University of Basrah

College of Engineering

Department of Mechanical Engineering

Advanced Vibration Lecture-3: Transient Vibration

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Transient Vibration

Analytical Method-Laplace Transform

For undamped system, the equation of motion:

$$m\ddot{x} + kx = F(t)$$

$$\ddot{x} + \omega_n^2 x = \frac{1}{m} F(t)$$

Assuming zero initial conditions, Laplace transform is:

$$(s^2 + \omega_n^2) X(s) = \frac{1}{m} F(s)$$

The solution is:

$$x(t) = \frac{1}{m} L^{-1} \left\{ \frac{1}{(s^2 + \omega_n^2)} F(s) \right\}$$

For damped system

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} F(t)$$

Assuming zero initial conditions, Laplace transform is:

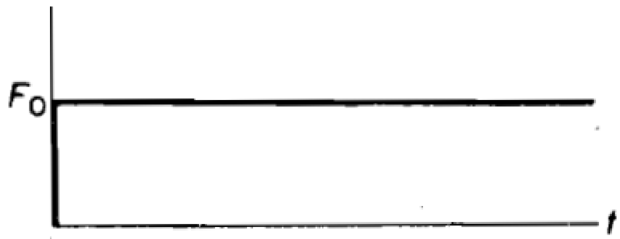
$$(s^2 + 2\zeta\omega_n s + \omega_n^2) X(s) = \frac{1}{m} F(s)$$

The solution is:

$$x(t) = \frac{1}{m} L^{-1} \left\{ \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} F(s) \right\}$$

Example

Find the response of undamped mass-spring system subjected to a force $F = F_0 u(t)$ assuming zero initial conditions.



Solution:

$$m\ddot{x} + kx = F_0 u(t)$$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m} u(t)$$

Using Laplace transform

$$(s^2 + \omega_n^2) X(s) = \frac{F_0}{ms}$$

$$X(s) = \frac{F_0}{m} \frac{1}{s(s^2 + \omega_n^2)}$$

Taking inverse Laplace transform, using the following Laplace transform table:

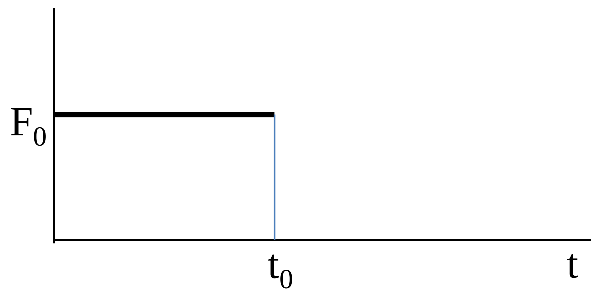
$\frac{1}{s(s^2 + a^2)}$	$\frac{1}{a^2}(1 - \cos at)$

$$x(t) = \frac{F_0}{m\omega_n^2}\left(1-\cos\omega_n t\right)$$

$$= \frac{F_0}{k}\left(1-\cos\omega_n t\right)$$

Example

Find the response of undamped mass-spring system subjected to the force shown, assuming zero initial conditions.



Solution:

$$m\ddot{x} + kx = F_0\left[u(t) - u(t-t_0)\right]$$

$$\ddot{x} + \omega_n^2 x = \frac{F_0}{m}\left[u(t) - u(t-t_0)\right]$$

Using Laplace transform

$$(s^2 + \omega_n^2)X(s) = \frac{F_0}{ms}\left(1 - e^{-t_0 s}\right)$$

$$X(s) = \frac{F_0}{m}\frac{1}{s(s^2+\omega_n^2)}\left(1 - e^{-t_0 s}\right)$$

Taking inverse Laplace transform, using the following Laplace transform table and applying shifting in time theory:

$\frac{1}{s(s^2+a^2)}$	$\frac{1}{a^2}(1-\cos at)$

Note: $\mathcal{L}\left\{f(t-a)u(t-a)\right\} = F(s)e^{-as}$

$$x(t)=\frac{F_0}{k}(1-\cos\omega_n t)-\frac{F_0}{k}[1-\cos\omega_n(t-t_0)]u(t-t_0)$$

Example

Find the response of undamped mass-spring system subjected to a force $F=F_0\sin\omega t$ assuming zero initial conditions.

Solution:

$$m\ddot{x}+kx=F_0\sin\omega t$$

$$\ddot{x}+\omega_n^2 x=\frac{F_0}{m}\sin\omega t$$

Using Laplace transform

$$(s^2+\omega_n^2)X(s)=\frac{F_0}{m}\frac{\omega}{s^2+\omega^2}$$

$$X(s)=\frac{F_0}{m}\frac{1}{s^2+\omega_n^2}\frac{\omega}{s^2+\omega^2}$$

Taking inverse Laplace transform, using the following Laplace transform table:

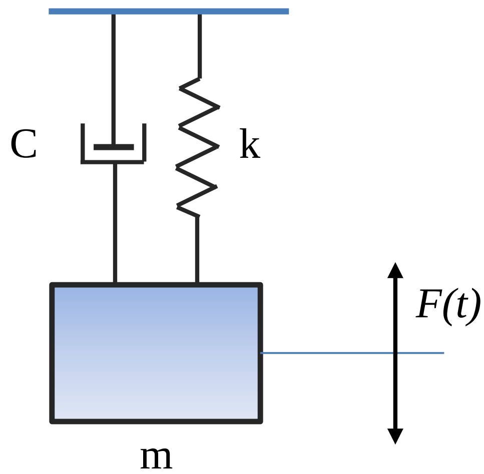
$\frac{1}{(s^2+a^2)(s^2+b^2)}$	$\frac{1}{b^2-a^2}\left(\frac{\sin at}{a}-\frac{\sin bt}{b}\right)$

$$x(t)=\frac{F_0}{m}\frac{\omega}{\omega^2-\omega_n^2}\left(\frac{\sin\omega_n t}{\omega_n}-\frac{\sin\omega t}{\omega}\right)$$

$$=\frac{F_0}{m}\frac{\omega}{\omega_n}\frac{1}{\omega^2-\omega_n^2}\sin\omega_n t-\frac{F_0}{m}\frac{1}{\omega^2-\omega_n^2}\sin\omega t$$

Example

Find the response of a damped mass-spring system subjected to a force $F = F_0 \sin \omega t$ assuming zero initial conditions.



Solution:

Applying Laplace transform yields:

$$X(s) = \frac{F_0}{m} \frac{1}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \frac{\omega}{(s^2 + \omega^2)}$$

Using the following Laplace transform table:

37. $\dfrac{1}{(s^2+\omega^2)[(s+a)^2+b^2]}$	$\dfrac{(1/\omega)\sin(\omega t+\phi_1)+(1/b)e^{-at}\sin(bt+\phi_2)}{[4a^2\omega^2+(a^2+b^2-\omega^2)^2]^{\frac{1}{2}}}$ $\phi_1 = \text{atan2}(-2a\omega, a^2+b^2-\omega^2)$ $\phi_2 = \text{atan2}(2ab, a^2-b^2+\omega^2)$

Where

$$a = \zeta\omega_n, \quad b = \sqrt{1-\zeta^2}\,\omega_n$$

$$x(t)=\frac{F_0\omega/m}{\sqrt{\left(2\zeta\omega_n\omega\right)^2+\left(\omega_n{}^2-\omega^2\right)^2}}\left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}\,\omega_n}\sin(\sqrt{1-\zeta^2}\,\omega_n t+\phi_2)+\frac{1}{\omega}\sin\left(\omega t+\phi_1\right)\right)$$

Where:

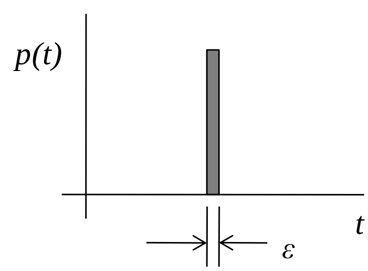
$$\phi_2=\tan^{-1}\frac{2\zeta\omega\sqrt{1-\zeta^2}\,\omega_n}{\left(2\zeta^2-1\right)\omega_n^2+\omega^2}$$

$$\phi_1=\tan^{-1}\frac{-2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$$

Numerical Method-Duhamel Integral

Impulse force

$$\hat{F}=\int p(t)dt$$



Delta function

$$\delta(t-\tau)=\begin{cases} non\ zero & t=\tau \\ 0 & t\neq\tau \end{cases}$$

$$\int_0^\infty \delta(t-\tau)dt=1$$

$$and\ \int_0^\infty p(t)\delta(t-\tau)dt=p(\tau)$$

Solution of undamped system:

$$x(t)=\frac{\dot{x}(0)}{\omega_n}\sin\omega_n t+x(0)\cos\omega_n t$$

Since an impulse force $\hat{F}$ acting at small time may present an initial velocity $\frac{\hat{F}}{m}$, then

$$x(t) = \frac{\hat{F}}{m\omega_n} \sin \omega_n t = \hat{F} h(t)$$

Where *h(t)* is the response for unit impulse input:

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

For damped systems:

$$x(t) = \frac{\hat{F} e^{-\zeta\omega_n t}}{m\omega_n \sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}\, t = \hat{F} h(t)$$

For Arbitrary Excitation:

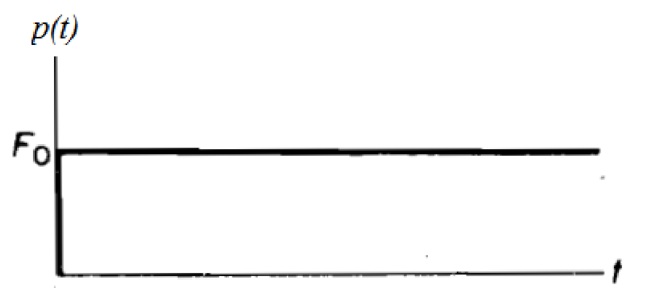
Given the response for unit impulse force is known *h(t)*, the response for an arbitrary force *p(t)* can be estimated by using superposition integral or convolution integral (Duhamel Integral):

$$x(t) = \int_0^t p(\tau) h(t-\tau) d\tau = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n (t-\tau) d\tau$$

The above integral state that the overall response is the sum of responses to forces each force is acting at different time τ and have duration $d\tau$.

Example:

Determine the response of an undamped system to the force shown in the figure:



Solution:

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

$$x(t) = \int_0^t p(t) h(t-\tau) d\tau = \int_0^t \frac{F_0}{m\omega_n} \sin \omega_n (t-\tau) d\tau$$
$$= \frac{F_0}{k}(1 - \cos \omega_n t)$$

Numerical Methods

When the integral is not possible in closed form, then numerical methods such as Euler and Runge-Kutta methods can be used.

Numerical Integration

The response to an arbitrary excitation p(t) with zero initial condition is given by

$$x(t) = \int_0^t p(\tau) h(t-\tau) d\tau = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n (t-\tau) d\tau$$
$$= \sin \omega_n t \frac{1}{m\omega_n} \int_0^t p(\tau) \cos \omega_n \tau d\tau - \cos \omega_n t \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n \tau d\tau$$
$$= A(t) \sin \omega_n t - B(t) \cos \omega_n t$$

Where:

$$A(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \cos \omega_n \tau d\tau$$
$$B(t) = \frac{1}{m\omega_n} \int_0^t p(\tau) \sin \omega_n \tau d\tau$$

These can be evaluated numerically. Consider ***A(t):***

$$A(t)=\frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau=\frac{1}{m\omega_n}\int_0^t y(\tau)d\tau=\frac{\Delta\tau}{m\omega_n}\frac{1}{\mu}\mathrm{T}_\mu^t$$

Where

$$\mathrm{T}_1^t=y_0+y_1+y_2+.....+y_{N-1} \qquad \textit{Simple Integration}\,(\mu=1)$$
$$\mathrm{T}_2^t=y_0+2y_1+2y_2+.....+2y_{N-1}+y_N \qquad \textit{Trapizoidal Integration}\,(\mu=2)$$
$$\mathrm{T}_3^t=y_0+4y_1+2y_2+.....+4y_{N-1}+y_N \qquad \textit{Simpson Integration}\,(\mu=3)$$

Using Incremental integration for numerical solution:

$$\mathrm{T}_1^t=\mathrm{T}_1^{t-\Delta\tau}+p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau) \qquad \textit{Simple Integration}$$
$$\mathrm{T}_2^t=\mathrm{T}_2^{t-\Delta\tau}+p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau)+p(t)\cos\omega_n t \qquad \textit{Trapizoidal}$$
$$\mathrm{T}_3^t=\mathrm{T}_3^{t-\Delta\tau}+[p(t-2\Delta\tau)\cos\omega_n(t-2\Delta\tau)+ \\ 4p(t-\Delta\tau)\cos\omega_n(t-\Delta\tau)+p(t)\cos\omega_n t] \qquad \textit{Simpson}$$

For damped vibration:

$$h(t)=\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$x(t)=A(t)e^{-\zeta\omega_n t}\sin\omega_n t-B(t)e^{-\zeta\omega_n t}\cos\omega_n t$$

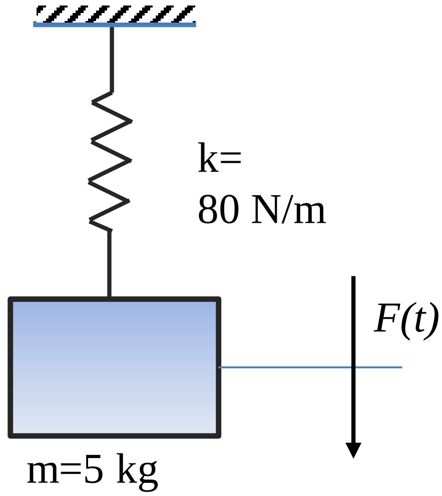
$$A(t)=\frac{1}{m\omega_n}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\cos\omega_n\tau d\tau$$
$$B(t)=\frac{1}{m\omega_n}\int_0^t p(\tau)e^{\zeta\omega_n\tau}\sin\omega_n\tau d\tau$$

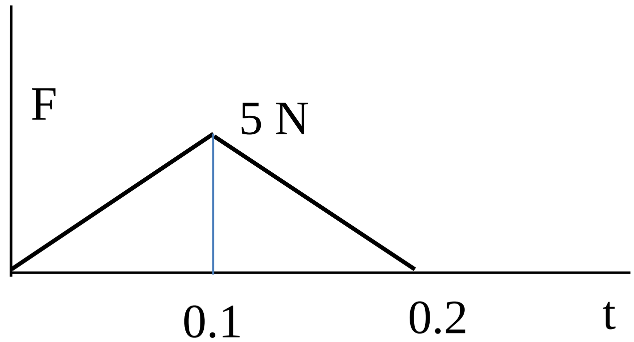
Example

Given the undamped mass-spring system shown below, determine the response when:

(a) t = 0.1 sec
(b) t = 0.2 sec

Use step 0.025 sec and apply trapezoidal method.





Solutions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{5}} = 4\ rad/s$$

$$x(t) = A(t)\sin\omega_n t - B(t)\cos\omega_n t$$

$$A(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau$$

$$B(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n\tau d\tau$$

$$A(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\cos\omega_n\tau d\tau = \frac{1}{m\omega_n}\int_0^t y(\tau)d\tau$$

$$B(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin\omega_n\tau d\tau = \frac{1}{m\omega_n}\int_0^t z(\tau)d\tau$$

i	τ	$p(\tau)$	$y(\tau) = p(\tau)\cos 4\tau$	$z(\tau) = p(\tau)\sin 4\tau$
0	0	0	0	0
1	0.025	1.25	1.2438	0.1248

2	0.05	2.5	2.4502	0.4967
3	0.075	3.75	3.5825	1.1082
4	0.1	5	4.6053	1.9471
5	0.125	3.75	3.2909	1.7978
6	0.15	2.5	2.0633	1.4116
7	0.175	1.25	0.9561	0.8053
8	0.2	0	0	0
…	…	0	0	0

(a) when t = 0.1 sec

$$A = \frac{1}{5\times 4}\frac{0.025}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right] = 0.0119738$$

$$B = \frac{1}{5\times 4}\frac{0.025}{2}\left[z_0 + 2z_1 + 2z_2 + 2z_3 + z_4\right] = 0.0033790$$

$$x(t) = 0.0119738\sin 4t - 0.003379\cos 4t$$

(b) when t = 0.2 sec (or more)

$$A = \frac{1}{5\times 4}\frac{0.025}{2}\left[y_0 + 2y_1 + 2y_2 + ... + 2y_7 + y_8\right] = 0.02274$$

$$B = \frac{1}{5\times 4}\frac{0.025}{2}\left[z_0 + 2z_1 + 2z_2 + ... + 2z_7 + z_8\right] = 0.0096144$$

$$x(t) = 0.02274\sin 4t - 0.0096144\cos 4t$$

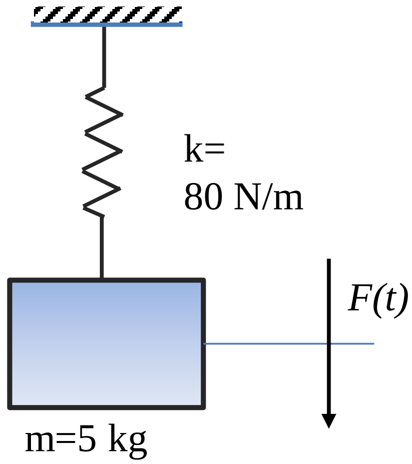
H.W:

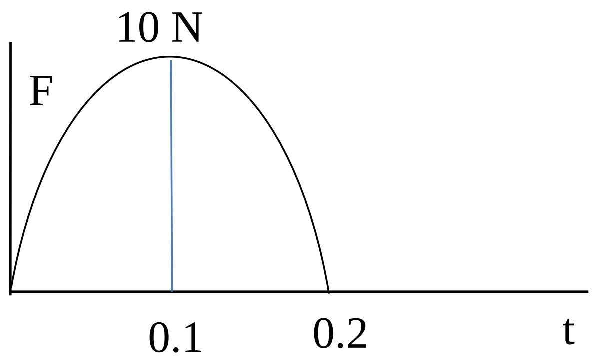
Q1: For the following system, determine the response when:

(a) t = 0.1 sec
(b) t = 0.2 sec

Use step 0.025 sec and apply trapezoidal method.

$$F(t) = \begin{cases} 10\sin(\pi t / 0.2) & 0 < t < 0.2 \\ 0 & otherwise \end{cases}$$





Q2: Re-solve Q1 assuming a damped system with damping factor $C = 8$ *N.s/m*