever beam

Boundary conditions:

At x = 0, the displacement and slope are zeros : $W=0,\ and\ \frac{dW}{dx}=0$

At x = L, the bending moment and shear force are zeros, then $\frac{d^2W}{dx^2}=0,\ \frac{d^3W}{dx^3}=0$

By substituting the boundary conditions into eq. (39) and after some manipulations, we obtain:

$$W_n(x) = A_n\left[\sin\beta_n x - \sinh\beta_n x - \frac{\sin\beta_n L + \sinh\beta_n L}{\cos\beta_n L + \cosh\beta_n L}\left(\cos\beta_n x - \cosh\beta_n x\right)\right]$$

And β_n can be obtained by solving:

$$\cos\beta_n L \cosh\beta_n L = -1$$

The above equation can be solved numerically to give Eigen values $\beta_n L$, with the first three Eigen values are:

$$\beta_1 = \frac{1.8751}{L},\ \beta_2 = \frac{4.6941}{L},\ \beta_3 = \frac{7.8548}{L}$$

The natural frequencies $\omega_n = \beta_n^2\sqrt{\frac{EI}{\rho A}}$ with first three values:

$$\omega_1 = \left(1.8751\right)^2\sqrt{\frac{EI}{\rho A L^4}}$$

$$\omega_2 = \left(4.6941\right)^2\sqrt{\frac{EI}{\rho A L^4}}$$

$$\omega_3 = \left(7.8548\right)^2\sqrt{\frac{EI}{\rho A L^4}}$$

The following table lists the common beam configurations and the associated mode shapes and Eigen values:

SUPPORTS	MODE n	(A) SHAPE AND NODES (NUMBERS GIVE LOCATION OF NODES IN FRACTION OF LENGTH FROM LEFT END)	(B) BOUNDARY CONDITIONS EQ (7.16)	(C) FREQUENCY EQUATION	(D) CONSTANTS EQ (7.16)	(E) kl EQ (7.14) $\omega_n = k^2\sqrt{\frac{EIg}{A\gamma}}$	(F) R RATIO OF NON-ZERO CONSTANTS COLUMN (D)
HINGED-HINGED	1		$x=0 \begin{cases} X=0 \\ X''=0 \end{cases}$	SIN kl=0	A = 0	3.1416	1.0000
	2	0.50			B = 0	6.283	1.0000
	3	0.333 0.667			$\frac{C}{D}$ = 1	9.425	1.0000
	4	0.25 0.50 0.75	$x=l \begin{cases} X=0 \\ X''=0 \end{cases}$			12.566	1.0000
	n>4					≈ $n\pi$	1.0000
CLAMPED-CLAMPED	1		$x=0 \begin{cases} X=0 \\ X'=0 \end{cases}$	(COS kl) (COSH kl) = 1	A = 0	4.730	-0.9825
	2	0.50			C = 0	7.853	-1.0008
	3	0.359 0.641			$\frac{D}{B}$ = R	10.996	-1.0000-
	4	0.278 0.50 0.722	$x=l \begin{cases} X=0 \\ X'=0 \end{cases}$			14.137	-1.0000+
	n>4					$\approx \frac{(2n+1)\pi}{2}$	-1.0000-
CLAMPED-HINGED	1		$x=0 \begin{cases} X=0 \\ X'=0 \end{cases}$	TAN kl = TANH kl	A = 0	3.927	-1.0008
	2	0.558			C = 0	7.069	-1.0000+
	3	0.386 0.692			$\frac{D}{B}$ = R	10.210	-1.0000
	4	0.294 0.529 0.765	$x=l \begin{cases} X=0 \\ X''=0 \end{cases}$			13.352	-1.0000
	n>4					$\approx \frac{(4n+1)\pi}{4}$	-1.0000
CLAMPED-FREE	1		$x=0 \begin{cases} X=0 \\ X'=0 \end{cases}$	(COS kl) (COSH kl) = -1	A = 0	1.875	-0.7341
	2	0.783			C = 0	4.694	-1.0185
	3	0.504 0.868			$\frac{D}{B}$ = R	7.855	-0.9992
	4	0.358 0.644 0.906	$x=l \begin{cases} X''=0 \\ X'''=0 \end{cases}$			10.996	-1.0000+
	n>4					$\approx \frac{(2n-1)\pi}{2}$	-1.0000-
FREE-FREE	1	0.224 0.776	$x=0 \begin{cases} X''=0 \\ X'''=0 \end{cases}$	(COS kl) (COSH kl) = 1	B = 0	0 (REPRESENTS TRANSLATION)	
	2	0.132 0.50 0.868			D = 0	4.730	-0.9825
	3	0.094 0.356 0.644 0.906			$\frac{C}{A}$ = R	7.853	-1.0008
	4	0.0734 0.277 0.50 0.723 0.927	$x=l \begin{cases} X''=0 \\ X'''=0 \end{cases}$			10.996	-1.0000-
	5					14.137	-1.0000+
	n>5					$\approx \frac{(2n-1)\pi}{2}$	-1.0000-

Notes about boundary conditions:

(1) Clamped: both displacement and slope are zeros: $W = 0$, and $\frac{dW}{dx} = 0$

(2) Hinged (or simply supported) both displacement and bending moment are zeros:

$$W = 0, \text{ and } \frac{d^2W}{dx^2} = 0$$

(3) Free: bending moment and shear force are zeros: $\frac{d^2W}{dx^2} = 0$, and $\frac{d^3W}{dx^3} = 0$