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Mathematics and Statistics\ first stage

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2020\2021

Functions:

Def : ***is a rule that assigns to each element in a nonempty set X one and only one element in set Y. (X is the domain of the function, while Y is the range of the function).in another meaning to denote the dependence of one quantity on another.***

- ***Domain:***
- ***Range: the set of all images of points in the domain*** $(f(x), x \in X)$ **.**

Arithmetic Operations on functions:

- ***Sum:*** $(f+g)(x) = f(x) + g(x)$.
- ***Difference :*** $(f-g)(x) = f(x) - g(x)$.
- ***Product:*** $(f * g)(x) = f(x) * g(x)$.
- ***Quotient::*** $(f/g)(x) = \frac{f(x)}{g(x)}$ ***,where*** $g(x) \neq 0$ **.**

The Limit of function at point:

the expression $lim_{x \to b} f(x)$ ***,where a is b real number andf is a function. This is read as: "the limit of*** $f(x)$ ***as x approaches b.***

#Now ,we will discuss four cases:

1.Evaluating the Limit of a Polynomial Function at a Point:

Example1 $f(x) = x^2 + x - 2$***.Calculate*** $lim_{x \to 2} f(x)$***?***

Solution $lim_{x\to 2} f(x) = 2^2 + 2 - 2 = 4.$

2. Evaluating the Limit of a Rational Function at a Point:

Example2 $f(x) = \frac{3x+2}{x-1}$.*Calculate* $lim_{x\to 2}\ f(x)$?

Note that f is a rational function with implied domain $Dom(f) = \{x \in R, x \neq 1\}$.
Solution $lim_{x\to 2} f(x) = \frac{3.2+2}{2-1}$= 8.

Example3 $f(x) = \frac{x^2-4}{x-2}$, *Calculate* $lim_{x\to 2} f(x)$?

Notice that the function $f(x) = \frac{x^2-4}{x-2}$ *is not defined when x =2,because the denominator is zero we use the simplification first and then we substitute the value of x.*

Solution $lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2}$

$$= \lim_{x\to 2}(x+2) = 4$$

Example4: *Let* $f(x) = \frac{1}{x^2}$ *Find* $lim_{x\to 0} f(x)$ *if it exists?*

Solution: *The values of f (x) can be made arbitrarily large by taking x close enough to 0. Thus the values of f (x) do not approach 0 number, so* $\lim_{x\to 0} \frac{1}{x^2}$ *does not exist.*

3.Evaluating the Limit of a Constant Function at a Point:

Example 5 *Let* $f(x) = 5$ *Find* $lim_{x\to 1} f(x)$?
Solution: $lim_{x\to 1}$ 5=5. *since there is no x in* $f(x)$ *anyway.*

4. One and Two sided limits:

$\lim_{x\to b}$ *is a two-sided limit operator in* $\lim_{x\to b} f(x)$ *, because we must consider the* *behavior of f as x approaches b from both the left and the right.* $\lim\limits_{x\to b^-} f(x)$ *is read as :the limit of* $f(x)$ *as x approaches b from the left.* $\lim\limits_{x\to b^+} f(x)$*is read as: the limit of* $f(x)$*as x approaches b from the right.*

Example 6 **Let** $f(x) = \begin{cases} 2x - 1 & , x \geq -1 \\ 3x & , x < -1 \end{cases}$,

Find the left and the right $\lim_{x\to -1} f(x)$*?and explain if* $\lim_{x\to -1} f(x)$*if it exists?*

Solution :

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (2x - 1) = -3 = L_1$$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 3x = -3 = L_2$$

$\therefore L_1 = L_2 \to \lim_{x\to -1} f(x)$*is exists and equal -3.*

Example 7 **Let** $f(x) = \begin{cases} -3x + 1 & , x < 1 \\ x^2 - 2 & , x \geq 1 \end{cases}$ *, Find the left and the right* $\lim_{x\to 1} f(x)$*? and explain if* $\lim_{x\to 1} f(x)$ *if it exists?*

Solution :

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 - 2) = -1 = L_1$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-3x + 1) = -2 = L_2$$

We notice $L_1 \neq L_2 \to \lim_{x\to -1} f(x)$*does not exist.*

The Limit Laws

If b and k are real numbers and $\lim_{x\to b} f(x) = L_1$ *and* $\lim_{x\to b} g(x) = L_2$ *,then*

1.Sum Rule: $\lim(_{x\to b} f(x) + g(x)) = L_1 + L_2$

2.Difference Rule: $\lim(_{x\to b} f(x) - g(x)) = L_1 - L_2$

3.Constant Multiple Rule: $\lim(k._{x\to b} f(x)) = k. L_1$

4.Product Rule: $\boldsymbol{lim(_{x\to b} f(x).g(x)) = L_1.L_2}$

5.Quotiesnt Rule: $\boldsymbol{lim_{x\to b}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}, L_2 \neq 0}$

6.Power Rule: $\boldsymbol{\lim_{x\to b}\left(f(x)\right)^n = L_1{}^n}$

7.Root Rule: $\boldsymbol{\lim_{x\to b}\sqrt[n]{f(x)} = \sqrt[n]{L_1 = L_1{}^{1/n}}, n\ apositive\ integer.}$

Example 8 _*Calculate the following :*

a) $\boldsymbol{\lim_{x\to b}(x^3+4x^2-3)}$

b) $\boldsymbol{\lim_{x\to b}\frac{x^2+3x+2}{x+1}}$

c) $\boldsymbol{\lim_{x\to 2}\sqrt{2x^2+1}}$

d) $\boldsymbol{\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}}$

f) $\boldsymbol{\lim_{x\to 9}\frac{\frac{1}{\sqrt{x}}-\frac{1}{3}}{x-9}}$

Solution

a) $\boldsymbol{\lim_{x\to b}(x^3+4x^2-3) = \lim_{x\to b}x^3 + \lim_{x\to b}4x^2 - \lim_{x\to b}3}$

$$\boldsymbol{= b^3+4b^2-3}$$

b) $\boldsymbol{\lim_{x\to b}\frac{x^2+3x+2}{x+1} = \frac{\lim_{x\to b}(x^2+3x+2)}{\lim_{x\to b}(x+1)} = \frac{b^2+3b+2}{b+1}}$

$$c) \lim_{x\to 2}\sqrt{2x^2+1} = \sqrt{\lim_{x\to 2}(2x^2+1)} = \sqrt{2.2^2+1} = 3$$

$$d) \lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$$

We can Multiply both numerator and denominator by the conjugate radical expression $\sqrt{x^2+100}+10$ *(obtained by changing the sign after the square root).*

$$\frac{\sqrt{x^2+100}-10}{x^2} = \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} = \frac{x^2}{x^2(\sqrt{x^2+100}+10)} = \frac{1}{(\sqrt{x^2+100}+10)}$$

Therefore,

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{1}{(\sqrt{x^2+100}+10)} = \frac{1}{20}$$

$$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-3)(x-2)} = \frac{4}{-1} = -4$$

$$f) \lim_{x\to 9}\frac{\frac{1}{\sqrt{x}}-\frac{1}{3}}{x-9} = \lim_{x\to 9}\frac{\frac{3-\sqrt{x}}{3\sqrt{x}}}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9}\frac{3-\sqrt{x}}{3\sqrt{x}} * \frac{1}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x\to 9}\frac{-(\sqrt{x}-3)}{3\sqrt{x}} * \frac{1}{(\sqrt{x}-3)(\sqrt{x}+3)} = \frac{-1}{54}$$

Differentiation:

Derivative definition *:the function* $f' = lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ *is called the derivative with respect to x of the function* f*. The domain of f0consists of all the points for which the limit exists. The domain of* f'*consists of all the points for which the limit exists.*

Geometric interpretation of the derivative:
Slope of the tangent Rate of change interpretation. function whose value at x is the instantaneous rate of change of y with respect to x at the point x.

Notes

- ***A function that has derivatives at a point*** x ***is said to be differentiable at*** x ***.***
- ***A function that is differentiable at every point of its' domain is called differentiable.***
- ***The differentiation operation is often denoted by*** $\frac{d}{dx}(f(x))$ ***, which read (the derivative of*** $f(x)$***with respect c.***

Example1: ***Find the derivative of*** $f(x) = \sqrt{x}$***by definition ?***
Sol. ***:-***

$$f'(x) = lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} * \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

- ***Example2:*** ***Find the derivative of*** $f(x) = x^2 + 3x + 2$ ***by definition ?***
  Sol:

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 + 3(x+h) + 2 - (x^2 + 3x + 2}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2hx + h^2 + 3x + 3h + 2 - x^2 - 3x - 2}{h}$$

$$= \lim_{h\to 0}\frac{2hx + h^2 + 3h}{h} = \lim_{h\to 0}\frac{h(2x + h + 3)}{h}$$

$$= 2x + 0 + 3 = 2x + 3.$$

- ***Example3:*** ***Find the derivative of*** $f(x) = \frac{1}{x}$ ***by definition ?***
- ***Sol:***

$$f'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{x - (x+h)}{x(x+h)}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2}.$$

- ***Example3:*** ***Find the derivative of*** $f(x) = x^2 + 3$ ***by definition ?***

Sol:

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 + 3 - (x^2 + 3)}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2hx + h^2 + 3 - x^2 - 3}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h} = 2x + 0 = 2x.$$

Slopes and Tangent Lines:

When the value $f'(x)$ ***is exists is called slope of the curve*** $y = f(x)$ ***at*** x***. The line through the point*** $(x, f(x))$ ***with slope*** $f'(x) = m$ ***is the tangent to the curve at*** x***.***

Now, steps to find the equation of the tangent :-

1. ***Find a contact point*** (x_1, y_1)***.***
2. ***Find the slope of the curve*** $m = f'(x)$ ***.***

3.Apply following relation $y - y_1 = m(x - x_1)$***.***

Example1: *Find the equation for the tangent to the curve* $f(x)=(2-x^2)^2$ *at* $x=2$?

Sol. *:- from steps a bove:*

1.Find a contact point $y_1=f(2)=(2-2^2)^2=(-2)^2=4.$

$(x_1, y_1)=(2,4)$

2.Find the slope $m = f'(x)$

$\rightarrow m = f'(x) = 2(2-x^2)*(-2x) = -4x(2-x^2)$

$\rightarrow m = f'(2) = -4(2)(2-2^2) = -8*(-2) = 16$

3.Apply the relation $y - y_1 = m(x-x_1)$

$y - 4 = 16(x-2) = 16x - 32$

$16x - y - 32 + 4 = 0$

$16x - y - 28 = 0$

$y = 16x - 28$

$\rightarrow$ *The equation for tangent is* $y = 16x - 28$

- ***Example 2 :-*** *Find equation for the tangent to the curve* $y=\sqrt{x}$ *at* $x=4$

Sol. :-

1. firstly find (x_1, y_1) *by*

$y_1 = \sqrt{x_1} = \sqrt{4} = 2 \rightarrow (x_1, y_1) = (4,2)$

2. Then, find slope $m = f'(x)$

$\rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

$\rightarrow m = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

3.Now, Apply the equation :-

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x-4)$

$y = \frac{1}{4}x - 1 + 2$

$\rightarrow$ *The equation for tangent is* $y = \frac{1}{4}x + 1$

Example 3:- *find equation for the tangent to the curve* $y=\sqrt[3]{3x+5}$ *at* $x=1$?

Sol. :-

1. find (x_1, y_1) *by* $y_1 = f(x_1) = f(1) = \sqrt[3]{3(1)+5} = \sqrt[3]{8} = 2$

$\rightarrow (x_1, y_1) = (1,2)$

2. find slope $m = f'(x)$

$\rightarrow f'(x) = \frac{1}{3}\,(3x+5)^{-2/3} * 3 = \frac{1}{\sqrt[3]{(3x+5)^2}}$

$m = f'(1) = \frac{1}{\sqrt[3]{(3(1)+5)^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$

3. Apply equation

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x-1) \rightarrow 4y - 8 = x - 1$

$x - 4y + 8 - 1 = 0$

$x - 4y + 7 = 0$

So, equation of tangent is $y = \frac{1}{4}(x+7).$

Example 4:- *find equation for the tangent to the curve*

$y = x^2$ *at the point* $\left(\frac{-1}{2}, \frac{1}{4}\right)$

Sol. :-

firstly, find slope of curve:

$f'(x) = 2x \rightarrow m = f'(x) = 2\left(\frac{-1}{2}\right) = -1$

Apply equation $y - y_1 = m(x - x_1)$

$y - \frac{1}{4} = -1\left(x - \left(\frac{-1}{2}\right)\right)$

$y - \frac{1}{4} = -x - \frac{1}{2}$

$y = -x - \frac{1}{4}$

$\rightarrow$*equation of tangent is* $y = -x - \frac{1}{4}$