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Advanced Vibration Lecture-4 and 5

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Symbols

[●]: matrix

{●}: column vector

<●>: row vector

Bold Letter or symbol: matrix or vector

Multi Degree of Freedom Systems

When the system requires more than one coordinate to describe its motion, the system is said to be multi degree of freedom or N-DOF system. This system differs from SDOF by the fact that it has N natural frequencies. For each natural frequency, there exist a specific state of vibration with displacement configuration known as *normal mode*.

Natural frequencies are associated with Eigenvalues and normal modes (mode shapes) are associated with Eigenvectors of the equation of motions.

Eigenvalues and Eigenvectors

Consider a square matrix $\mathbf{A}$. If we multiply any vector by this matrix, the result is another vector. There is a special-case vector $\mathbf{X}$ (Eigenvector) which when multiplied by $\mathbf{A}$ will result in a vector in the same direction of $\mathbf{X}$ (with different or same length). This can be mathematically represented by: $\mathbf{AX} = \lambda\mathbf{X}$

Where λ is a scalar and called Eigenvalue. And $\mathbf{x}$ is the Right Eigenvector associated with the Eigenvalue.

<u>Example-1:</u>

Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. This matrix has two Eigenvalues and two Eigenvectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(2-\lambda)x_1 + x_2 = 0 \quad ....(1)$$

$$x_1 + (2-\lambda)x_2 = 0 \quad ....(2)$$

Substituting x_2 from eq. (1) into eq.(2)

$$x_1 - (2-\lambda)(2-\lambda)x_1 = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0$$

$$either \ \lambda = 1 \ or \ \lambda = 3$$

Substituting values of λ in either equation (in eq. (1) for example) will result in:

$$when \ \lambda = 1: \ x_1 + x_2 = 0 \ \Rightarrow x_1 = -x_2$$

$$when \ \lambda = 3: -x_1 + x_2 = 0 \ \Rightarrow x_1 = x_2$$

Eigenvectors are $\phi_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, and \ \phi_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The above Eigenvectors are normalized such that $x_1 = 1$

Generally, it can be shown that, for any square matrix, Eigenvalues and Eigenvectors satisfy the following equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{X} = 0 \quad .... (3)$$

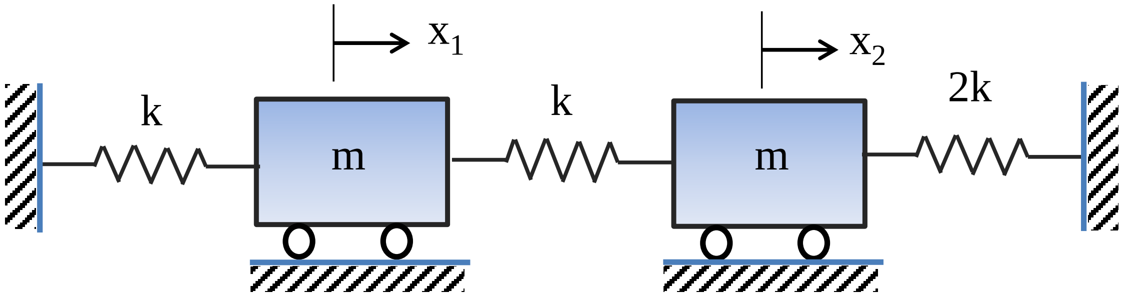
Where $\mathbf{I}$ is the identity matrix. As shown in Cramer's rule, a linear system of equations has nontrivial solutions if the determinant of $(\mathbf{A} - \lambda \mathbf{I})$ vanishes, so the solutions of the above equation are:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad .... (4)$$

Equation (4) is named the characteristics equation of the system. Eigenvectors can be found by substituting Eigenvalues in eq. (3).

Two Degree of Freedom

Example-2:



$$m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0 \qquad \ldots(5)$$
$$m\ddot{x}_2 + 2kx_2 + k(x_2 - x_1) = 0 \qquad \ldots(6)$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assuming sinusoidal motion, the response can be written as:

$$x_1 = X_1 \sin \omega t$$
$$x_2 = X_2 \sin \omega t$$

Substituting in the equations of motions:

$$(2k - m\omega^2)X_1 - kX_2 = 0 \qquad \ldots(7)$$
$$(3k - m\omega^2)X_2 - kX_1 = 0 \qquad \ldots(8)$$

In matrix form

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above equation is an Eigenvalue problem. The characteristics equation is given by

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0$$

By letting $\lambda = \omega^2$:

$$(2k - m\lambda)(3k - m\lambda) - k^2 = 0$$

$$\lambda^2 - 5\left(\frac{k}{m}\right)\lambda + 5\left(\frac{k}{m}\right)^2 = 0$$

$$\lambda = \frac{5\left(\frac{k}{m}\right) \pm 2.236\left(\frac{k}{m}\right)}{2}$$

$$either \ \lambda = 3.618\left(\frac{k}{m}\right) \ \ or \ \ \lambda = 1.382\left(\frac{k}{m}\right)$$

$$\omega_1 = \sqrt{3.618\left(\frac{k}{m}\right)} \ \ and \ \ \omega_2 = \sqrt{1.382\left(\frac{k}{m}\right)}$$

by substituting the first value of λ in equation (7) :

$$\frac{X_1}{X_2} = \frac{k}{2k - \lambda m}$$

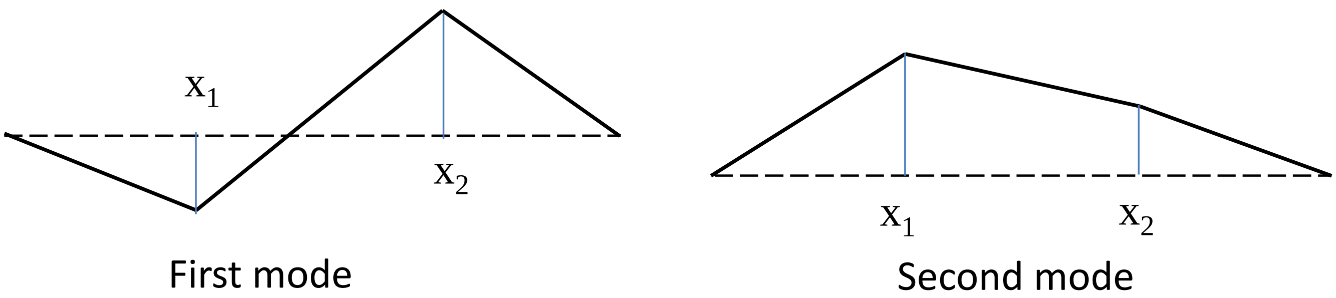
$$\frac{X_1}{X_2} = \frac{k}{2k - 3.618k} = \frac{1}{-1.618} = -0.618 \Rightarrow \{\psi\}_1 = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_1 = \begin{Bmatrix} \frac{X_1}{X_2} \\ \frac{X_2}{X_2} \end{Bmatrix}_1 = \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix}$$

by substituting the second value of λ in equation (7) :

$$\frac{X_1}{X_2} = \frac{k}{2k - \lambda m}$$

$$\frac{X_1}{X_2} = \frac{k}{2k - 1.382k} = \frac{1}{0.618} = 1.618 \;\Rightarrow \{\psi\}_2 = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_2 = \begin{Bmatrix} \dfrac{X_1}{X_2} \\ \dfrac{X_2}{X_2} \end{Bmatrix}_2 = \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$$

It should be noted that both $\{\psi\}_1$ and $\{\psi\}_2$ are arbitrarily scaled modal shapes. These mode shapes are unique in direction (shape) but not in value.



Response to Initial Conditions

For the last example, if the system is subjected to certain initial conditions, the motion must satisfy the normal modes (for each mode) such that the ratios of different coordinates remain same as the normal modes ratios for each mode. Generally:

$$x_1 = c_1(\psi_1)_1 \sin\omega_1 t + d_1(\psi_1)_1 \cos\omega_1 t + c_2(\psi_1)_2 \sin\omega_2 t + d_2(\psi_1)_2 \cos\omega_2 t$$
$$x_2 = c_1(\psi_2)_1 \sin\omega_1 t + d_1(\psi_2)_1 \cos\omega_1 t + c_2(\psi_2)_2 \sin\omega_2 t + d_2(\psi_2)_2 \cos\omega_2 t$$

This can simply be written as:

Or alternatively

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \sum_{r=1}^{2} \left(c_r \sin \omega_r t + d_r \cos \omega_r t \right) \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r$$

$$= \sum_{r=1}^{2} q_r(t) \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_r$$

Where $q_r(t)$ is a function of time depends on the initial conditions.

Example-3:

Find the response of the above system to the following excitation

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \end{Bmatrix}, \quad \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \sum_{i=1}^{2} c_i \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_i \sin \omega_i t + d_i \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}_i \cos \omega_i t$$

$$x_1 = -0.618 c_1 \sin \omega_1 t - 0.618 d_1 \cos \omega_1 t + 1.618 c_2 \sin \omega_2 t + 1.618 d_2 \cos \omega_2 t$$

$$x_2 = c_1 \sin \omega_1 t + d_1 \cos \omega_1 t + c_2 \sin \omega_2 t + d_2 \cos \omega_2 t$$

In order to apply initial velocity conditions, we need to differentiate with respect to time

$$\dot{x}_1 = -0.618 \omega_1 c_1 \cos \omega_1 t + 0.618 \omega_1 d_1 \sin \omega_1 t + 1.618 \omega_2 c_2 \cos \omega_2 t - 1.618 \omega_2 d_2 \sin \omega_2 t$$

$$\dot{x}_2 = \omega_1 c_1 \cos \omega_1 t - \omega_1 d_1 \sin \omega_1 t + \omega_2 c_2 \cos \omega_2 t - \omega_2 d_2 \sin \omega_2 t$$

Applying the initial conditions:

$$2 = -0.618 d_1 + 1.618 d_2$$

$$4 = d_1 + d_2$$

$$0 = -0.618 \omega_1 c_1 + 1.618 \omega_2 c_2$$

$$0 = \omega_1 c_1 + \omega_2 c_2$$

From the above equations:

$$d_1 = 2$$

$$d_2 = 2$$

$$c_1 = c_2 = 0$$

So far

$$x_1 = -1.236\cos\omega_1 t + 3.236\cos\omega_2 t$$
$$x_2 = 2\cos\omega_1 t + 2\cos\omega_2 t$$

Facts about response of M-DOF system:

1. All modes exist in the response
2. The amount of participation of each mode in the final response depends on the initial conditions
3. Phase angles between different modes (values of c_i with respect to d_i) depend on the initial conditions and natural frequencies.

Modal Mass and Modal Stiffness Matrices

Defining mode shape matrix

$$[\psi] = \left[\{\psi\}_1 \quad \{\psi\}_2 \right]$$

The modal mass and modal stiffness matrices are diagonal matrices and they can be found by post multiplying mass matrix and stiffness matrix by the mode shape matrix and pre multiplying them by the transpose of mode shape matrix.

They are diagonal matrices. They are also sometimes named generalized mass and generalized stiffness matrices.

$$[\psi]^T [M][\psi] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \left[\cdot m_{r.} \right] \quad (9)$$

$$[\psi]^T [K][\psi] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \left[\cdot k_{r.} \right] \quad (10)$$

Or equivalently, to determine individual elements of the modal mass and stiffness:

$$m_r = \{\psi\}_r^T [M] \{\psi\}_r \quad (11)$$

$$k_r = \{\psi\}_r^T [K] \{\psi\}_r \quad (12)$$

The roots of the system can be found by multiplying the modal stiffness by the inverse of modal mass matrix:

$$\left[\,^{\cdot}\lambda_{r.}\right]=\left[\,^{\cdot}k_{r.}\right]\left[\,^{\cdot}m_{r.}\right]^{-1} \qquad (13)$$

By diagonalizing the mass and stiffness matrices, the following equivalent system is obtained for free vibration:

$$m_r\ddot{q}_r + k_r q_r = 0 \quad r = 1, 2, \ldots N \qquad (14)$$

Where q_r are named the modal (or generalized) coordinates.

Mass Normalized Mode Shapes

The mass-normalized mode shape $\{\phi\}_r$ can be found as follows:

$$\{\phi\}_r = \frac{1}{\sqrt{m_r}}\{\psi\}_r \qquad (15)$$

Using mass normalized mode shapes, it can be shown that:

$$[\phi]^T[M][\phi] = \mathbf{I} \qquad (16)$$

$$[\phi]^T[K][\phi] = \left[\,^{\cdot}\lambda_{r.}\right] \qquad (17)$$

<u>Example-4:</u>

Find the mass-normalized mode shapes for Example-2.

Solution:

The equations of motions are given by:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And we have the arbitrarily scaled mode shapes:

$\{\psi\}_1 = \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix}$, and $\{\psi\}_2 = \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$, so far: $[\psi] = \begin{bmatrix} -0.618 & 1.618 \\ 1 & 1 \end{bmatrix}$

Hence:

$$[\psi]^T [M][\psi] = \begin{bmatrix} -0.618 & 1 \\ 1.618 & 1 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} -0.618 & 1.618 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.382m & 0 \\ 0 & 3.618m \end{bmatrix}$$

$$\{\phi\}_1 = \frac{1}{\sqrt{m_1}} \{\psi\}_1 = \frac{1}{\sqrt{1.382m}} \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix}$$

$$\{\phi\}_2 = \frac{1}{\sqrt{m_2}} \{\psi\}_2 = \frac{1}{\sqrt{3.618m}} \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$$

Forced Vibration of 2-DOF

Consider a 2-DOF system given by:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}\sin\omega t \qquad \text{….(18)}$$

The solution is given by:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}\sin\omega t$$

Substituting the solution into the equations of motion

$$\begin{bmatrix} k_{11}-m_1\omega^2 & k_{12} \\ k_{21} & k_{22}-m_2\omega^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$or\ \ simply\ \ [Z(\omega)]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \qquad \text{….(19)}$$

This can be re-written as:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=[Z(\omega)]^{-1}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}=\frac{Adj\,(Z(\omega))}{|Z(\omega)|}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

$$=\frac{\begin{bmatrix} k_{22}-m_2\omega^2 & -k_{12} \\ -k_{21} & k_{11}-m_1\omega^2 \end{bmatrix}}{|Z(\omega)|}\begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

But $|Z(\omega)|=m_1 m_2(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)$

Therefore:

$$X_1 = \frac{(k_{22} - m_2\omega^2)F_1}{m_1 m_2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{-k_{21}F_1}{m_1 m_2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \qquad \text{....(20)}$$

Example-5:

For the system of Example-2, find the response if a sinusoidal force is applied at the first mass.

Solution:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}\sin\omega t$$

Using equations (17):

$$X_1 = \frac{(3k - m\omega^2)F_1}{m^2\ (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \qquad \text{...(21)}$$

Forced Response of 2-DOF in terms of Normal Modes

Considering X_1 from equation (21);

$$X_1 = \frac{(3k - m\omega^2)F_1}{m^2\ (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} = \frac{C_1}{(\omega_1^2 - \omega^2)} + \frac{C_2}{(\omega_2^2 - \omega^2)}$$

C_1 and C_2 can be evaluated using residues method:

Remembering that (from Example-1)

$$\omega_1^2 = 3.618\left(\frac{k}{m}\right) \; and \; \omega_2^2 = 1.382\left(\frac{k}{m}\right)$$

$$C_1 = \frac{(3k - m\omega^2)F_1}{m^2 (\omega_2^2 - \omega^2)} \downarrow_{\omega=\omega 1} = \frac{3k - 3.618k}{m^2(\omega_2^2 - \omega_1^2)} F_1 = \frac{0.618}{2.236} \frac{F_1}{m}$$

Similarly

$$C_2 = \frac{(3k - m\omega^2)F_1}{m^2 (\omega_1^2 - \omega^2)} \downarrow_{\omega=\omega 2} = \frac{3k - 1.382k}{m^2(\omega_1^2 - \omega_2^2)} F_1 = \frac{1.618}{2.236} \frac{F_1}{m}$$

Hence:

$$X_1 = \frac{F_1}{2.236m}\left(\frac{0.618}{(\omega_1^2 - \omega^2)} + \frac{1.618}{(\omega_2^2 - \omega^2)}\right)$$

Similarly, we can show that:

$$X_2 = \frac{F_1}{2.236m}\left(\frac{-1}{(\omega_1^2 - \omega^2)} + \frac{1}{(\omega_2^2 - \omega^2)}\right)$$

Response of Undamped MDOF to Sinusoidal Force

Consider a forced M-DOF system

$$[M]\{\ddot{x}\} + [K]\{x\} = \{f(t)\}$$

Assuming sinusoidal excitation, the above function can be written as:

$$\left([K] - \omega^2 [M]\right)\{X\} = \{F\}$$

$$or \; simply \; [Z(\omega)]\{X\} = \{F\} \quad \ldots(22)$$

The matrix $[Z(\omega)]$ is named **dynamic stiffness** matrix

$$\{X\} = [Z(\omega)]^{-1}\{F\} = [\alpha(\omega)]\{F\} \quad \ldots(23)$$

The inverse of dynamic stiffness matrix is named **Receptance FRF** matrix. For example, for a 2-DOF system, eq. (20) can be written as:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11}(\omega) & \alpha_{12}(\omega) \\ \alpha_{21}(\omega) & \alpha_{22}(\omega) \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \ldots(24)$$

The values of receptance matrix elements can be evaluated in terms of normalized modal constants as follows:

We have $\left([K]-\omega^2[M]\right)=[\alpha(\omega)]^{-1}$

Pre-multiply by the transpose of normalized modal matrix and post multiply by it;

$$[\phi]^T\left([K]-\omega^2[M]\right)[\phi]=[\phi]^T[\alpha(\omega)]^{-1}[\phi]$$

Utilizing equations (13) and (14)

$$\left(\left[{}^{\cdot}\omega_r^2{}_{\cdot}\right]-\omega^2[I]\right)=[\phi]^T[\alpha(\omega)]^{-1}[\phi]$$

$$\left[{}^{\cdot}(\omega_r^2-\omega^2){}_{\cdot}\right]=[\phi]^T[\alpha(\omega)]^{-1}[\phi]$$

$$\begin{bmatrix} \omega_1^2-\omega^2 & 0 & 0 & 0 \\ 0 & \omega_2^2-\omega^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega_n^2-\omega^2 \end{bmatrix} = [\phi]^T[\alpha(\omega)]^{-1}[\phi] \qquad \ldots(25)$$

by taking inverse for both sides then pre-multiply by mode shape and post-multiply by its transpose:

$$[\alpha(\omega)]=[\phi]\left[{}^{\cdot}(\omega_r^2-\omega^2){}_{\cdot}\right]^{-1}[\phi]^T \qquad (26)$$

Where:

$$\left[\cdot(\omega_r^2-\omega^2)_{.} \right]^{-1} = \begin{bmatrix} \frac{1}{\omega_1^2-\omega^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\omega_2^2-\omega^2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{\omega_n^2-\omega^2} \end{bmatrix} \quad (27)$$

Hence, it can easily be shown that:

$$\alpha_{ij}(\omega) = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{jr}}{\omega_r^2-\omega^2} \quad (28)$$

Example-6:

Re-solve example-5 using modal decomposition method.

Solution:

Since $F_2 = 0$, then we have

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11}(\omega) & \alpha_{12}(\omega) \\ \alpha_{21}(\omega) & \alpha_{22}(\omega) \end{bmatrix} \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix}$$

We need to find only $\alpha_{11}(\omega)$ and $\alpha_{21}(\omega)$

We have (from example-4):

$$\{\phi\}_1 = \frac{1}{\sqrt{1.382m}} \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix} \text{ and } \{\phi\}_2 = \frac{1}{\sqrt{3.618m}} \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$$

$$\alpha_{11}(\omega) = \frac{\phi_{11}\phi_{11}}{\omega_1^2-\omega^2} + \frac{\phi_{12}\phi_{12}}{\omega_2^2-\omega^2} = \frac{1}{1.382m} \frac{(-0.618)^2}{(\omega_1^2-\omega^2)} + \frac{1}{3.618m} \frac{(1.618)^2}{(\omega_2^2-\omega^2)}$$

$$\alpha_{21}(\omega) = \frac{\phi_{21}\phi_{11}}{\omega_1^2-\omega^2} + \frac{\phi_{22}\phi_{12}}{\omega_2^2-\omega^2} = \frac{1}{1.382m} \frac{-0.618}{(\omega_1^2-\omega^2)} + \frac{1}{3.618m} \frac{1.618}{(\omega_2^2-\omega^2)}$$

Hence:

$$X_1 = \alpha_{11}(\omega)F_1 = \frac{F_1}{2.236m}\left(\frac{0.618}{(\omega_1^2-\omega^2)}+\frac{1.618}{(\omega_2^2-\omega^2)}\right)$$

$$X_2 = \alpha_{21}(\omega)F_1 = \frac{F_1}{2.236m}\left(\frac{-1}{(\omega_1^2-\omega^2)}+\frac{1}{(\omega_2^2-\omega^2)}\right)$$

Response of Undamped MDOF to General Excitation

In previous section, the response of the MDOF system to a sinusoidal excitation which a special case is presented. In this section, the response to any input force will be discussed.

Consider the *N*-DOF system:

$$[M]\{\ddot{x}\}+[K]\{x\}=\{f(t)\} \qquad (29)$$

The displacement vector can be written as sum of modal contributions as seen in previous lecture:

$$\{x\}=\sum_{r=1}^{N} q_r(t)\{\psi\}_r = [\psi]\{q\} \qquad (30a)$$

Or more clearly:

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{Bmatrix} = \sum_{r=1}^{N} q_r(t)\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{Bmatrix}_r = \left[\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{Bmatrix}_1 \quad \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{Bmatrix}_2 \quad \cdots \quad \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{Bmatrix}_N\right]\begin{Bmatrix} q \\ q_2 \\ \vdots \\ q_N \end{Bmatrix} \qquad (30b)$$

Substituting eq. (30) in eq. (29):

$$\sum_{r=1}^{N}[M]\{\psi\}_r\,\ddot{q}_r(t)+\sum_{r=1}^{N}[K]\{\psi\}_r\,q_r(t)=\{f(t)\} \qquad (31)$$

Pre-multiplying all terms of eq. (31) by $\{\psi\}_n^T$ gives:

$$\sum_{r=1}^{N}\{\psi\}_n^T[M]\{\psi\}_r\,\ddot{q}_r(t)+\sum_{r=1}^{N}\{\psi\}_n^T[K]\{\psi\}_r\,q_r(t)=\{\psi\}_n^T\{f(t)\} \qquad (32)$$

Due to orthogonality of the mode shapes, all terms of the summations are vanishing except when $n = r$, hence:

$$\{\psi\}_n^T [M]\{\psi\}_n \ddot{q}_n(t) + \{\psi\}_n^T [K]\{\psi\}_n q_n(t) = \{\psi\}_n^T \{f(t)\} \qquad (33)$$

Hence:

$$m_n \ddot{q}_n(t) + k_n q_n(t) = \{\psi\}_n^T \{f(t)\} \qquad (34)$$

Where m_n, k_n are the modal mass and modal stiffness for mode n. The force $\boldsymbol{\psi}_n^T \mathbf{f}(t)$ is called modal (generalized) force for mode n. The above equation can be interpreted as an equation for a single DOF system with mass m_n, stiffness k_n and force $\boldsymbol{\psi}_n^T \mathbf{f}(t)$. Thus the set of N coupled equations with displacement $\mathbf{x}$ is transformed into a set of N uncoupled equations with modal displacement $\mathbf{q}$. Once each modal displacement is evaluated by solving the corresponding SDOF equation, the actual displacements can be calculated from eq. (30).

**If the applied force is sinusoidal, the solution of eq. (34) is straightforward. For any input force however, eq. (34) can be solved by Laplace transform or even numerically by Duhamel integral for more complex input forces.

If the initial conditions are generally not zero, the corresponding initial conditions of the uncoupled modal coordinates can be evaluated by multiplying eq. (30) by $\boldsymbol{\psi}^T \mathbf{M}$:

$$\begin{aligned} &[\psi]^T [M]\{x\} = [\psi]^T [M][\psi]\{q\} = \left[{}^{\cdot} m_{r.} \right]\{q\} \\ &\Rightarrow \{q\} = \left[{}^{\cdot} m_{r.} \right]^{-1} [\psi]^T [M]\{x\} \end{aligned} \qquad (35)$$

Note: when mass normalized mode shapes are used then:

$$\ddot{q}_n(t) + \omega_n q_n(t) = \{\phi\}_n^T \{f(t)\}$$

$$\{q\} = [\phi]^T [M]\{x\}$$

Example-7:

Re-solve example-5 using uncoupled equations of motion.

Solution:

Since $F_2 = 0$, then we have

$$\{f(t)\} = \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t$$

We have (from example-4):

$$\{\psi\}_1 = \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix} \; and \; \{\psi\}_2 = \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix}$$

Hence we have:

For the first mode: $\{\psi\}_1^T \{f(t)\} = \langle -0.618 \quad 1 \rangle \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t = -0.618 F_1 \sin \omega t$

For the second mode: $\{\psi\}_2^T \{f(t)\} = \langle 1.618 \quad 1 \rangle \begin{Bmatrix} F_1 \\ 0 \end{Bmatrix} \sin \omega t = 1.618 F_1 \sin \omega t$

Therefore, the uncoupled equation of motions are:

$$m_1 \ddot{q}_1(t) + k_1 q_1(t) = -0.618 F_1 \sin \omega t$$

$$m_2 \ddot{q}_2(t) + k_2 q_2(t) = 1.618 F_1 \sin \omega t$$

Solutions of the above equations are:

$$q_1(t) = \frac{-0.618 F_1 \omega_1^2}{k_1 \left(\omega_1^2 - \omega^2\right)} \sin \omega t$$

$$q_2(t) = \frac{1.618 F_1 \omega_2^2}{k_2 \left(\omega_2^2 - \omega^2\right)} \sin \omega t$$

The modal stiffness factors can be evaluated as follows:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = [\psi]^T [K][\psi] = \begin{bmatrix} -0.618 & 1 \\ 1.618 & 1 \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{bmatrix} -0.618 & 1.618 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5k & 0 \\ 0 & 5k \end{bmatrix}$$

Using the results from Example-2: $\omega_1 = \sqrt{3.618 \left(\frac{k}{m}\right)}$ and $\omega_2 = \sqrt{1.382 \left(\frac{k}{m}\right)}$

$$q_1(t) = \frac{-0.618F_1\left(3.618\frac{k}{m}\right)}{5k\left(\omega_1^2-\omega^2\right)}\sin\omega t = \frac{-F_1}{2.236m\left(\omega_1^2-\omega^2\right)}\sin\omega t$$

$$q_2(t) = \frac{1.618F_1\left(1.382\frac{k}{m}\right)}{5k\left(\omega_2^2-\omega^2\right)}\sin\omega t = \frac{F_1}{2.236m\left(\omega_2^2-\omega^2\right)}\sin\omega t$$

Finally, the actual displacement can be evaluated from eq. (30) as follows:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \sum_{r=1}^{2}\{\psi\}_r\, q_r(t)$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -0.618 \\ 1 \end{Bmatrix} q_1(t) + \begin{Bmatrix} 1.618 \\ 1 \end{Bmatrix} q_2(t)$$

$$x_1(t) = -0.618q_1(t) + 1.618q_2(t) = \frac{F_1}{2.236m}\left(\frac{0.618}{\left(\omega_1^2-\omega^2\right)} + \frac{1.618}{\left(\omega_2^2-\omega^2\right)}\right)\sin\omega t$$

$$x_2(t) = q_1(t) + q_2(t) = \frac{F_1}{2.236m}\left(\frac{-1}{\left(\omega_1^2-\omega^2\right)} + \frac{1}{\left(\omega_2^2-\omega^2\right)}\right)\sin\omega t$$

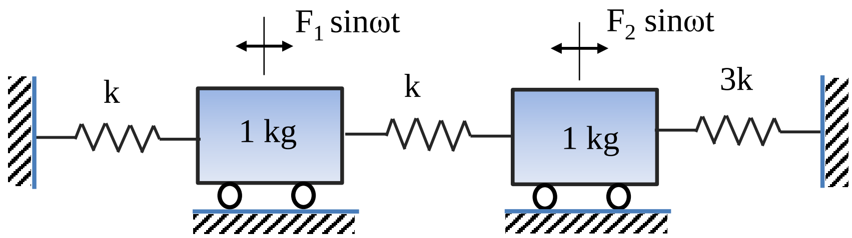
The result is compatible with that obtained from Example-6 since $x_n(t) = X_n \sin\omega t$.

H.W:

Solve problems 5.1 , 5.3, 5.4, 5.5

Q1: Find mass-normalized mode shapes for problems 5.1 and 5.5

Q2: for the system shown below, k = 50 N/m



Find:

(a) Natural frequencies and mode shapes
(b) Modal mass and modal stiffness matrices
(c) Mass-normalized mode shapes
(d) Responses using modal decomposition
(e) Response using uncoupled equations of motion.