9. 8/3/2014

18/3/2015

Integration

1. indefinite integrals

Rules for indefinite integrals

1. $\int dx = x + c$
2. $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$
3. $\int [g(x)]^n \cdot g'(x) \cdot dx = \frac{[g(x)]^{n+1}}{n+1} + c$
4. $\int k \cdot f(x) \cdot dx = k \int f(k) \cdot dx$ where k is a constant
5. $\int [f(x) \mp g(x)] \cdot dx = \int f(x) \cdot dx \mp \int g(x) \cdot dx$

Example: evalute the following integrals

1. $\int x^5 dx$

$\int x^5 \cdot dx = \frac{x^6}{6} + c$

2. $\int (x+5)^5 dx$

$\int (x+5)^5 dx = \frac{(x+5)^6}{6} + c$

3. $\int \sqrt{4x-1}\, dx = \int (4x-1)^{1/2} dx = \int (4x-1)^{1/2} \frac{4}{4} dx$

$= \frac{1}{4} \int (4x-1)^{1/2} 4 \cdot dx$

$= \frac{1}{4} \frac{(4x-1)^{3/2}}{3/2} + c = \frac{1}{6} (4x-1)^{3/2} + c$

مشتقة $(2x+2)\,dx$

$$4.\quad \int (x^2+2x-3)^2 (x+1)\,dx$$

$$= \int (x^2+2x-3)^2 (x+1)\cdot\frac{2}{2}\,dx$$

$$= \frac{1}{2}\int (x^2+2x-3)^2 \boxed{(x+1)\cdot 2\,dx}$$ المشتقة

$$= \frac{1}{2}\,\frac{(x^2+2x-3)^3}{3} + c = \frac{1}{6}(x^2+2x-3)^3 + c$$

$$5.\quad \int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int (z^2+1)^{-1/3}\cdot 2z\,dz = \frac{(z^2+1)^{2/3}}{2/3} + c$$

$$= \frac{3}{2}(z^2+1)^{2/3} + c$$

$$6.\quad \int \frac{(1+x)^2}{\sqrt{x}}\,dx = \int \frac{1+2x+x^2}{\sqrt{x}}\,dx = \int \left[\frac{1}{\sqrt{x}} + \frac{2x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right]$$

$$= \int \left(x^{-1/2} + 2x^{1/2} + x^{3/2}\right)dx$$

$$= \frac{x^{1/2}}{1/2} + \frac{2x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + c = 2x^{1/2} + \frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2} + c$$

عندما درجة البسط أكبر أو تساوي درجة المقام نستخدم القسمة الطويلة.

$$7.\quad \int \frac{x^2+2x}{x^2+2x+1}\,dx$$

الناتج: 1

المقسوم عليه: x^2+2x+1) x^2+2x

$$\mp x^2 \mp 2x \mp 1$$

$$0 + 0 \; -1$$ المتبقي

$$\frac{\text{المتبقي}}{\text{المقسوم عليه}} + \text{الناتج}$$

$$= \int \left(1 - \frac{1}{x^2+2x+1}\right)dx$$

$$= \int \left(1 - \frac{1}{(x+1)^2}\right)dx$$

$$= \int \left(1 - (x+1)^{-2}\right)dx = x - \frac{(x+1)^{-1}}{-1} + c = x + \frac{1}{x+1} + c$$

$$= \frac{x^2+x+1}{x+1} + c$$

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9.

Integration of trigonometric functions:

1. $\int \sin u \cdot du = -\cos u + c$
2. $\int \cos u \cdot du = \sin u + c$
3. $\int \sec^2 u \cdot du = \tan u + c$
4. $\int \csc^2 u \cdot du = -\cot u + c$
5. $\int \sec u \tan u \, du = \sec u + c$
6. $\int \csc u \cot u \, du = -\csc u + c$

Example: evaluate the following integrals.

1. $\int \sin 2x \, dx$ ($2dx$)

مشتقة

$= \int \sin 2\cancel{x} \cdot \frac{2}{2} \, dx = \frac{1}{2} \int \sin 2x \cdot 2dx$

$= \frac{1}{2}(-\cos 2x) + c = \frac{-\cos 2x}{2} + c$

2. $\int \cos(7x+5) \, dx$

$= \int \cos(7x+5) \, dx = \int \cos(7x+5) \frac{7}{7} dx$

$= \frac{1}{7} \int \cos(7x+5) \cdot 7dx = \frac{1}{7} \sin(7x+5) + c$

3. $\int x^2 \sin x^3 \, dx$

x^3 مشتقة

$= \int x^2 \sin x^3 \, \frac{3}{3} dx = \frac{1}{3} \int 3x^2 \sin x^3 \, dx = \frac{-\cos x^3}{3} + c$

4. $\int \frac{1}{\cos^2 2x} dx = \int \sec^2 2x \, dx = \int \sec^2 2x \frac{2}{2} dx$

مشتقة

$= \frac{1}{2} \int \sec^2 2x \cdot 2dx = \frac{1}{2} \tan 2x + c$

$\sec^2 = \tan$ مشتقة

9.*

5. $\int \tan x \sec^2 x \, dx = \frac{\tan^2 x}{2} + c$

or $= \int \tan x \sec x \sec x \, dx = \frac{\sec^2 x}{2} + c$

sec مشتقة

15/3 ٩.

مسائل p.179

6. $\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$
$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

7. $\int \cos^2 3x\,dx = \int \frac{1}{2}(1+\cos 6x)\,dx$
$= \frac{1}{2}x + \frac{1}{12}\sin 6x + c$

$\sin^2 + \cos^2 = 1$

8. $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$
$= -\cos x + \frac{1}{3}\cos^3 x + c \quad (\cos x)^2$

مشتقة cosx
$-\sin x\,dx$

$-\cos^2 x \sin x$
$-((\cos x)^2 \sin$
مشتقة
$\cos x = -\sin x$
$\therefore \int (\cos x)^2(-\sin x)\,dx$
$\therefore \frac{(\cos x)^3}{3}$

9. $\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx$
$[1 - 2\sin^2 x + \sin^4 x]$

$(\sin x)^2$ مشتقة

$= \int \cos x\,dx - 2\int \sin^2 x \cos x\,dx + \int \sin^4 x \cos x\,dx$

$(\sin x)^2$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$

10. $\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{8}\int (1-\cos 4x)\,dx$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$

$\sin^2 x \cos^2 x = (\sin x \cos x)^2$
$= (\frac{1}{2}\sin 2x)^2$
$= \frac{1}{4}\sin^2 2x$

$\sin^2 x + \cos^2 x = 1$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin x * \cos x = \frac{1}{2}\sin 2x$

٩. 17/3 2016
٧. 16/3 2016

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2- definite integrals:

rules for definite integrals

1. $\int_a^a f(x)\,dx = 0$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3- $\int_a^b k\,f(x)\,dx = K\int_a^b f(x)\,dx$ where K is a constant

when $K = -1$, then, $\int_a^b (-1)\,f(x)\,dx = -\int_a^b f(x)\,dx$

4. $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

5. If $f(x) \geq 0$ on $[a, b]$ then, $\int_a^b f(x)\,dx \geq 0$ on $[a, b]$

6- If $f(x) \geq g(x)$ on $[a, b]$ then, $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ on $[a, b]$

7- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$ where $a \leq c \leq b$

Examples: evalute the following integrals:

1- $\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{1}{3}(2^3-0) = \frac{1}{3}(8-0) = \frac{8}{3}$

2- $\int_0^{\pi} \cos x\,dx = [\sin x]_0^{\pi} = \sin\pi - \sin 0 = 0-0 = 0$

3- $\int_{-2}^{2} (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left[4(2) - \frac{(2)^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right]$

$= 8 - \frac{8}{3} + 8 - \frac{8}{3} = 16 - \frac{16}{3} = \frac{32}{3}$

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Area under curves

If an integrable function $y=f(x)$ is nonnegative throughout an interval $[a,b]$, and we want to estimate the area of the shaded region R that lies above the x-axis, below the graph of $y=f(x)$ from a to b. If we divide the region into $\underline{n}$ rectangles, then the area of each rectangle is $f(c_k)\Delta x_k$. finally we sum all of these subareas to get.

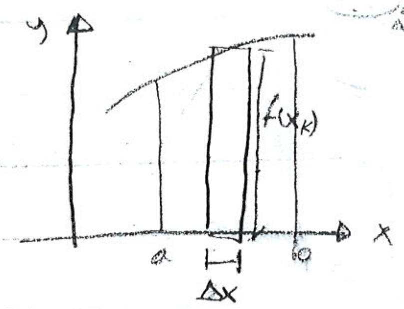
$$A \simeq \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

Since the rectangles give an increasing good approximation of the region as we use subdivisions with smaller and smaller subintervals, then

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(c_k)\cdot \Delta x_k$$

As $\Delta x \to 0 \Rightarrow n \to \infty$, thus

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \int_a^b f(x)\,dx.$$



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Example: Find the area between the x-axis and the curve:

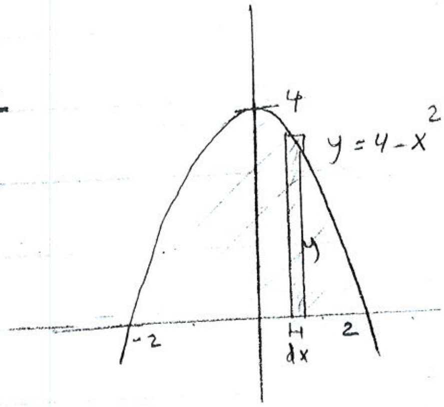
a. $y = 4 - x^2$. b. $y = x^2 - 4$

for $-2 \leq x \leq 2$.

a. Since $y = 4 - x^2 \geq 0$ on $[-2, 2]$, the area between the curve and x-axis from -2 to 2:

$$\int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$$

Area $= \frac{32}{3}$ square units.



b. since $y = x^2 - 4 \leq 0$ on $[-2, 2]$, the area between the curve and x-axis from -2 to 2:

$0 = y$

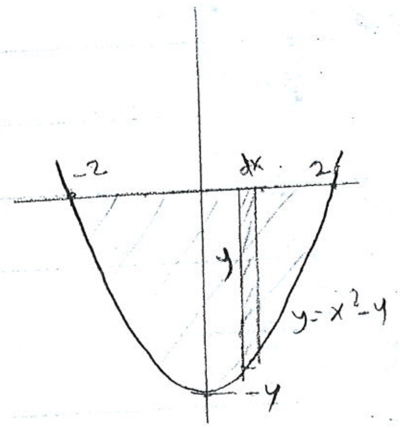
$0 = (x^2 - 4)$

$$\int_{-2}^{2} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = -\frac{32}{3}$$

$\frac{8}{3} - 8 - (-\frac{8}{3} + 8)$

$\frac{8}{3} - 8 + \frac{8}{3} - 8$

$\therefore$ Area $= \left| -\frac{32}{3} \right| = \frac{32}{3}$ square units.



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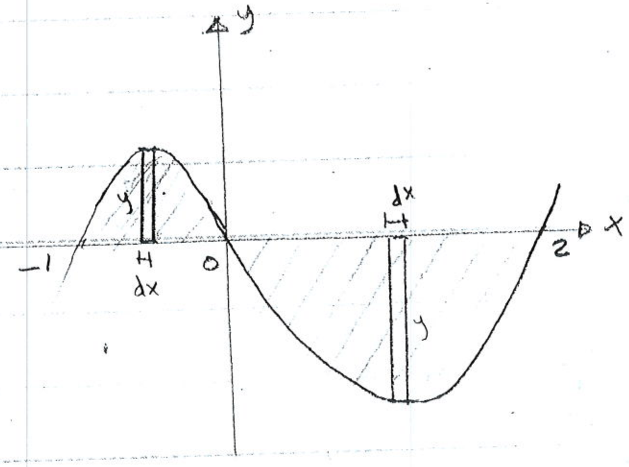
Example: Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, for $-1 \leq x \leq 2$.

Solution:

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x-2)(x+1)$$

, put $x(x-2)(x+1) = 0 \Rightarrow x = 0$ or $x = 2$ or $x = -1$



$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{-1}^{0}$$

$$= \left(\frac{0^4}{4} - \frac{0^3}{3} - 0^2\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right) = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{0}^{2}$$

$$= \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - 0 = -\frac{8}{3}$$

$$A = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \text{ square units.}$$

خد بالك لو مباشرة

$$\int_{-1}^{2} (x^3 - x^2 - 2x)\,dx = \frac{-9}{4}$$

هذه النتيجة هي تساوي حاصل جمع المساحتين

$$\frac{5}{12} - \frac{8}{3} = \frac{-9}{4}$$

∴ هي تساوي نتيجة المساحة في الأسفل مطروحة منها المساحة في الأعلى

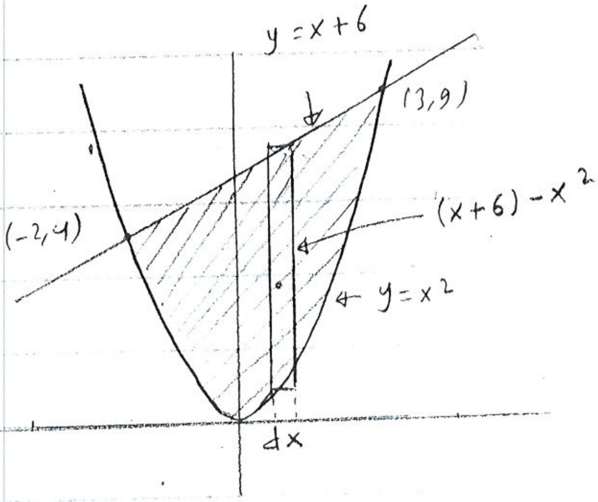
Calculus 400

Applications of diffinite integrals

1- Area between Curves :

Example: Find the area of the region that is enclosed between the curves $y=x^2$ and $y=x+6$

Solution:



to find the endpoints, equait the two equations

$x^2 = x + 6$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2$ and $x = 3$

the points of intersections ~~is~~ are $(-2,4)$ and $(3,9)$.

$$A = \int_{-2}^{3} \left[(x+6) - x^2\right] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}$$

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Calculus 401

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Example: Find the area of the region enclosed by $x = y^2$ and $y = x - 2$

Solution:

to find the intersection points, equaite the two equation

$x = y^2$ and $x = y + 2$

$y^2 = y + 2$

$y^2 - y - 2 = 0$

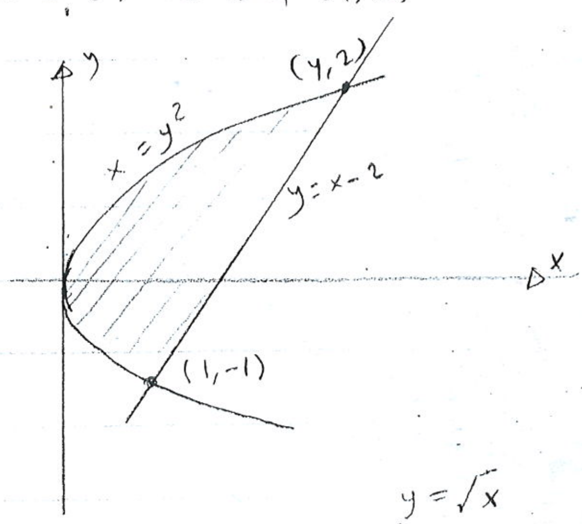
or $(y+1)(y-2) = 0$

$\therefore y = -1, y = 2$

substituting these values in the equation $x = y^2$ or $x = y + 2$

the corresponding x-value are $x = 1$ and $x = 4$, respectivel

so the points of intersection are $(1, -1)$ and $(4, 2)$.



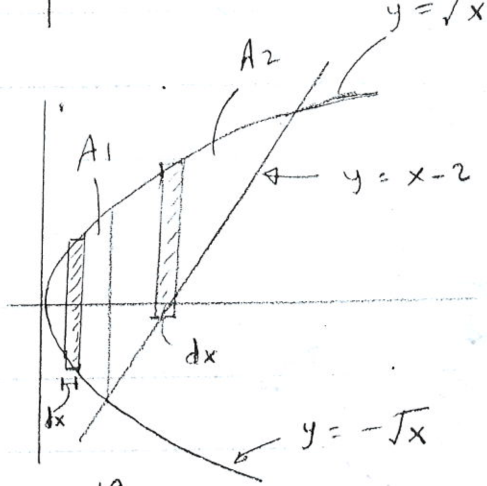
$x = y^2$

$\therefore y = \pm\sqrt{x}$

$A_1 = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx$

$= 2\int_0^1 \sqrt{x}\,dx = 2\left[\frac{2}{3}x^{3/2}\right]_0^1$

$= \frac{4}{3} - 0 = \frac{4}{3}$



$A_2 = \int_1^4 [\sqrt{x} - (x - 2)]\,dx$

$= \int_1^4 (\sqrt{x} - x + 2)\,dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4$

$= \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{19}{6}$

$\therefore A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{9}{2}$

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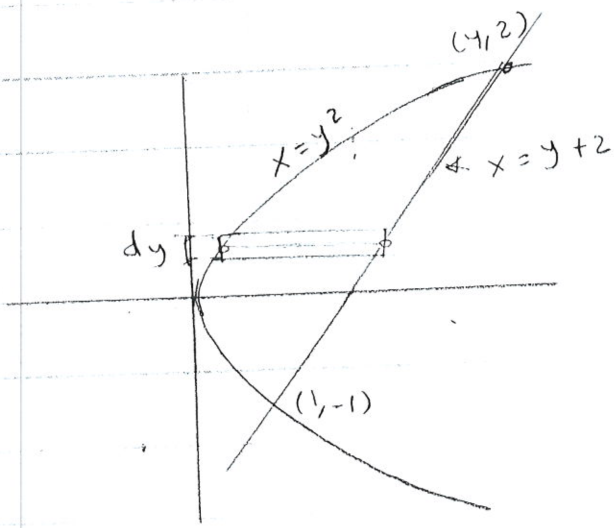
Other solution



$$A = \int_{-1}^{2} \left[(y+2) - y^2 \right] dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2} = \frac{9}{2}$$

$$\left\{ 2 + 4 - \frac{8}{3} \right\} - \left\{ \frac{1}{2} - 2 + \frac{1}{3} \right\}$$

$$\left(6 - \frac{8}{3} \right) - \frac{3 - 12 + 2}{6}$$

$$\frac{18-8}{3} - \frac{-7}{6}$$

$$\frac{10}{3} + \frac{7}{6}$$

$$\frac{20+7}{6} = \frac{27}{6}$$

$$= \frac{9}{2}$$

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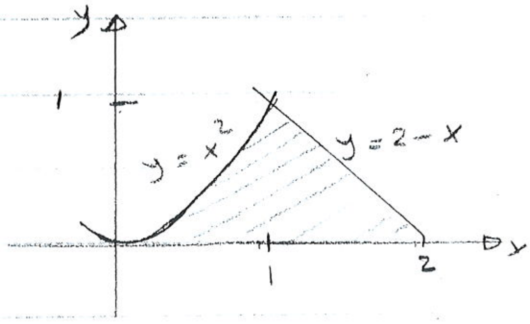
Caculus early p. 436

Example: Find the area bounded by the graphs of $y = x^2$, $y = 2 - x$ and $y = 0$

Solution:

$y = x^2$

$\Rightarrow x = \pm\sqrt{y}$



ايجاد نقاط التقاطع

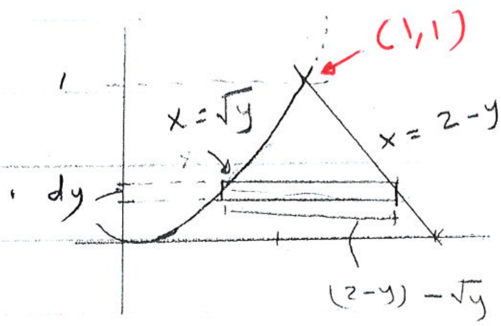
$\sqrt{y} = 2 - y$

$y = (2-y)^2$

$y = 4 - 4y + y^2$

$y^2 - 5y + 4 = 0$

$(y-1)(y-4) = 0$



So, the curves intersect at

$y = 1$ and $y = 4$

From the figure, it is clear that $y = 1$ is the solution we need.

لم نأخذ $y = 4$ لأن

$y = 4$ [illegible]

[illegible]

$y = 4 \Rightarrow x = -2$

(-2,4)

هذه المساحة غير مطلوبة

$$A = \int_0^1 [(2-y) - \sqrt{y}]\,dy$$

$$= \left[2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\right]_0^1$$

$$= 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$$

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\underbrace{}_{2x}$

$-x$

x

$\therefore x = -2, \quad x = 1$

$y = 4, \quad y = 1$

$(-2,4), (1,1)$

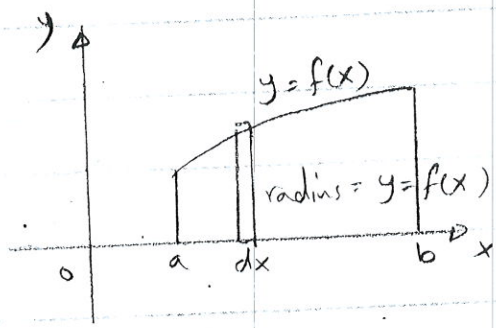
2. Volume of Solids of revolution

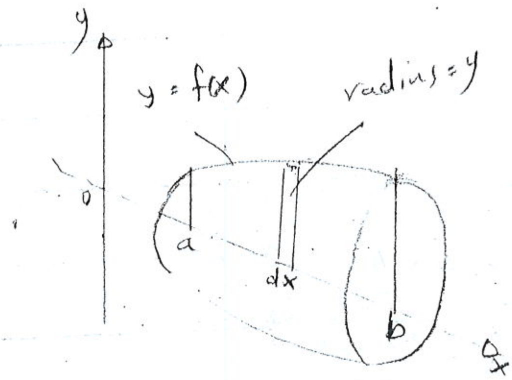
A. The disk method (the strip is perpendicular to the axis of revolution):

① rotation about x-axis:

$dv = \pi \cdot (\text{radius})^2 \cdot (\text{thickness}) = \pi \cdot y^2 dx$
$= \pi \cdot (f(x))^2 dx$

$\text{Volume} = \int dv = \int_a^b \pi (\text{radius})^2 dx = \int_a^b \pi (f(x))^2 dx$

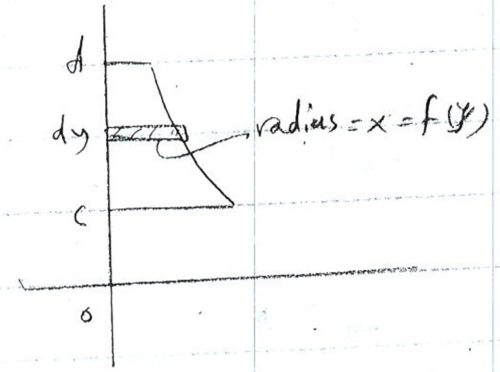


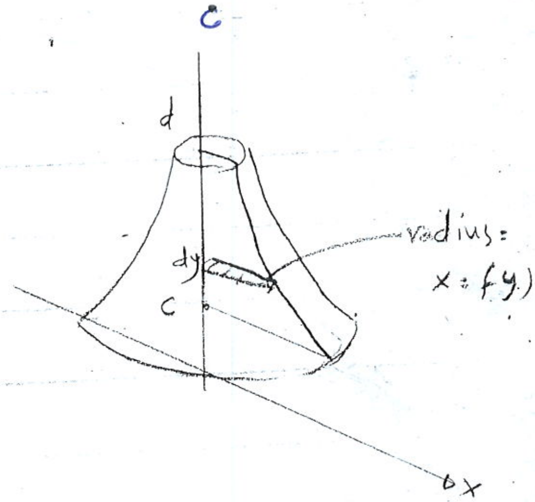


② rotation about y-axis:

$dv = \pi \cdot (\text{radius})^2 (\text{thickness}) = \pi x^2 dy$
$= \pi \cdot (f(y))^2 dy$

$\text{Volume} = \int dv = \int_c^d \pi (\text{radius})^2 dy = \int_c^d \pi (f(y))^2 dy$





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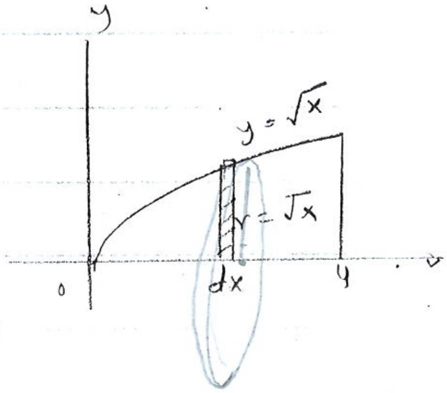
Example (1): The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution:



the volume of the disk is

$dv = \pi . (\text{radius})^2 (\text{thickness})$

$= \pi r^2 . t$

$r = y = f(x) = \sqrt{x}$ and $t = dx$

$\therefore dv = \pi (\sqrt{x})^2 dx = \pi x \, dx$

$\therefore v = \int dv = \int_0^4 \pi x \, dx = \pi \frac{x^2}{2} \Big|_0^4$

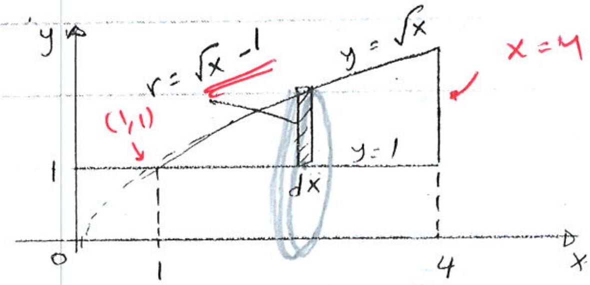
$= \frac{\pi}{2} [4^2 - 0^2] = \frac{16\pi}{2} = 8\pi$ cubic units

Example (2): Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$

Solution:

the volume of the disk is $dv = \pi (\text{radius})^2 (\text{thickness}) = \pi r^2 t$

$r = y - 1 = \sqrt{x} - 1$ and $t = dx$



$\therefore dv = \pi (\sqrt{x} - 1)^2 dx$

$v = \int dv = \int_1^4 \pi (\sqrt{x} - 1)^2 dx$

$= \int_1^4 \pi (x - 2\sqrt{x} + 1) dx$

$= \pi \left(\frac{x^2}{2} - \frac{2x^{3/2}}{3/2} + x \right) \Big|_1^4$

$= \pi \left[\left(\frac{4^2}{2} - \frac{4 * 4^{3/2}}{3} + 4 \right) - \left(\frac{1^2}{2} - \frac{4 * 1^{3/2}}{3} + 1 \right) \right]$

$= \frac{7\pi}{6}$ cubic units

$y = \sqrt{x}$; $y = 1$

$\sqrt{x} = 1$

$\therefore x = \pm 1$

$\therefore (1,1)$

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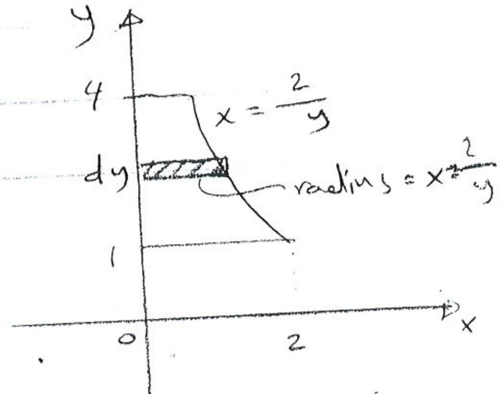
مهم ب جداً

★ Example ②: Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about y-axis

Solution:



$dv = \pi\,(\text{radius})^2\,(\text{thickness})$

$= \pi r^2 t$

$r = x = \frac{2}{y}$ and $t = dy$

$\therefore dv = \pi \left(\frac{2}{y}\right)^2 dy$

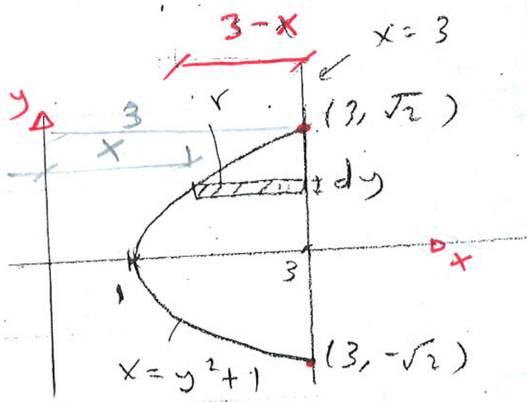
$= \frac{4\pi}{y^2}\, dy$

$\therefore v = \int dv = \int_1^4 \frac{4\pi}{y^2}\, dy = \left. -\frac{4\pi}{y} \right|_1^4$

$= 4\pi \left[-\frac{1}{4} - \left(-\frac{1}{1}\right) \right] = 4\pi \times \frac{3}{4} = 3\pi$ cubic units

Example ③: Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and line $x = 3$, about $x = 3$.

Solution:



$x = y^2 + 1$
$\Rightarrow y^2 = x - 1$
$y = \pm\sqrt{x-1}$
and $x = 3$
$\therefore y = \pm\sqrt{3-1}$
$y = \pm\sqrt{2}$
$\therefore$ the points
$(3, \sqrt{2})$
and $(3, -\sqrt{2})$

$dv = \pi\,(\text{radius})^2\,(\text{thickness})$

$= \pi r^2 t$

$r = \underline{3 - x} = 3 - (y^2 + 1) = \underline{2 - y^2}$

and $t = dy$

$\therefore dv = \pi\,(2 - y^2)^2\, dy$

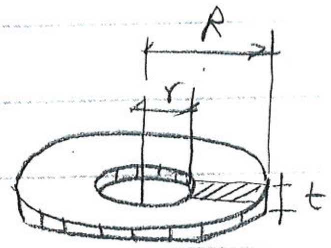
$\therefore v = \int dv = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\,(2 - y^2)^2\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \pi\,(4 - 4y^2 + y^4)\, dy$

or

$= 2\int_0^{\sqrt{2}} \pi\,(4 - 4y^2 + y^4)\, dy = 2\pi \left(4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right)\Big|_0^{\sqrt{2}}$

$= 2\pi \left[\left(4\sqrt{2} - \frac{4}{3}(\sqrt{2})^3 + \frac{(\sqrt{2})^5}{5}\right) - (0) \right] = \frac{64\pi\sqrt{2}}{15}$ c.u cubic unit

B. The washer method (the strip is perpendicular to the axis of revolution)

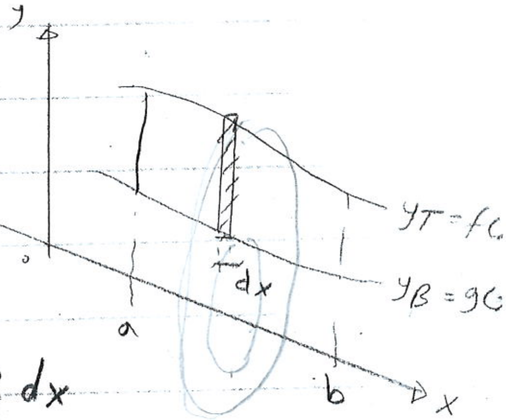


the washer's volume is

$$d\cancel{v} = \pi \{R^2 - r^2\} t$$

$\pi R^2 t - \pi r^2 t$
$= \pi \{R^2 - r^2\} t$

① Rotation about x-axis

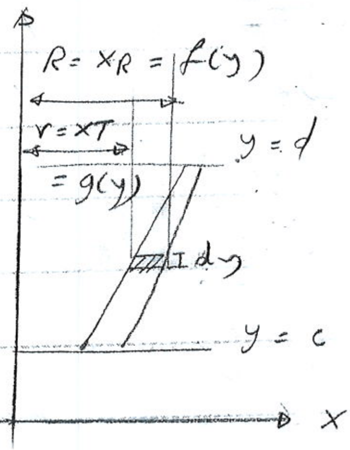


$$\cancel{v} = \int d\cancel{v} = \int_a^b \pi \{R^2 - r^2\} dx$$

$$= \int_a^b \pi \{(y_T)^2 - (y_B)^2\} dx$$

$$= \int_a^b \pi \{[f(x)]^2 - [g(x)]^2\} dx$$

② Rotation about y-axis



$$V = \int dv = \int_c^d \pi \{R^2 - r^2\} dy$$

$$= \int_c^d \pi \{(x_R)^2 - (x_T)^2\} dy$$

$$= \int_c^d \pi \{[f(y)]^2 - [g(y)]^2\} dy$$

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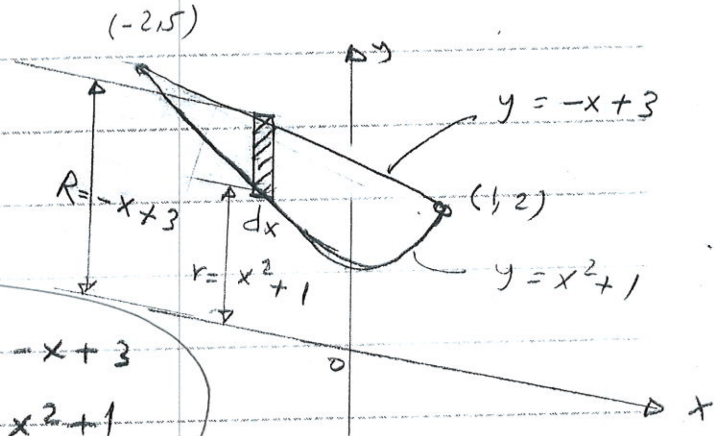
Example : The region bounded by the curve $y=x^2+1$ and the line $y=-x+3$ is revolved about the x-axis to generate a solid. Find its volume.



outer radius: $R = y_T = -x+3$

Inner radius: $r = y_B = x^2+1$

$t = dx$

$x^2+1 = -x+3$

$x^2+x-2=0 \Rightarrow (x+2)(x-1)=0$

$\therefore x=-2 \Rightarrow y=5$

or $x=1 \Rightarrow y=2$

$dv = \pi[R^2-r^2]t$

$\boxed{R = y_T = -x+3,\ r = y_B = x^2+1 \text{ and } t = dx}$

$\therefore dv = \pi[(-x+3)^2-(x^2+1)^2]dx$

$= \pi\{x^2-6x+9-(x^4+2x^2+1)\}dx$

$= \pi\{8-6x-x^2-x^4\}dx$

$\therefore v = \int dv = \int_{-2}^{1} \pi(8-6x-x^2-x^4)dx$

$= \pi\left(8x - \frac{6x^2}{2} - \frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-2}^{1}$

$= \frac{117\pi}{5}$ cubic units

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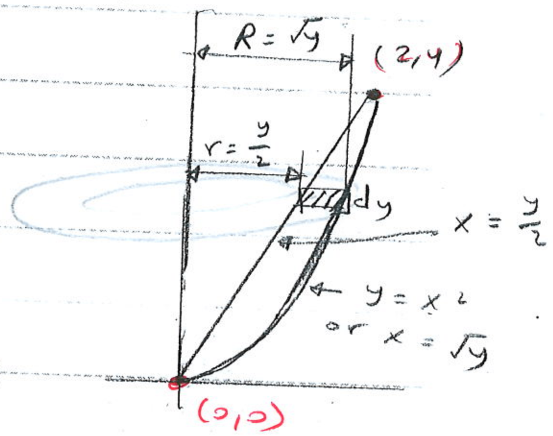
Example: The region bounded by the parabola $y = x^2$ and the line $y = 2x$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



$$\begin{cases} R = \sqrt{y} \\ r = \frac{y}{2} \\ t = dy \end{cases}$$

the points of intersection are $(2,4)$ and $(0,0)$

$$dv = \pi\,[R^2 - r^2]\,t = \pi\left[(\sqrt{y})^2 - (y/2)^2\right]dy = \pi\left[y - \frac{y^2}{4}\right]dy$$

$$\therefore\ v = \int dv = \int_0^4 \pi\left[y - \frac{y^2}{4}\right]dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{3*4}\right]\Bigg|_0^4$$

$$= \frac{8}{3}\pi \text{ cubic units.}$$

C. Volumes by cylindrical shells (the strip is parallel to the axis of revolution).

Example: The region bounded by the parabola $y = x^2$, the y-axis and the line $y = 1$ in the first quadrant is revolved about the line $x = 2$ to generate a solid. Find the volume of the solid by integrating with respect to x

$x^2 = 1$

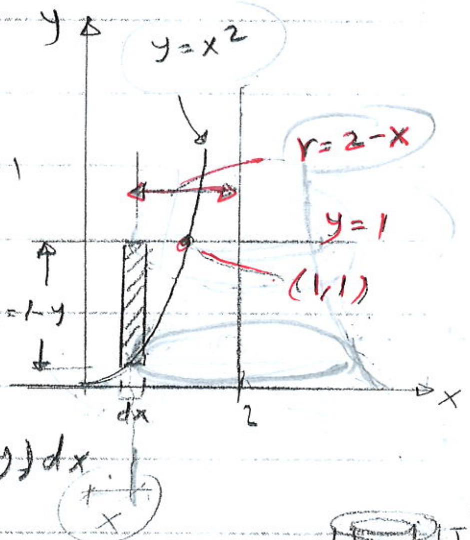
$\therefore x = \pm 1$

نقط التقاطع

$x = 1, \Rightarrow y = 1$

$\therefore (1,1)$

$(-1,1)$



$dv = 2\pi r \cdot l \cdot t$

where $l = 1 - y$, $r = 2 - x$

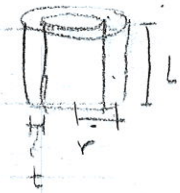
and $t = dx$

المحيط الارتفاع السمك

$dv = 2\pi r \cdot l \cdot t = 2\pi(2-x)(1-y)\,dx$

$= 2\pi(2-x)(1-x^2)\,dx$

مكرر $= 2\pi(2-x)(1-x^2)\,dx$



the limits of integration from

$x = 0$ to $x = 1$

$\therefore V = \int dv = \int_0^1 2\pi(2-x)(1-x^2)\,dx$

المحيط × الارتفاع × السمك يعطي حجم القشرة ثم نجمع القشرة لنعطي الحجم كامل

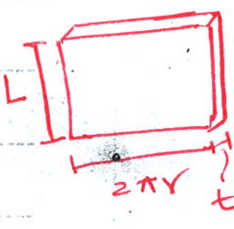
$= \int_0^1 2\pi(2 - 2x^2 - x + x^3)\,dx$

$= 2\pi\left(2x - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4}\right)\Big|_0^1$

$= 2\pi\left[\left(2*1 - \frac{2*1^3}{3} - \frac{1^2}{2} + \frac{1^4}{4}\right) - (0)\right]$

$= \frac{13}{6}\pi$ cubic units

نفتح الاسطوانة نشكل



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3- Volumes by slicing
صفائح ، شرائح

Example: A pyramid 3m high has a square base that is 3m on a side. The cross section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

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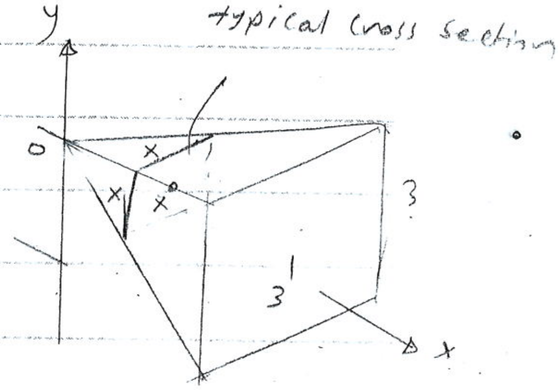
عمودي على الارتفاع



$d\forall = (\text{Area}) \cdot (\text{height})$

$= x \cdot x \cdot dx = x^2 \cdot dx$

$\forall = \int d\forall = \int_0^3 x^2 dx$

$= \left[\frac{x^3}{3} \right]_0^3 = 9$ cubic units

4- length of plane curves (arc length):

① if $y = f(x)$ is a smooth curve on the interval $[a,b]$

$$\therefore L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{—①}$$

② if $x = f(y)$ is a continuous from $y = c$ to $y = d$

$$\therefore L = \int_c^d \sqrt{1+[f'(y)]^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \quad \text{—②}$$

③ if the curve is represented by a parametric equations:

??t

$x = x(t)$, $y = y(t)$ and $a \leq x \leq b$ and if $\frac{dx}{dt}$, $\frac{dy}{dt}$ are continuous functions on

??t

$a \leq x \leq b$, then the arc-length of the curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \quad \text{—③}$$

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Example: Find the length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \; ; \quad 0 \leq x \leq 1$$

Solution:

$$\frac{dy}{dx} = \frac{3}{2} * \frac{4\sqrt{2}}{3} x^{1/2} = 2\sqrt{2}\, x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}\, x^{1/2}\right)^2 = 8x$$

$$\therefore L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1+8x}\, dx$$

$$= \frac{1}{8} \int_0^1 \sqrt{1+8x}\; 8dx = \frac{1}{8} \left. \frac{(1+8x)^{3/2}}{3/2} \right|_0^1$$

$$= \frac{1}{12} \left\{ (1+8*1)^{3/2} - (1+8*0)^{3/2} \right]$$

$$= \frac{13}{6} \text{ unit length.}$$

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Example: Find the length of the curve $y = \left(\frac{x}{2}\right)^{2/3}$ from $x=0$ to $x=2$

Solution:

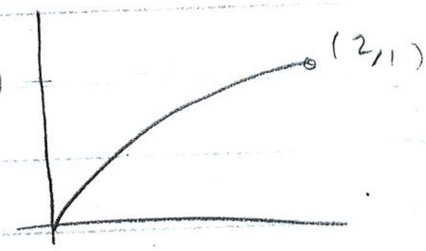
$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \times \frac{1}{2} = \frac{1}{3\sqrt[3]{2}}\,\frac{1}{\sqrt[3]{x}}$$

is not defined at $x=0$, So we can not find the curve's length with equation (1)

الرفع بالأس $\frac{3}{2}$

$$y = \left(\frac{x}{2}\right)^{2/3} \Rightarrow y^{3/2} = \frac{x}{2} \Rightarrow x = 2y^{3/2}$$

when $x=0 \Rightarrow y=0$; $x=2 \Rightarrow y=1$



$$\therefore \frac{dx}{dy} = 2 \times \frac{3}{2} y^{1/2} = 3y^{1/2}$$

$$\therefore L = \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$= \int_0^1 \sqrt{1+(3y^{1/2})^2}\,dy$$

$$= \int_0^1 \sqrt{1+9y}\,dy = \frac{1}{9}\,\frac{(1+9y)^{3/2}}{3/2}\Bigg|_0^1$$

$$= \frac{2}{27}\left[\left((1+9\times1)^{3/2}\right) - \left((1+9\times0)^{3/2}\right)\right]$$

$$= \frac{2}{27}\left(10\sqrt{10}-1\right) \approx 2.27 \text{ unit length.}$$

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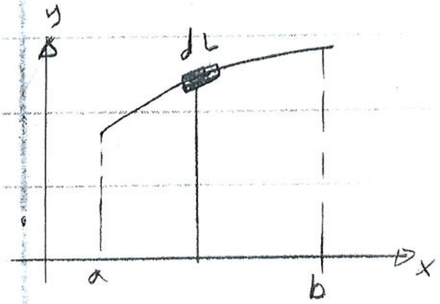
5- Area of surface of revolution



The surface area of typical is

$ds = \underset{\text{محيط}}{2\pi r} \cdot \underset{\text{الطول}}{dL}$

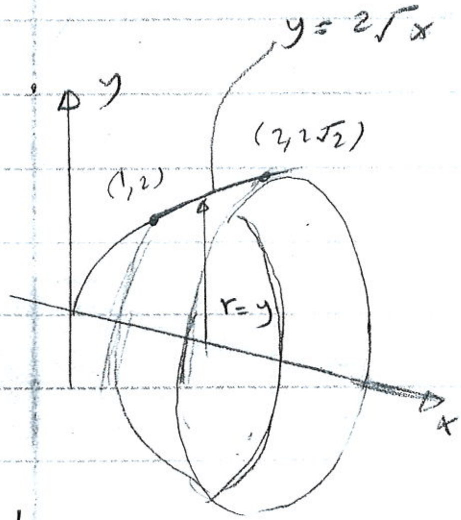
dL will be calculated from one of the following three relations:

① $dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

② $dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

③ $dL = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$

Example: Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$ about the x-axis.



Solution:

$$dS = 2\pi r\, dL$$

where $r = y = 2\sqrt{x}$

$2\pi r$ محيط الدائرة

$$dL = \sqrt{1 + [f'(x)]^2}\, dx$$

$$f'(x) = 2 \times \frac{1}{2} x^{-1/2} = \frac{1}{\sqrt{x}}$$

$$dS = 2\pi (2\sqrt{x}) \sqrt{1 + [f'(x)]^2}\, dx$$

$$= 4\pi \sqrt{x} \sqrt{1 + \left[\frac{1}{\sqrt{x}}\right]^2}\, dx = 4\pi \sqrt{x} \sqrt{1 + \frac{1}{x}}\, dx$$

$$= 4\pi \sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \sqrt{x+1}\, dx$$

$$\sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

$$\therefore S = \int dS = \int_1^2 4\pi \sqrt{x+1}\, dx = 4\pi \left.\frac{(x+1)^{3/2}}{3/2}\right|_1^2$$

$$= \frac{8\pi}{3}\left[(2+1)^{3/2} - (1+1)^{3/2}\right] \doteq 19.84 \text{ square units}$$

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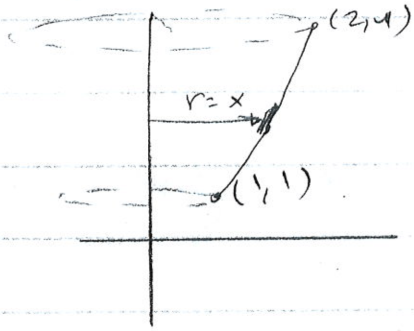
Example: Find the area of the surface generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis

solution:



$$dS = 2\pi r \cdot dL$$

where $r = x$

and $dL = \sqrt{1 + [f'(x)]^2}\, dx$

$y = x^2 \Rightarrow f'(x) = 2x$

$dL = \sqrt{1 + (2x)^2}\, dx = \sqrt{1 + 4x^2}\, dx$ ← مشتق ما داخل القوس = $8x$

$$\therefore S = \int dS = \int_1^2 2\pi x \sqrt{1 + 4x^2} \cdot dx = 2\pi \left[\frac{1}{8} \frac{(1 + 4x^2)^{3/2}}{3/2} \right]_1^2$$

$$= \frac{\pi}{6} (1 + 4x^2)^{3/2} \Big|_1^2 = \frac{\pi}{6} \left[(1 + 4 * 2^2)^{3/2} - (1 + 4 * 1^2)^{3/2} \right]$$

$$= \frac{\pi}{6} \left[17^{3/2} - 5^{3/2} \right] \simeq 30.85 \text{ square units.}$$

another solution: use $x = f(y)$

$$y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow \frac{dx}{dy} = \frac{1}{2\sqrt{y}} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \left(\frac{1}{2\sqrt{y}}\right)^2 = \frac{1}{4y}$$

$$\therefore dL = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{1}{4y}}\, dy$$

$$= \sqrt{\frac{4y + 1}{4y}}\, dy = \frac{\sqrt{4y + 1}}{2\sqrt{y}}\, dy$$

when $x = 1 \Rightarrow y = 1$ and when $x = 2 \Rightarrow y = 4$

$$\therefore S = \int dS = \int_1^4 2\pi x \frac{\sqrt{4y + 1}}{2\sqrt{y}}\, dy = \int_1^4 2\pi \cancel{\sqrt{y}} \frac{\sqrt{4y + 1}}{2\cancel{\sqrt{y}}}\, dy$$

$$= \int_1^4 \pi \sqrt{4y + 1}\, dy = \pi \frac{1}{4} \frac{(4y + 1)^{3/2}}{3/2} \Big|_1^4$$

$$= \frac{\pi}{3 * 2} \left[(4 * 4 + 1)^{3/2} - (4 * 1 + 1)^{3/2} \right]$$

$$= \frac{\pi}{6} \left[17^{3/2} - 5^{3/2} \right] \simeq 30.85 \text{ square units}$$

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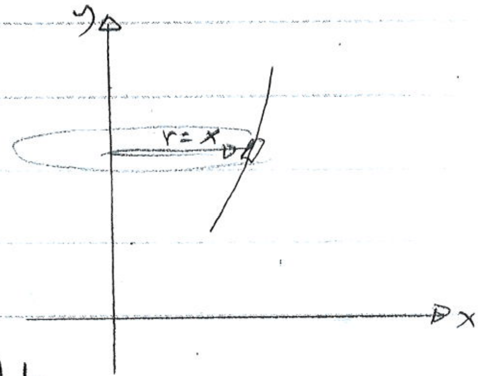
Example: Find the area of the surface generated by revolving the parametric curve $x = \cos^2 t$, $y = \sin^2 t$, $0 \le t \le \pi/2$ about y-axis.



Solution:

$$ds = 2\pi r\, dL$$

where $r = x = \cos^2 t$

and $dL = \sqrt{dx^2 + dy^2}$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$x = \cos^2 t \Rightarrow \frac{dx}{dt} = -2\cos t \sin t$$ $\Leftarrow 2\cos t \cdot (-\sin t)$ ... $(\cos t)^2$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 4\cos^2 t \sin^2 t$$

$$y = \sin^2 t \Rightarrow \frac{dy}{dt} = 2\sin t \cos t$$

$$\Rightarrow \left(\frac{dy}{dt}\right)^2 = 4\sin^2 t \cos^2 t$$

$$dL = \sqrt{8\sin^2 t \cos^2 t}\, dt = 2\sqrt{2}\sin t \cos t\, dt$$

$$\therefore S = \int ds = \int_0^{\pi/2} 2\pi \cos^2 t\,(2\sqrt{2}\sin t \cos t)\,dt$$

مشتقة داخل القوس وهو cos موجودة وهي sin

$$= \int_0^{\pi/2} 4\sqrt{2}\,\pi \cos^3 t \sin t\, dt$$

مشتقة cos هي -sin

$$= -4\sqrt{2}\,\pi \left.\frac{\cos^4 t}{4}\right|_0^{\pi/2} = -\sqrt{2}\,\pi\left[\cos^4\frac{\pi}{2} - \cos^4 0\right]$$

$$= -\sqrt{2}\pi\,[0 - 1] = \sqrt{2}\,\pi \text{ square units}$$

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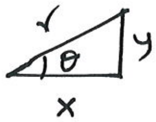
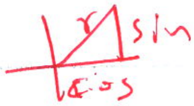
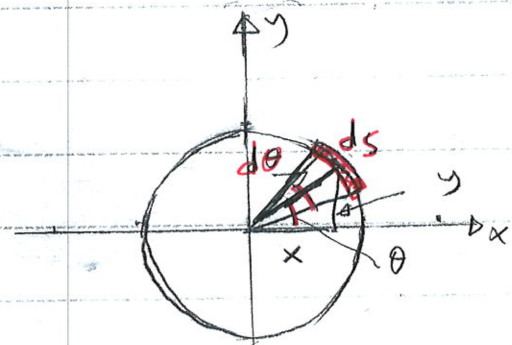


$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$
$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$



Example: The circle
$$x^2 + y^2 = r^2$$
is revolved about the x-axis. Find the area of the sphere generated.



Solution:

$$ds = 2\pi r\, dL$$

$r = y$

$dL = \sqrt{dx^2 + dy^2}$

أشتقنا لـ θ
r ثابت

$x = r\cos\theta$, $y = r\sin\theta$

اشتقاق $\frac{dx}{d\theta}$

$dx = -r\sin\theta\, d\theta$, $dy = r\cos\theta\, d\theta$

$\therefore dL = r\, d\theta$

$$dL = \sqrt{dx^2 + dy^2} = \sqrt{(-r\sin\theta\, d\theta)^2 + (r\cos\theta\, d\theta)^2} = \sqrt{r^2\sin^2\theta\, d^2\theta + r^2\cos^2\theta\, d^2\theta} = \sqrt{r^2 d^2\theta(\sin^2 + \cos^2)} = \sqrt{r^2 d^2\theta}$$

$\therefore dL = r\, d\theta$

$ds = 2\pi y\, dL = 2\pi(r\sin\theta)\, r\, d\theta$

$ds = 2\pi r^2 \sin\theta\, d\theta$

$$\therefore S = \int_0^{\pi} 2\pi r^2 \sin\theta\, d\theta$$

$$= 2\pi r^2 \left[-\cos\theta\right]_0^{\pi} = 4\pi r^2$$

$-[\cos\pi - \sin 0]$
$-[-1 - 1]$
$-[-2] = +2$

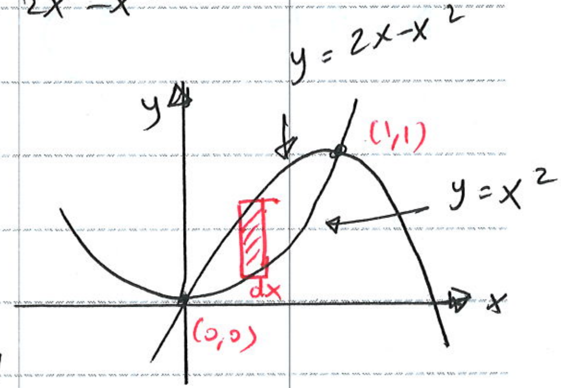
ع 7/4 2016
ع 6/5 2016

س ٤١

من المحاضرات

Q1: Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$

<u>solution</u>:



$x^2 = 2x - x^2 \Rightarrow x^2 + x^2 - 2x = 0$

$\Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$

either $2x = 0 \Rightarrow x = 0 \Rightarrow y = 0$

or $x - 1 = 0 \Rightarrow x = 1 \Rightarrow y = 1$

the points of intersection are $(0,0)$ and $(1,1)$

the area of a typical rectangle is

$dA = (2x - x^2) - (x^2) = 2x - x^2 - x^2 = 2x - 2x^2$

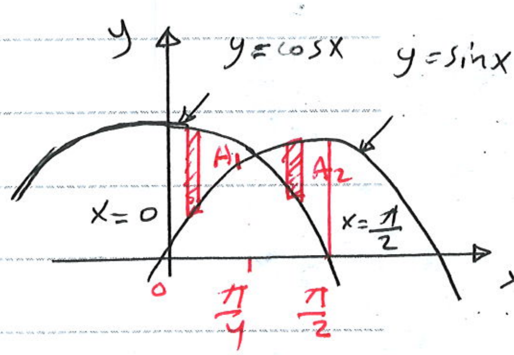
and the region lies between $x = 0$ and $x = 1$

$A = \int dA = \int_0^1 (2x - 2x^2)\, dx = \frac{2x^2}{2} - \frac{2x^3}{3} \Big|_0^1$

$= [1^2 - \frac{2(1)^3}{3}] - 0 = \frac{1}{3}$ square units.

من بازرات **Q2:** Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.

<u>solution</u>:



$\sin x = \cos x$, when $x = \frac{\pi}{4}$

observe that $\cos x \geq \sin x$

when $0 \leq x \leq \frac{\pi}{4}$ but $\sin x \geq \cos x$

when $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

$A = \int_0^{\frac{\pi}{2}} [\cos x - \sin x]\, dx = A_1 + A_2$

$= \int_0^{\frac{\pi}{4}} \{\cos x - \sin x\} dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \{\sin x - \cos x\} dx$

$= \{\sin x + \cos x\}_0^{\pi/4} + \{-\cos x - \sin x\}_{\pi/4}^{\pi/2}$

$= (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1) + (-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}})$

$= 2\sqrt{2} - 2$.

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \frac{\pi}{4}$ and so.

$$A = 2A_1 = 2\int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx.$$

Calculus p433

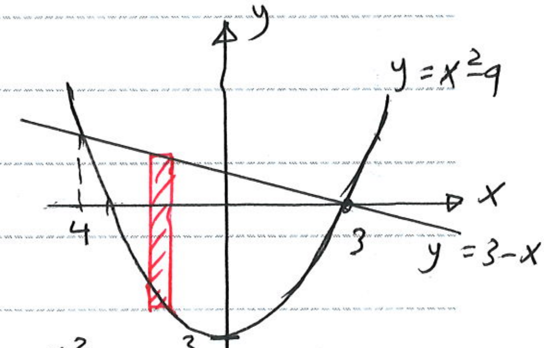
Q3: Find the area bounded by the graphs of $y = 3 - x$ and $y = x^2 - 9$

<u>Solution</u>:

$3 - x = x^2 - 9$ or $x^2 + x - 12 = 0$

$(x-3)(x+4) = 0$

$x = -4$ and $x = 3$



$$A = \int_{-4}^{3} [(3-x) - (x^2 - 9)]\,dx$$

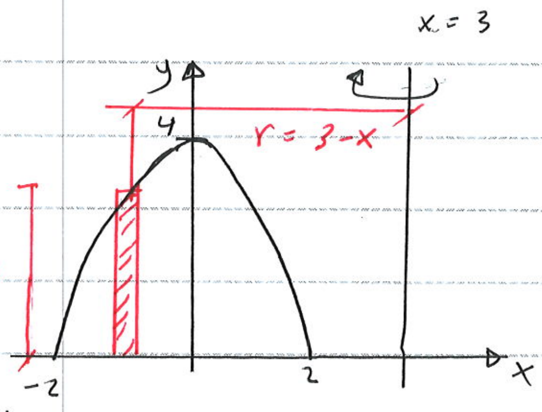
$$= \int_{-4}^{3} (-x^2 - x + 12) = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 12x \right]_{-4}^{3}$$

$$= \left[-\frac{3^3}{3} - \frac{3^2}{2} + 12(3) \right] - \left[-\frac{(-4)^3}{3} - \frac{(-4)^2}{2} + 12(-4) \right]$$

$$= \frac{343}{6}.$$

calculus
p 459 Q2: Find the volume of the solid formed by revolving the region bounded by the graph of $y = 4 - x^2$ and the x-axis about the line $x = 3$.

Solution:



$r = 3 - x$

$$V = \int_{-2}^{2} 2\pi \underbrace{(3-x)}_{r}\underbrace{(4-x^2)}_{L}\underbrace{dx}_{t}$$

$$= 2\pi \int_{-2}^{2} (x^3 - 3x^2 - 4x + 12)\,dx$$

$$= 64\pi.$$

Calculus P449

H.W Volume

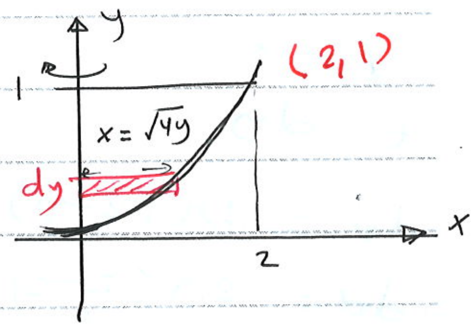
CP : let R be the region bounded by the graphs of $y = \frac{1}{4}x^2$, $x = 0$ and $y = 1$. compute the volume of the solid formed by revolving R about (a) the y-axis (b) the x-axis and (c) the line $y = 2$.

solution :

(a)

$$V = \int_0^1 \underbrace{\pi(\sqrt{4y})^2}_{\pi r^2} dy$$

$$= \pi \frac{4}{2} y^2 \Big|_0^1 = 2\pi$$



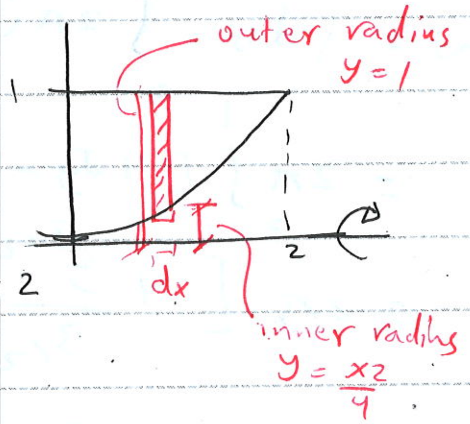
(b)

$$V = \int_0^2 \underbrace{\pi(1)^2}_{\pi R^2} dx - \int_0^2 \underbrace{\pi\left(\frac{1}{4}x^2\right)^2}_{\pi r^2} dx$$

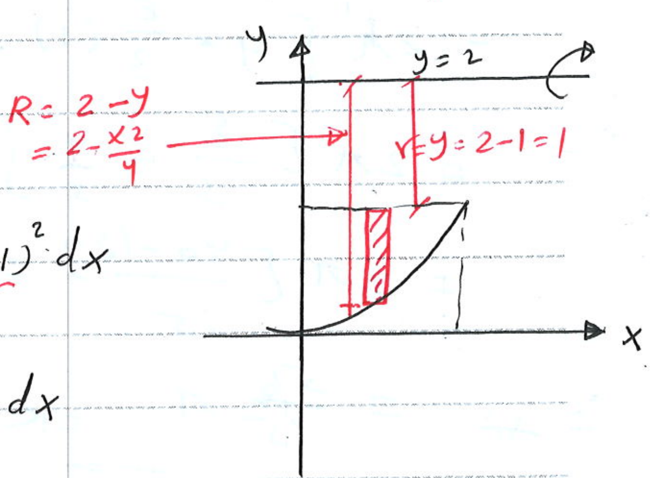
$$= \int_0^2 \left(1 - \frac{x^4}{16}\right) dx = \pi\left(x - \frac{1}{80}x^5\right)\Big|_0^2$$

$$= \pi\left(2 - \frac{32}{80}\right) = \frac{8}{5}\pi.$$

$$= 1.6\pi$$



(c)

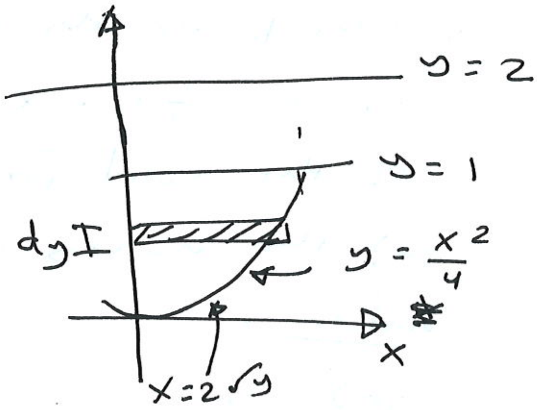


$$V = \int_0^2 \underbrace{\pi\left(2 - \frac{1}{4}x^2\right)^2}_{\pi R^2} dx - \int_0^2 \underbrace{\pi(2-1)^2}_{\pi r^2} dx$$

$$= \pi \int_0^2 \left[\left(4 - x^2 + \frac{x^4}{16}\right) - 1\right] dx$$

$$= \pi\left[3x - \frac{1}{3}x^3 + \frac{1}{80}x^5\right]_0^2 = \pi\left(6 - \frac{8}{3} + \frac{32}{80}\right) = \frac{56}{15}\pi$$

by shell method



$d\forall = 2\pi r l t$

$r = 2 - y$

$l = x$

$t = dy$

$d\forall = 2\pi(2-y)\cdot x \cdot dy$

$\Rightarrow d\forall = 2\pi(2-y)(2\sqrt{y})\,dy$

$\forall = \int d\forall = \int_0^1 2\pi(2-y)(2\sqrt{y})\,dy$

$= 2\pi\int_0^1 [4\sqrt{y} - 2y^{3/2}]\,dy$

$= 2\pi\left[4\frac{y^{3/2}}{3/2} - 2\frac{y^{5/2}}{5/2}\right]_0^1$

$= 2\pi\left[4 * \frac{2}{3}(1) - 2 * \frac{2}{5}(1)\right]$

$= 2\pi\left[\frac{8}{3} - \frac{4}{5}\right]$

$= 2\pi\left[\frac{40-12}{15}\right] = 2\pi\,\frac{28}{15}$

$\Rightarrow \forall = \frac{56}{15}\pi$ ok

$\cong 3.733\pi$

$y = \frac{x^2}{4}$

$x^2 = 4y$

$\cancel{\Rightarrow x = \pm\sqrt{y}}$

$\Rightarrow x = \pm 2\sqrt{y}$

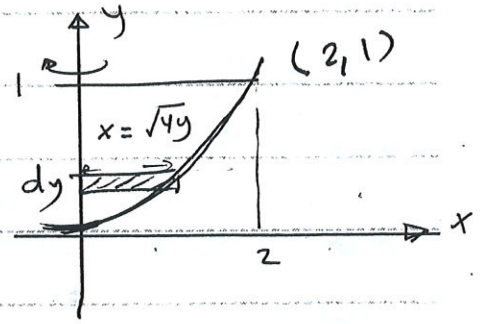
Calculus H.W Volume

P449

CP: let R be the region bounded by the graphs of $y=\frac{1}{4}x^2$, $x=0$ and $y=1$. Compute the volume of the solid formed by revolving R about (a) the y-axis (b) the x-axis and (c) the line $y=2$.

solution:

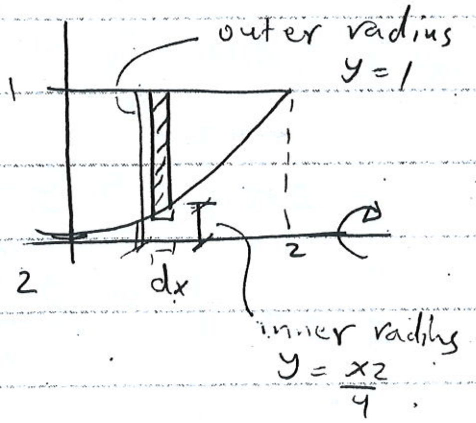
(a)



$$V=\int_0^1 \underbrace{\pi(\sqrt{4y})^2}_{\pi r^2}\,dy$$

$$=\pi\frac{4}{2}y^2\Big|_0^1 = 2\pi$$

(b)



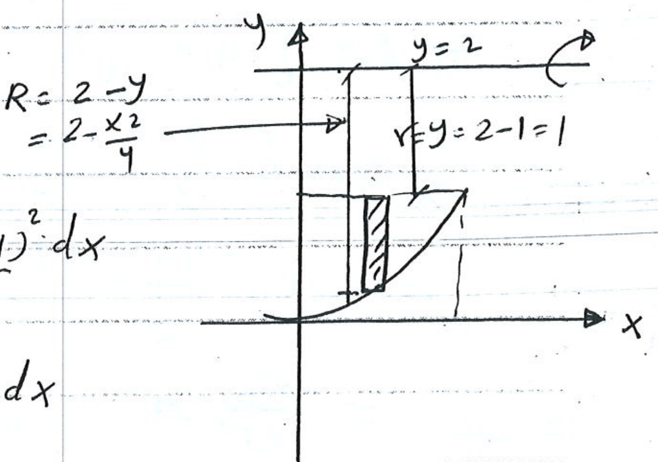
$$V=\int_0^2 \underbrace{\pi(1)^2}_{\pi R^2}dx-\int_0^2\underbrace{\pi\left(\frac{1}{4}x^2\right)^2}_{\pi r^2}dx$$

$$=\int_0^2\left(1-\frac{x^4}{16}\right)dx=\pi\left(x-\frac{1}{80}x^5\right)\Big|_0^2$$

$$=\pi\left(2-\frac{32}{80}\right)=\frac{8}{5}\pi$$

$$=1.6\pi$$

(c)

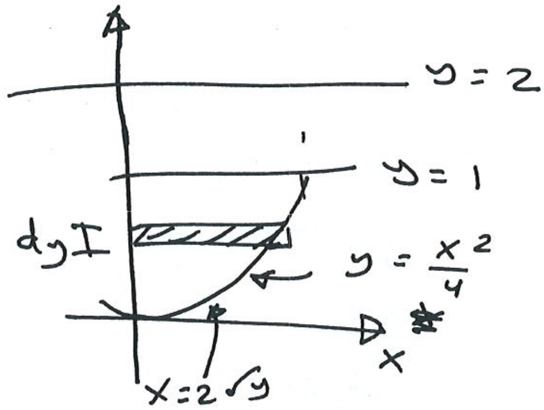


$$V=\int_0^2\underbrace{\pi\left(2-\frac{1}{4}x^2\right)^2}_{\pi R^2}dx-\int_0^2\underbrace{\pi(2-1)^2}_{\pi r^2}dx$$

$$=\pi\int_0^2\left[\left(4-x^2+\frac{x^4}{16}\right)-1\right]dx$$

$$=\pi\left[3x-\frac{1}{3}x^3+\frac{1}{80}x^5\right]_0^2=\pi\left(6-\frac{8}{3}+\frac{32}{80}\right)=\frac{56}{15}\pi$$

by shell method



$d\forall = 2\pi r l t$

$r = 2 - y$

$l = x$

$t = dy$

$d\forall = 2\pi(2-y) \cdot x \cdot dy$

$\Rightarrow d\forall = 2\pi(2-y)(2\sqrt{y})\,dy$

$\forall = \int d\forall = \int_0^1 2\pi(2-y)(2\sqrt{y})\,dy$

$= 2\pi \int_0^1 [4\sqrt{y} - 2y^{3/2}]\,dy$

$= 2\pi \left[4\frac{y^{3/2}}{3/2} - 2\frac{y^{5/2}}{5/2} \right]_0^1$

$= 2\pi \left[4 * \frac{2}{3}(1) - 2 * \frac{2}{5}(1) \right]$

$= 2\pi \left[\frac{8}{3} - \frac{4}{5} \right]$

$= 2\pi \left[\frac{40-12}{15} \right] = 2\pi \frac{28}{15}$

$\Rightarrow \forall = \frac{56}{15}\pi$ ok

$\cong 3.733\pi$

$y = \frac{x^2}{4}$

$x^2 = 4y$

$\cancel{\Rightarrow x = \pm\sqrt{y}}$

$\Rightarrow x = \pm 2\sqrt{y}$

10/4/2014

9.

Method of Integrations (Techniques of Integration)

(1) Integration by substitution

<u>Examples</u> :

① $\int (x+2)^5\,dx$, let $u = x+2$, $\frac{du}{dx} = 1 \Rightarrow du = dx$

$\therefore \int (x+2)^5\,dx = \int u^5\,du = \frac{u^6}{6} + c = \frac{(x+2)^6}{6} + c$

② $\int \sqrt{4x-1}\,dx$, let $u = 4x-1$, $du = 4dx \Rightarrow dx = \frac{1}{4}d$

$\therefore \int \sqrt{4x-1}\,dx = \int \sqrt{u} \cdot \frac{1}{4}\,du = \frac{1}{4}\int u^{\frac{1}{2}}du$

$= \frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right) + c = \frac{1}{6}u^{3/2} + c = \frac{1}{6}(4x-1)^{3/2} +$

P 2.

③ $\int x^2 \sin x^3\,dx$, let $u = x^3$, $du = 3x^2\,dx$

$\Rightarrow x^2\,dx = \frac{du}{3}$

$\therefore \int x^2 \cdot \sin x^3\,dx = \int \sin u \cdot \frac{du}{3}$

$= \frac{1}{3}\int \sin u \cdot du = \frac{1}{3}(-\cos u) + c = -\frac{1}{3}\cos x^3 + c$

(4) $\int \frac{2z}{\sqrt[3]{z^2+1}}\,dz$, let $u = z^2+1$, $\frac{du}{dz} = 2z$

$\Rightarrow dz = \frac{du}{2z}$

$\therefore \int \frac{2z}{\sqrt[3]{z^2+1}}\,dz = \int \frac{\cancel{2z}}{\sqrt[3]{u}} \times \frac{du}{\cancel{2z}} \Rightarrow \int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du$

$= \frac{u^{2/3}}{2/3} + C = \frac{3}{2} u^{2/3} + C = \frac{3}{2}(z^2+1)^{2/3} + C$

(5) $\int \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}\,dx$, let $u = \sqrt{x+1}$, $u^2 = x+1$

$2u\,du = dx$

$\therefore \int \frac{\sin u}{\cancel{u}}\, 2\cancel{u}\,du = -2\cos u + C$

$= -2\cos\sqrt{x+1} + C$

(6) $\int \frac{3x^2-7x}{3x+2}\,dx$ (degree of numenator greater than or equal to degree of denominator)

$\therefore \int \frac{3x^2-7x}{3x+2}\,dx = \int \left(x-3+\frac{6}{3x+2}\right)dx$

$$\begin{array}{r l} & x-3 \\ 3x+2 & \overline{)\,3x^2-7x} \\ & \mp 3x^2 \mp 2x \\ \hline & -9x \\ & \pm 9x \pm 6 \\ \hline & 6 \end{array}$$

$= \int (x-3)\,dx + \int \frac{6}{3x+2}\,dx$

$= \frac{x^2}{2} - 3x + 2\ln|3x+2| + C$

3dx

$= \frac{1}{3}\int \frac{6(3dx)}{3x+2}$ $\quad \frac{3dx}{3}$

$2\int \frac{3dx}{3x+2} = 2\ln(3x+2)$

2

P.

⑦ 6

$$\int x \cdot \sqrt[3]{1-x}\, dx \qquad \text{let } u = \sqrt[3]{1-x}, \quad u^3 = 1-x$$

$$\Rightarrow \boxed{x = 1-u^3}$$

$$dx = -3u^2\, du$$

$$\therefore \int x \cdot \sqrt[3]{1-x}\, dx = \int \overset{x}{(1-u^3)} \cdot \overset{\sqrt[3]{1-x}}{u} \cdot \overset{dx}{(-3u^2)\, du}$$

$$= \int (u-u^4)(-3u^2)\, du$$

$$= 3\int (u^6-u^3)\, du$$

$$= 3\left[\frac{u^7}{7} - \frac{u^4}{4}\right] + c$$

$$= \frac{3}{7}u^7 - \frac{3}{4}u^4 + c$$

$$= \frac{3}{7}\left(\sqrt[3]{1-x}\right)^7 - \frac{3}{4}\left(\sqrt[3]{1-x}\right)^4 + c$$

⑧ 7

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx$$

$$\text{let } u = 1-x^2$$

$$du = -2x\, dx$$

$$x\, dx = -\tfrac{1}{2}\, du$$

$$= \int \frac{3x}{\sqrt{1-x^2}}\, dx + \int \frac{2}{\sqrt{1-x^2}}\, dx$$

$$\therefore \int \frac{3x}{\sqrt{1-x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\, du}{\sqrt{u}}$$

$$= -\frac{3}{2}\int u^{-1/2}\, du = \frac{-3}{2}\,\frac{u^{1/2}}{1/2} + c_1 = -3\sqrt{1-x^2} + c$$

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1}x + c_2$$

$$\therefore \int \frac{3x+2}{\sqrt{1-x^2}}\, dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + c$$

(8) $\int \sec x \, dx = \int (\sec x)(1) \, dx$

$$= \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} \, dx$$

$u = \tan x + \sec x$

$du = (\sec^2 x + \sec x \tan x) \, dx$

$$= \int \frac{du}{u} = \ln|u| + c$$

$$= \ln|\sec x + \tan x| + c$$

$\int \csc x \, dx$

$$= \int \csc x \, \frac{-\csc x + \cot x}{-\csc x + \cot x}$$

$$= \int \frac{-\csc^2 x + \csc x \cdot \cot x}{-\csc x + \cot x}$$

$u = \cot x - \csc x$

$du = -\csc^2 - (-\csc x \cdot \cot x)$

$\therefore du = -\csc^2 x + \csc x \cdot \cot x.$

$\therefore \int \frac{du}{u} = \ln|u|$

$\therefore \quad = \ln|\cot - \csc|$

(2) integration by parts

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$d(uv) = u\,dv + v\,du$$

$$u\,dv = d(uv) - v\,du$$

$$\therefore \int u\,dv = uv - \int v\,du$$

Example : evaluate the following :

(1) $\int x \cos x \, dx$

let $u = x$ and $dv = \cos x\,dx$

$\therefore du = dx$ and $v = \sin x$

$$\therefore \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

(2) $\int x^3 e^{x^2} dx$

let $u = x^2$ $\quad dv = x e^{x^2} dx$

$du = 2x\,dx$ $\quad v = \frac{1}{2} e^{x^2}$

$$\therefore \int x^3 e^{x^2} dx = \frac{1}{2} x^2 e^{x^2} - \frac{1}{2}\int x e^{x^2} \cdot 2x\,dx$$

$$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + c$$

لو فرضنا $u = x^3$
كنا $dv = e^{x^2} dx$
وبهذه الحالة لا نستطيع أن نكامل dv لأن مشتقة e^{x^2} غير موجودة

مشتقة e^{x^2}
$2x e^{x^2} dx$
نضرب ونقسم على 2

$\frac{1}{2}\int x e^{x^2}(2dx)$

$= \frac{1}{2} e^{x^2}$

ن. 7/4 2016

6/5/2015
P

③ $\int \ln x \, dx$

let $u = \ln x \qquad dv = dx$

$du = \dfrac{dx}{x} \longrightarrow v = x$

$$\therefore \int \ln x \, dx = x\ln x - \int \frac{\cancel{x}\, dx}{\cancel{x}} = x\ln x - \int dx$$

$$= x\ln x - x + c$$

④ $\int x^2 e^{-x} dx$

let $u = x^2 \qquad dv = e^{-x} dx$

$du = 2x\,dx \longrightarrow v = -e^{-x}$

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x}(2x\,dx)$$

$$\therefore \int x^2 e^{-x} dx = -x^2 e^{-x} + 2\int x e^{-x} dx$$

another integration by Parts applied to $\int xe^{-x} dx$ will complete the problem

$\int x e^{-x} dx$

let $u = x \qquad dv = e^{-x} dx$

$du = dx \longrightarrow v = -e^{-x}$

$$\int x^2 e^{-x} dx = \boxed{-x^2 e^{-x} + 2}\left[-xe^{-x} - \int (-e^{-x})\, dx\right]$$

$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -e^{-x}(x^2 + 2x + 2) + C$$

مشتقة e^{-x}

$e^{-x}(-1)\,dx$

نضرب ونقسم -1

⑤ $\int e^x \cos x \, dx$

let $u = e^x$ $\qquad dv = \cos x \, dx$

$du = e^x dx \qquad v = \sin x$

$\therefore \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

again use integration by parts to integrate $\int e^x \sin x \, dx$

let $u = e^x$ $\qquad dv = \sin x \, dx$

$du = e^x \, dx \qquad v = -\cos x$

$\int e^x \cos x \, dx = e^x \sin x - \left[-e^x \cos x - \int (-\cos x) e^x dx \right]$

$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$

$\therefore \int e^x \cos x \, dx = \frac{e^x}{2} (\sin x + \cos x) + C$

$\int e^x \cos x\,dx$

$u = \cos x \qquad dv = e^x\,dx$

$du = -\sin x\,dx \qquad v = e^x$

$\int e^x \cos x\,dx = e^x \cos x - \int e^x(-\sin x)\,dx$

$\int e^x \cos x\,dx = e^x \cos x + \underline{\int e^x \sin x\,dx}$ another integrale.

$\int e^x \sin x\,dx$

$u = \sin x \qquad dv = e^x\,dx$

$du = \cos x\,dx \qquad v = e^x$

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$

$\therefore \int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$

$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$

10/5/2015

9.

دكتور [illegible]

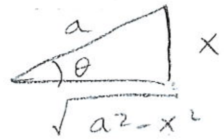
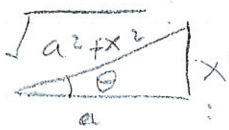
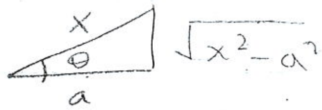
13/4

9.

③ trigonometric substitutions.

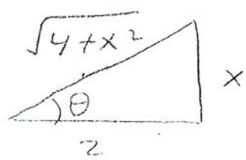
we will concern with integrals contain expressions of the form:

$a^2 + x^2$, $a^2 - x^2$, and $x^2 - a^2$

Expression in the integrand	Reference triangle
$a^2 - x^2$	
$a^2 + x^2$	
$x^2 - a^2$	

Example : evalute the following

① $\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{dx}{\sqrt{2^2+x^2}}$



let $x = 2\tan\theta \Rightarrow dx = 2\sec^2 d\theta$

$\cos\theta = \frac{2}{\sqrt{4+x^2}} \Rightarrow \sqrt{4+x^2} = \frac{2}{\cos\theta} = 2\sec\theta$

??

$\therefore \int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$

$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c = \ln|(\sqrt{4+x^2} + x)| - \ln 2 + c$

$= \ln|(\sqrt{4+x^2} + x)| + c_1$ where $c_1 = c - \ln 2$

كلو مقسوم ع 2

$\ln\left|\frac{\sqrt{4+x^2}+x}{2}\right| + c$

8

$$\int \sec\theta \, d\theta = \int \sec\theta \, \frac{\sec + \tan}{\sec + \tan} \, d\theta$$

$$= \int \frac{\sec^2 + \sec \tan}{\sec + \tan} \, d\theta$$

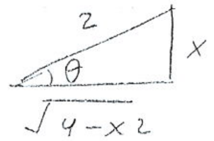
$$= \ln|\sec\theta + \tan\theta|$$

10/5/2015

② $\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

$\sin\theta = \frac{x}{2} \Rightarrow x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$



$\cos\theta = \frac{\sqrt{4-x^2}}{2} \Rightarrow \sqrt{4-x^2} = 2\cos\theta$

$\cot = \frac{1}{\tan}$

$\tan = \frac{x}{\sqrt{4-x^2}}$

$\therefore \cot = \frac{\sqrt{4-x^2}}{x}$

$\therefore \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \int_1^{\sqrt{2}} \frac{2\cos\theta\, d\theta}{4\sin^2\theta * 2\cos\theta}$

$= \frac{1}{4}\int_1^{\sqrt{2}} \frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int_1^{\sqrt{2}} \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta \Big|_1^{\sqrt{2}} \cancel{+c}$

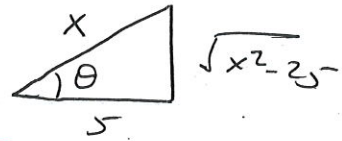
$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} \Big|_1^{\sqrt{2}} \cancel{+c}$

$\therefore \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{-1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = \frac{-1}{4}\left[\frac{\sqrt{4-2}}{\sqrt{2}} - \frac{\sqrt{4-1}}{1}\right] = \frac{\sqrt{3}-1}{4}$

③ $\int \frac{dx}{\sqrt{x^2-25}}$

let $x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\, d\theta$



* and $x^2 - 25 = 25\sec^2\theta - 25$

$= 25(\sec^2\theta - 1) = 25\tan^2\theta$

$= \int \frac{5\sec\theta\tan\theta\, d\theta}{\sqrt{25\tan^2\theta}} = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + c$

$= \ln\left|\frac{x}{5} + \frac{\sqrt{x^2-25}}{5}\right| + c$

$\cos\theta = \frac{5}{x}$

$\therefore x = \frac{5}{\cos\theta}$

$\therefore x = 5\sec\theta$

$* \tan\theta = \frac{\sqrt{x^2-25}}{5}$

$\sqrt{x^2-25} = 5\tan\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sec\tan d\theta}{5\tan\theta}$

$= \int \sec\theta\, d\theta$

الدوال النسبية

(4) Integrating Rational Functions

القسمة الطويلة

A) by long division: For improper fraction (the degree of the numerator is ~~simplify the integral.~~ equal to or greater than the degree of the denominator. Use the long division to simplify the integral.

1. $\int \frac{5x+2}{x+1}\,dx$

$$\begin{array}{r|l} & 5 \\ \hline x+1 & 5x+2 \\ & \mp 5x \mp 5 \\ \hline & -3 \end{array}$$

solution : $\int \frac{5x+2}{x+1}\,dx$

$\cancel{= \int (5}$

$$= \int \left[5 - \frac{3}{x+1}\right] dx = 5x - 3\ln|x+1| + c$$

2. $\int \frac{x^2+2x+2}{x+2}\,dx$

$$\begin{array}{r|l} & x \\ \hline x+2 & x^2+2x+2 \\ & \mp x^2 \mp 2x \\ \hline & 2 \end{array}$$

solution : $\int \frac{x^2+2x+2}{x+2}\,dx$

$$= \int \left[x + \frac{2}{x+2}\right] dx$$

$$= \frac{x^2}{2} + 2\ln|x+2| + c$$

R.

B) By partial Fractions :

1. Two linear factors in the denominator

Example: $\int \frac{x+4}{x^2+5x-6}\,dx$

solution : $\int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{x+4}{(x+6)(x-1)}\,dx$

$$\frac{x+4}{(x+6)(x-1)} = \frac{A}{x+6} + \frac{B}{x-1}$$

$$\frac{x+4}{\cancel{(x+6)(x-1)}} = \frac{A(x-1)+B(x+6)}{\cancel{(x+6)(x-1)}}$$

$$x+4 = A(x-1) + B(x+6)$$

at $x=-6 \Rightarrow A = \frac{2}{7}$

at $x=1 \Rightarrow B = \frac{5}{7}$

عوّض
اما ان نحل او
نأخذ القيم التي تجعل المقام = صفر

$$\therefore \int \frac{x+4}{x^2+5x-6}\,dx = \int \frac{2/7\,dx}{x+6} + \int \frac{5/7\,dx}{x-1}$$

$$= \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + c$$

$x+4 = Ax - A + Bx + 6B$

$x+4 = x(A+B) + (-A+6B)$

$\therefore A+B = 1$

$\Rightarrow A = 1-B \rightarrow (1)$

$-A+6B = 4 \rightarrow (2)$

$-(1-B)+6B = 4$

$-1+B+6B = 4$

and then $A = 1 - \frac{5}{7}$

$\therefore A = \frac{7-5}{7} = \frac{2}{7}$

17/4/2014 Q.4 , 188

مثال في ... ص $\frac{x+1}{x^2(x-1)}$
هذا يعني ان x^2 يعتبر ... فهي متكررة

2: A repeated linear factors in the denominator

Example: $\int \frac{x^2\, dx}{(x-1)(x^2+2x+1)}$ $\quad x^2+2x+1$

$$\Rightarrow \int \frac{x^2\, dx}{(x-1)(x^2+2x+1)} = \int \frac{x^2}{(x-1)(x+1)^2}\, dx$$

قوس مكرر

$$\therefore \frac{x^2}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

at $x = 1 \Rightarrow A = \frac{1}{4}$

at $x = -1 \Rightarrow C = \frac{-1}{2}$

at $x = 0 \Rightarrow B = \frac{3}{4}$

$x = 2$

$4 = \frac{1}{4} \times (2+1)^2 + 3B + \frac{-1}{2}(2-1)$

$$\therefore \int \frac{x^2\, dx}{(x-1)(x^2+2x+1)} = \int \frac{1/4\, dx}{x-1} + \int \frac{3/4\, dx}{x+1} - \int \frac{1/2\, dx}{(x+1)^2}$$

$$= \frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2}\left(\frac{1}{x+1}\right) + C$$

$4 = \frac{1}{4} \times 9 + 3B - \frac{1}{2}$

$2.25 = 3B$

$\therefore B = 0.75$

$\int \frac{1}{(x+1)^2}\, dx$

$= \int (x+1)^{-2}\, dx = \frac{(x+1)^{-1}}{-1}$

$= -\frac{1}{x+1}$

4. P

كسر غير فعلي

3. An improper fraction (degree of the numerator greater than or equal to the degree of denominator.

Example: $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\,dx = \int \left(2x + \frac{5x-3}{x^2-2x-3}\right)dx$

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & \mp 2x^3 \pm 4x^2 \pm 6x \\ \hline & 5x-3 \end{array}$$

$= \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx$

$= \int 2x\,dx + \int \frac{5x-3}{(x-3)(x+1)}\,dx$

put $\frac{5x-3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$

$\frac{5x-3}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$

$\therefore 5x-3 = A(x+1) + B(x-3) \Rightarrow 5x - 3 = (A+B)x + (A-3B)$

$\therefore A+B = 5$ — (1)

$A-3B = -3$ — (2)

from equation (1) $\Rightarrow A = 5-B$ — (3)

sub. equation (3) into equation (2):

$\therefore 5-B-3B = -3 \Rightarrow \therefore 8 = 4B \Rightarrow B = 2 \Rightarrow A = 3$

$\therefore \int 2x\,dx + \int \frac{5x-3}{x^2-2x-3}\,dx = \int 2x\,dx + \int \frac{3\,dx}{(x-3)} + \int \frac{2\,dx}{(x+1)}$

$= x^2 + 3\ln|x-3| + 2\ln|x+1| + c$

14/3 2016

H.W

تمرينات

① $\int \frac{4x+1}{2x^2+4x+10}\,dx$

Ans. $\ln\left[\left(\frac{x+1}{2}\right)^2+1\right]-\frac{3}{4}\tan^{-1}\left(\frac{x+1}{2}\right)+c$

② $\int_1^2 x^3 \ln x\,dx$

Ans. $4\ln 2-\frac{15}{16}$

③ $\int \frac{1}{\sqrt{9+x^2}}\,dx$

Ans. $\ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2}+\frac{x}{3}\right|+c$

④ $\int \sin^4 x\cos^5 x\,dx$

Ans. $\frac{1}{5}\sin^5 x-\frac{2}{7}\sin^7 x+\frac{1}{9}\sin^9+c$

⑤ $\int \frac{dx}{x\sqrt{1-x}}$

Ans. $\ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right|+c$

⑥ $\int \frac{dx}{x^2\sqrt{4+x^2}}$

Ans. $-\frac{\sqrt{4+x^2}}{4x}+c$

هذه غير معادة

(7) $$\int \frac{dx}{(4x^2-24x+27)^{3/2}}$$

Ans.

$$-\frac{1}{9}\,\frac{x-3}{\sqrt{4x^2-24x+27}}+c$$

(8) $$\int \frac{(3x+5)\,dx}{x^3-x^2-x+1}$$

Ans.

$$-\frac{4}{x-1}+\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+c$$

(9) $$\int \sin^3 3x\,\cos^5 3x\,dx$$

Ans.

$$\frac{1}{12}\sin^4 3x-\frac{1}{9}\sin^6 3x+\frac{1}{24}\sin^8 3x+c$$

(10) $$\int \tan^5 x\,dx$$

Ans.

$$\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c$$

نیز مله

Calculus P512

① $\int \frac{4x+1}{2x^2+4x+10}\,dx$

Ans.

$\ln\left[\left(\frac{x+1}{2}\right)^2+1\right]-\frac{3}{4}\tan^{-1}\left(\frac{x+1}{2}\right)+c$

Calculus ② P518

$\int_1^2 x^3\ln x\,dx$

Ans.

$=4\ln 2-\frac{15}{16}$

Calculus ③ P527

$\int \frac{1}{\sqrt{9+x^2}}\,dx$

Ans.

$=\cancel{\ln|\sec\theta+\tan\theta|}+c$

$=\ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2}+\frac{x}{3}\right|+c$

Calculus
شمس
④ P536

$\int \sin^4 x\cos^5 x\,dx$

Ans.

$=\frac{1}{5}\sin^5 x-\frac{2}{7}\sin^7 x+\frac{1}{9}\sin^9 x+c$

سلسلة ⑤ P191 Q1

$\int \frac{dx}{x\sqrt{1-x}}$

Ans.

$\ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right|+c$

سلسلة ⑥ P183 Q1

$\int \frac{dx}{x^2\sqrt{4+x^2}}$

Ans. $-\frac{\sqrt{4+x^2}}{4x}+c$

سلسلة ⑦ P185 Q7

$\int \frac{dx}{(4x^2-24x+27)^{3/2}}$

Ans. $-\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}}+c$

حل
P188
Q3

(8) $\displaystyle\int \frac{(3x+5)\,dx}{x^3-x^2-x+1}$

Ans.

$$-\frac{4}{x-1}+\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+c$$

حل
P179
Q7

(9) $\displaystyle\int \sin^3 3x \cos^5 3x \; dx$

Ans.

$$\frac{1}{12}\sin^4 3x-\frac{1}{9}\sin^6 3x+\frac{1}{24}\sin^8 3x+c$$

حل
P180
Q19

(10) $\displaystyle\int \tan^5 x\, dx$

Ans.

$$\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c$$

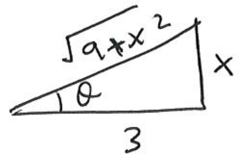
Calculus

P527

H.w

3

$$\int \frac{1}{\sqrt{9+x^2}}\,dx$$



$$\tan\theta = \frac{x}{3} \Rightarrow x = 3\tan\theta$$

$$dx = 3\sec^2 d\theta$$

$$\cos\theta = \frac{3}{\sqrt{9+x^2}} \Rightarrow \sqrt{9+x^2} = \frac{3}{\cos\theta} = 3\sec\theta$$

$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2 d\theta}{3\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta|$$

$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + c$$

$$\tan\theta = \frac{x}{3}$$

$$\sec = \frac{1}{\cos} = \frac{1}{\frac{3}{\sqrt{9+x^2}}}$$

$$\sec = \frac{\sqrt{9+x^2}}{3}$$

or $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{x}{3}\right)^2}$

$$\therefore \int \frac{dx}{\sqrt{9+x^2}} = \ln\left|\sqrt{1+\left(\frac{x}{3}\right)^2} + \frac{x}{3}\right| + c$$

6/4/2014

TRANSCENDENTAL FUNCTIONS

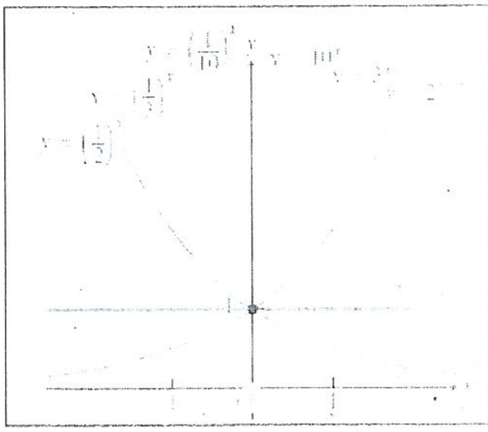
1. Exponential and logarithmic functions:

$y = f(x) = a^x$, $(a > 0, a \neq 1)$ is called an exponential function.

a: The base x: The exponent

* Domain: $-\infty < x < \infty$

* Range: $0 < y < \infty$



$y = f(x) = \log_a x$, $(a > 0, a \neq 1)$ is called a logarithmic function.

Logarithmic functions are the inverse of the exponential functions.

$10^2 = 100 \Rightarrow \log_{10} 100 = 2$

$10^{-2} = 1/100 \Rightarrow \log_{10}(1/100) = -2$

$2^3 = 8 \Rightarrow \log_2 8 = 3$

$a^0 = 1 \Rightarrow \log_a 1 = 0$

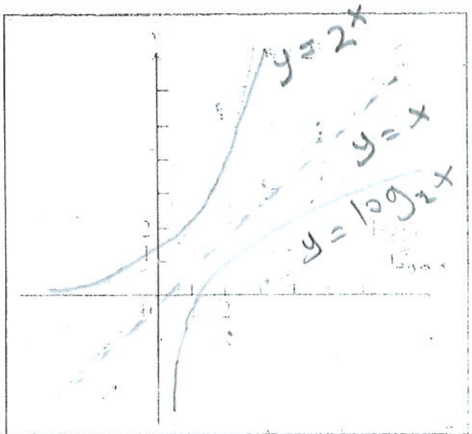
$a^1 = a \Rightarrow \log_a a = 1$

$2^3 = 8$

* Domain: $0 < x < \infty$

* Range: $-\infty < y < \infty$

2 log 8



Special cases

If $a = e \approx 2.71828 \Rightarrow y = f(x) = e^x$ is called the natural exponential function.

$y = f(x) = \log_e x = \ln x$ is called the natural logarithmic function.

$e^0 = 1 \Rightarrow \ln 1 = 0$

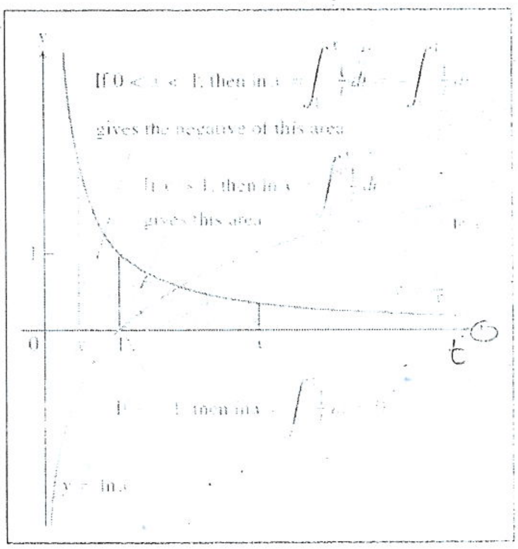
$e^1 = e \Rightarrow \ln e = 1$

The natural logarithm of positive number x, written as $\ln x$, is the value of integral:

$$\ln x = \int_1^x \frac{1}{t}\, dt \; ; \; x > 0$$

We can see from the figure

- $\ln x > 0$ if $x > 1$
- $\ln x = 0$ if $x = 1$
- $\ln x < 0$ if $0 < x < 1$
- $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to 0^+} \ln x = -\infty$



Definition The number e,

The number e is that the number in the domain of the natural logarithm satisfy $\ln(e) = 1$. Geometrically, the number e corresponds to the point on the x-axis for which the area under graph of $(y = 1/t)$ and above the interval $[1, e]$ is the exact area of the unit square.

Rules:

$* \; e^{-x} = \dfrac{1}{e^x}$ $\qquad * \; e^m . e^n = e^{m+n}$ $\qquad * \; \dfrac{e^m}{e^n} = e^{m-n}$ $\qquad * \; (e^m)^n = e^{mn}$

$* \; \ln mn = \ln m + \ln n$ $\qquad * \; \dfrac{\ln m}{\ln n} = \ln m - \ln n$ $\qquad * \; \ln m^n = n \ln m$

$* \; a^{\log_a x} = x \quad \Rightarrow \quad \log_a a^x = x$ $\qquad * \; e^{\ln x} = x \quad \Rightarrow \quad \ln e^x = x$

$* \; \log_a u = \dfrac{\ln u}{\ln a}$ $\qquad * \; \ln u = \dfrac{\log u}{\log e} \approx 2.303 \log u$

Examples:

$* \; \log 10^x = x$ $\qquad * \; 2^{\log_2 x} = x$ $\qquad * \; \ln e^{3x} = 3x$ $\qquad * \; e^{\ln x^2} = x^2$

$* \; a^x = e^{\ln a^x} = e^{x \ln a}$

Examples: Solve for x

$* \; \ln(2x+1) = 2$

Sol.: $e^{\ln(2x+1)} = e^2 \quad \Rightarrow \quad 2x+1 = e^2 \quad \Rightarrow \quad x = \dfrac{e^2 - 1}{2} \approx 3.19$

$* \; 4^{x^2} = 5$

Sol.: $\ln 4^{x^2} = \ln 5 \Rightarrow x^2 \ln 4 = \ln 5 \quad \Rightarrow \quad x = \pm\sqrt{\dfrac{\ln 5}{\ln 4}} \approx 1.077$

Differentiation and integration:

If u is a function of x, then

$* \; \dfrac{d}{dx}(a^u) = a^u . \ln a . \dfrac{du}{dx} \quad \Rightarrow \quad \int a^u . \ln a . du = a^u + C \quad \Rightarrow \quad \int a^u . du = \dfrac{a^u}{\ln a} + C$

$$\boxed{* \; \dfrac{d}{dx}(e^u) = e^u . \dfrac{du}{dx} \quad \Rightarrow \quad \int e^u . du = e^u + C}$$

$* \; \dfrac{d}{dx}(\log_a u) = \dfrac{1}{u \ln a} . \dfrac{du}{dx} \quad \Rightarrow \quad \int \dfrac{du}{u \ln a} = \log_a u + C$

$* \; \dfrac{d}{dx}(\ln u) = \dfrac{1}{u} . \dfrac{du}{dx} \quad \Rightarrow \quad \int \dfrac{du}{u} = \ln|u| + C$

Examples: Find $\dfrac{dy}{dx}$ for:

$* \; y = 3^{\sin x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 3^{\sin x} . \ln 3 . \cos x$

$* \; y = e^{x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = e^{x^2} . (2x) = 2x e^{x^2}$

$* \; y = x^2 e^{-x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = x^2 [e^{-x^2} . (-2x)] + e^{-x^2}(2x) \quad \Rightarrow \quad \dfrac{dy}{dx} = 2x e^{-x^2}(-x^2 + 1)$

$* \; y = \ln(3x^2 + 4) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3x^2 + 4} . (2 * 3x) = \dfrac{6x}{3x^2 + 4}$

$\log_a u = \frac{\ln u}{\ln a}$ or $\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}\frac{du}{dx}$

$= \frac{1}{e^x \ln 10} \cdot e^x$

* $y = \log e^x = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 10}$

* $y = \log_5 (x^2+1)^2 = \frac{\ln(x^2+1)^2}{\ln 5} = \frac{2\ln(x^2+1)}{\ln 5} \Rightarrow \frac{dy}{dx} = \frac{2}{\ln 5}\cdot\frac{2x}{x^2+1} = \frac{4x}{\ln 5.(x^2+1)}$

* $y = \ln\sqrt{\frac{1+x}{1-x}} \Rightarrow y = \ln\left(\frac{1+x}{1-x}\right)^{1/2} \Rightarrow y = \frac{1}{2}\ln\frac{1+x}{1-x}$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{\frac{1+x}{1-x}}\cdot\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right] = \frac{1}{2}\left[\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2}\right] = \frac{1}{1-x^2}$$

$\frac{d}{dx}(\log_a u) = \frac{1}{u \ln a}$ $= \frac{1}{(x^2+1)^2 \ln 5}[2(x^2+1)\ldots]$ $\frac{4x}{\ln 5 (x^2+1)}$

Examples: Evaluate the following integrals:

(1) * $\int a^{2x}dx = \int a^{2x}.\frac{\ln a}{\ln a}.\frac{2}{2}.dx = \frac{1}{2\ln a}\int a^{2x}.\ln a.(2).dx = \frac{a^{2x}}{2\ln a} + C$ $\quad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

(2) * $\int e^{-x}dx = \int e^{-x}.\frac{-1}{-1}.dx = -\int e^{-x}.(-1).dx = -e^{-x} + C$

(3) * $\int e^x(e^x+3)^2dx = \frac{(e^x+3)^3}{3} + C$ $\quad e^{x+e^x}$ $\quad e^m.e^n = e^{m+n}$

(4) * $\int e^{x+e^x}dx = \int e^x.e^{e^x}dx = e^{e^x} + C$ $\quad \int e^{x+e^x}$

(5) * $\int \cos 2x.e^{\sin 2x}dx = \frac{1}{2}e^{\sin 2x} + C$

(6) * $\int \frac{1}{1+x}dx = \ln|1+x| + C$

(7) * $\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}.\frac{1}{x}.dx = \ln|\ln x| + C$

(8) * $\int_0^2 \frac{x}{x^2-5}dx = \frac{1}{2}\left[\ln|x^2-5|\right]_0^2 = \frac{1}{2}\left[\ln|4-5| - \ln|0-5|\right] = \frac{1}{2}\left[\ln|-1| - \ln|-5|\right] = \frac{-\ln 5}{2}$

(9) * $\int \tan x.dx = \int \frac{\sin x}{\cos x}dx = -\ln|\cos x| + C = \ln|(\cos x)^{-1}| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

(10) * $\int \cot x.dx = \int \frac{\cos x}{\sin x}dx = \ln|\sin x| + C$

(11) * $\int_0^{\pi/6} \tan 2x dx = \frac{1}{2}\ln|\sec 2x|\Big|_0^{\pi/6} = \frac{1}{2}\left[\ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0|\right] = \frac{1}{2}[\ln 2 - \ln 1] = \frac{1}{2}\ln 2$

$\ln 1 - \ln\cos x$ $\quad \frac{1}{\ln\cos x}$

(12) * $\int \frac{\log(x+2)}{x+2}dx = \int \frac{\ln(x+2)}{\ln 10.(x+2)}dx = \frac{1}{\ln 10}\int \ln(x+2).\frac{1}{(x+2)}dx$

$\log_a u = \frac{\ln u}{\ln a}$ $\quad = \frac{1}{\ln 10}.\frac{\ln^2(x+2)}{2} + C = \frac{\ln^2(x+2)}{2\ln 10} + C$

Logarithmic differentiation:

The derivatives of positive function given by formulas that involve products, quotients and powers can often be found more quickly if we take natural logarithm of both sides before differentiating. This enables us to use laws of logarithms to simplify the formulas before differentiating. The process is called **logarithmic differentiation.**

Examples: Find $\frac{dy}{dx}$ for:

1. $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$

Sol.: We take the natural logarithm of both sides and simplify the result with properties of logarithms:

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y = \ln\left[\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right] = \ln[(x^2+1)(x+3)^{1/2}] - \ln(x-1)$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

Then we differentiate by implicit differentiation

$$\frac{1}{y} * \frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2} * \frac{1}{x+3} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left[\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right]$$

$$\frac{dy}{dx} = \left[\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right]\left[\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right]$$

2. $y = x^x$

Sol.: $\ln y = \ln x^x \quad \Rightarrow \quad \ln y = x \ln x \quad \Rightarrow \quad \frac{1}{y}.\frac{dy}{dx} = x\frac{1}{x} + \ln x.(1)$

$$\frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x).$$

3. $y = (\sin x)^{\tan x}$

Sol.: $\ln y = \ln(\sin x)^{\tan x} \quad \Rightarrow \quad \ln y = \tan x.\ln(\sin x)$

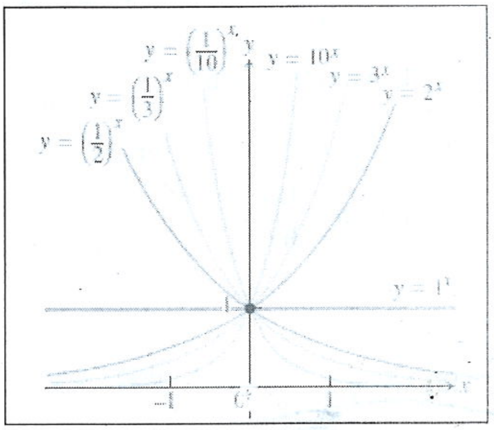
$$\frac{1}{y}.\frac{dy}{dx} = \tan x.(\frac{1}{\sin x}.\cos x) + \ln(\sin x).\sec^2 x$$

$$\frac{dy}{dx} = y[\frac{\sin x}{\cos x}.\frac{\cos x}{\sin x} + \sec^2 x.\ln(\sin x)] = (\sin x)^{\tan x}[1+\sec^2 x.\ln(\sin x)].$$

6/4/2014

TRANSCENDENTAL FUNCTIONS

1. Exponential and logarithmic functions:



$y=f(x)=a^x$, $(a>0, a\neq 1)$ is called an exponential function.

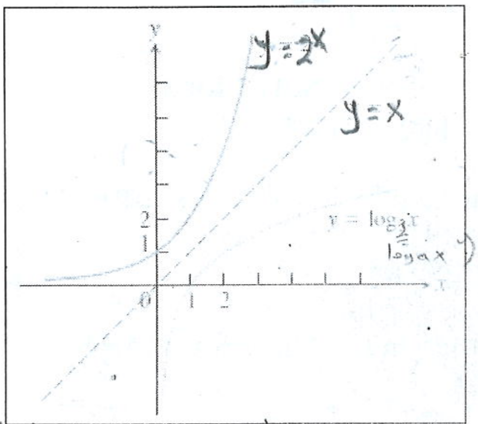
a: The base $\qquad$ x: The exponent

* Domain: $-\infty<x<\infty$

* Range: $0<y<\infty$

$y=f(x)=\log_a x$, $(a>0, a\neq 1)$ is called a logarithmic function.

Logarithmic functions are the inverse of the exponential functions.



$10^2=100 \quad \Rightarrow \log_{10}100=2$

$10^{-2}=1/100 \quad \Rightarrow \log_{10}(1/100)=-2$

$2^3=8 \quad \Rightarrow \log_2 8=3$

$a^0=1 \quad \Rightarrow \log_a 1=0$

$a^1=a \quad \Rightarrow \log_a a=1$

$2^3=8$

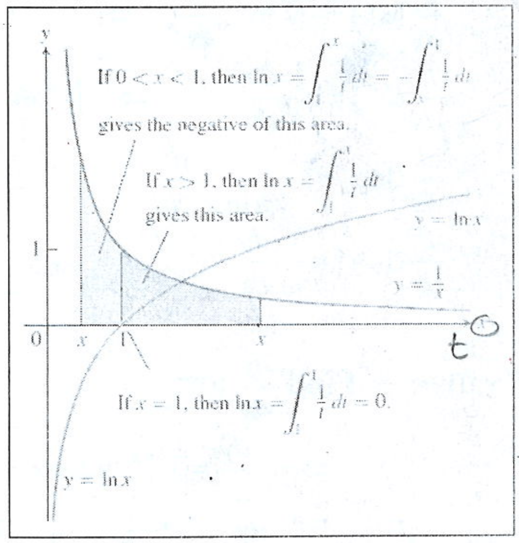
* Domain: $0<x<\infty$

* Range: $-\infty<y<\infty$

Special cases

If $a=e\approx 2.71828 \Rightarrow y=f(x)=e^x$ is called the natural exponential function.

$y=f(x)=\log_e x=\ln x$ is called the natural logarithmic function.



$e^0=1 \quad \Rightarrow \ln 1=0$

$e^1=e \quad \Rightarrow \ln e=1$

The natural logarithm of positive number x, written as $\ln x$, is the value of integral:

$$\ln x=\int_1^x \frac{1}{t}dt\ ;\ x>0$$

We can see from the figure

- $\ln x>0$ if $x>1$
- $\ln x=0$ if $x=1$
- $\ln x<0$ if $0<x<1$
- $\lim_{x\to\infty}\ln x=\infty$ and $\lim_{x\to 0^+}\ln x=-\infty$

Definition The number e,

The number e is that the number in the domain of the natural logarithm satisfy $\ln(e)=1$. Geometrically, the number e corresponds to the point on the x-axis for which the area under graph of ($y=1/t$) and above the interval $[1, e]$ is the exact area of the unit square.

Rules:

$* \; e^{-x} = \dfrac{1}{e^x}$ $\qquad * \; e^m.e^n = e^{m+n}$ $\qquad * \; \dfrac{e^m}{e^n} = e^{m-n}$ $\qquad * \; (e^m)^n = e^{mn}$

$* \; \ln mn = \ln m + \ln n$ $\qquad * \; \dfrac{\ln m}{\ln n} == \ln m - \ln n$ $\qquad * \; \ln m^n = n \ln m$

$* \; a^{\log_a x} = x \quad \Rightarrow \quad \log_a a^x = x$ $\qquad * \; e^{\ln x} = x \quad \Rightarrow \quad \ln e^x = x$

$* \; \log_a u = \dfrac{\ln u}{\ln a}$ $\qquad * \; \ln u = \dfrac{\log u}{\log e} \approx 2.303 \log u$

Examples:

$* \; \log 10^x = x$ $\qquad * \; 2^{\log_2 x} = x$ $\qquad * \; \ln e^{3x} = 3x$ $\qquad * \; e^{\ln x^2} = x^2$

$* \; a^x = e^{\ln a^x} = e^{x \ln a}$

Examples: Solve for x

$* \; \ln(2x+1) = 2$

Sol.: $e^{\ln(2x+1)} = e^2 \quad \Rightarrow \quad 2x + 1 = e^2 \quad \Rightarrow \quad x = \dfrac{e^2 - 1}{2} \approx 3.19$

$* \; 4^{x^2} = 5$

Sol.: $\ln 4^{x^2} = \ln 5 \Rightarrow x^2 \ln 4 = \ln 5 \quad \Rightarrow \quad x = \pm\sqrt{\dfrac{\ln 5}{\ln 4}} \approx 1.077$

Differentiation and integration:

If u is a function of x, then

$* \; \dfrac{d}{dx}(a^u) = a^u.\ln a.\dfrac{du}{dx} \quad \Rightarrow \quad \int a^u.\ln a.du = a^u + C \quad \Rightarrow \quad \int a^u.du = \dfrac{a^u}{\ln a} + C$

$* \; \dfrac{d}{dx}(e^u) = e^u.\dfrac{du}{dx} \quad \Rightarrow \quad \int e^u.du = e^u + C$

$* \; \dfrac{d}{dx}(\log_a u) = \dfrac{1}{u \ln a}.\dfrac{du}{dx} \quad \Rightarrow \quad \int \dfrac{du}{u \ln a} = \log_a u + C$

$* \; \dfrac{d}{dx}(\ln u) = \dfrac{1}{u}.\dfrac{du}{dx} \quad \Rightarrow \quad \int \dfrac{du}{u} = \ln|u| + C$

Examples: Find $\dfrac{dy}{dx}$ for:

$* \; y = 3^{\sin x} \quad \Rightarrow \quad \dfrac{dy}{dx} = 3^{\sin x}.\ln 3.\cos x$

$* \; y = e^{x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = e^{x^2}.(2x) = 2xe^{x^2}$

$* \; y = x^2 e^{-x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = x^2[e^{-x^2}.(-2x)] + e^{-x^2}(2x) \quad \Rightarrow \quad \dfrac{dy}{dx} = 2xe^{-x^2}(-x^2 + 1)$

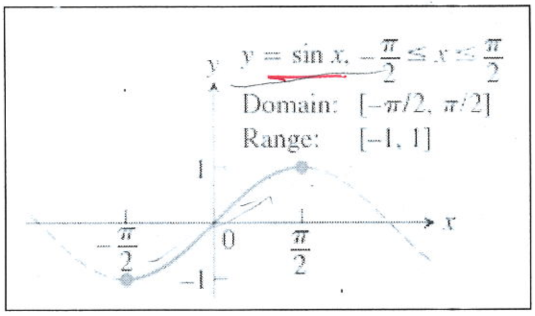
$* \; y = \ln(3x^2 + 4) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3x^2 + 4}.(2 * 3x) = \dfrac{6x}{3x^2 + 4}$

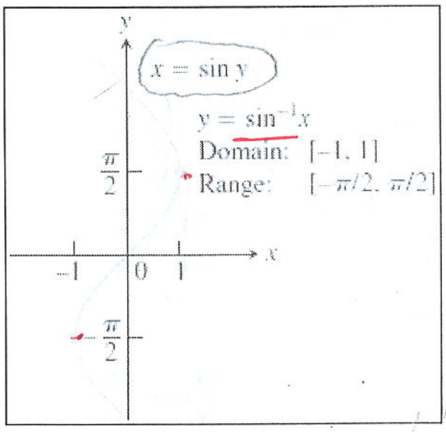
17/5/2015

2. Inverse of trigonometric functions:

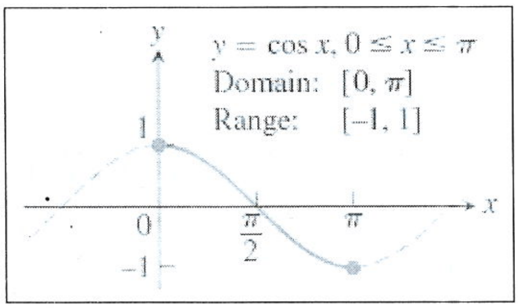
The trigonometric functions are not one-to-one, and therefore they do not have inverse functions. If we restrict their domain to become one-to-one, then they will have inverse functions.

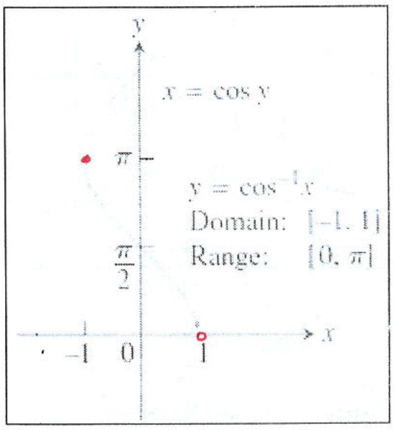
a. $y = \sin^{-1} x$ (or $y = \arcsin x$) is the angle in $[-\pi/2, \pi/2]$ whose sine is x.





b. $y = \cos^{-1} x$ (or $y = \arccos x$) is the angle in $[0, \pi]$ whose cosine is x.



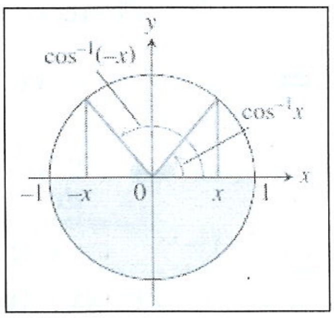


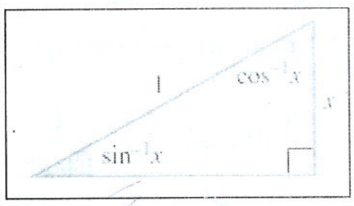
Identities:

1. We can see from the shown unit circle

$$\cos^{-1} x + \cos^{-1}(-x) = \pi$$

$$\therefore \cos^{-1}(-x) = \pi - \cos^{-1} x$$





2. We can see from the shown triangle

$$\sin^{-1} x + \cos^{-1} x = \pi/2$$

$$\therefore \cos^{-1} x = (\pi/2) - \sin^{-1} x$$

3. $\sec^{-1} x = \cos^{-1}(\frac{1}{x})$

4. $\csc^{-1} x = (\pi/2) - \sec^{-1} x = (\pi/2) - \cos^{-1}(\frac{1}{x}) = \sin^{-1}(\frac{1}{x})$

5. $\cot^{-1} x = (\pi/2) - \tan^{-1} x$

Differentiation and integration:

If u is a function of x, then

* $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}.\frac{du}{dx}$ $\quad \Rightarrow \quad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$
* $\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}}.\frac{du}{dx}$
* $\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}.\frac{du}{dx}$ $\quad \Rightarrow \quad \int \frac{du}{1+u^2} = \tan^{-1} u + C$
* $\frac{d}{dx}(\cot^{-1} u) = \frac{-1}{1+u^2}.\frac{du}{dx}$
* $\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$ $\quad \Rightarrow \quad \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
* $\frac{d}{dx}(\csc^{-1} u) = \frac{-1}{|u|\sqrt{u^2-1}}.\frac{du}{dx}$

Examples: Find $\frac{dy}{dx}$ for:

1. $y = \sin^{-1} x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(x^2)^2}}.(2x) = \frac{2x}{\sqrt{1-x^4}}$

2. $y = \tan^{-1}\sqrt{x+1} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{1+(\sqrt{x+1})^2}.\frac{1}{2}(x+1)^{-1/2} = \frac{1}{1+x+1}.\frac{1}{2\sqrt{x+1}} = \frac{1}{2(x+2).\sqrt{x+1}}$

3. $y = \sec^{-1} 3x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{|3x|.\sqrt{(3x)^2-1}}.(3) = \frac{1}{|x|\sqrt{9x^2-1}}$

4. $y = \cos^{-1}(\cos x)$ $\quad y : \frac{\cos}{\cos} = 1$

Either $y = \cos^{-1}(\cos x) \quad \Rightarrow \quad y = x \quad \Rightarrow \quad \frac{dy}{dx} = 1$

Or $y = \cos^{-1}(\cos x) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-(\cos x)^2}}.(-\sin x) = \frac{\sin x}{\sqrt{\sin^2 x}} = 1$ $\quad \sin^2 + \cos^2 = 1$, $\sin^2 = 1 - \cos^2$

5. $y^2 \sin x + y = \tan^{-1} y \quad \Rightarrow \quad y^2 \cos x + \sin x.(2y.\frac{dy}{dx}) + \frac{dy}{dx} = \frac{1}{1+y^2}.\frac{dy}{dx}$

$\frac{dy}{dx}(2y\sin x + 1 - \frac{1}{1+y^2}) = -y^2\cos x \quad \Rightarrow \quad \frac{dy}{dx}(\frac{2y\sin x(1+y^2)+(1+y^2)-1}{1+y^2}) = -y^2\cos x$

$= -y^2 \cos x$

$\frac{dy}{dx} = (\frac{2y\sin x(1+y^2)+(1+y^2)-1}{1+y^2}) = \frac{-y^2\cos x(1+y^2)}{2y\sin x + 2y^3\sin x + 1 + y^2 - 1} = \frac{-y\cos x(1+y^2)}{2\sin x + 2y^2\sin x + y}$

Examples: Evaluate the following integrals:

1. $\int_0^1 \frac{dx}{1+x^2} \quad \Rightarrow \quad \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

2. $\int\frac{x^2dx}{\sqrt{1-x^6}} \Rightarrow \int\frac{x^2dx}{\sqrt{1-x^6}}=\int\frac{x^2dx}{\sqrt{1-(x^3)^2}}=\frac{1}{3}\sin^{-1}x^3+C$

3. $\int\frac{dx}{\sqrt{9-x^2}} \Rightarrow \int\frac{dx}{\sqrt{9-x^2}}=\int\frac{dx}{\sqrt{9(1-x^2/9)}}=\frac{1}{3}\int\frac{dx}{\sqrt{1-(x/3)^2}}.\frac{1/3}{1/3}=\sin^{-1}(x/3)+C$

4. $\int\frac{dx}{e^x\sqrt{1-e^{-2x}}} \Rightarrow \int\frac{dx}{e^x\sqrt{1-e^{-2x}}}=\int\frac{e^{-x}dx}{\sqrt{1-(e^{-x})^2}}=-\sin^{-1}e^{-x}+C$

5. $\int\frac{\sin^{-1}3xdx}{\sqrt{1-9x^2}} \Rightarrow \int\frac{\sin^{-1}3xdx}{\sqrt{1-9x^2}}=\int\sin^{-1}3x.\frac{dx}{\sqrt{1-(3x)^2}}.\frac{3}{3}=\frac{1}{3}\frac{(\sin^{-1}3x)^2}{2}+C=\frac{(\sin^{-1}3x)^2}{6}+C$

6. $\int\frac{\cot xdx}{\sqrt{\sin^2x-9}} \Rightarrow \int\frac{\cot xdx}{\sqrt{\sin^2x-9}}=\int\frac{\cos xdx}{\sin x\sqrt{9(\frac{\sin^2x}{9}-1)}}=\int\frac{\cos xdx}{3\sin x\sqrt{(\frac{\sin x}{3})^2-1}}$

$=\frac{1}{3}\int\frac{\cos xdx}{3(\frac{\sin x}{3})\sqrt{(\frac{\sin x}{3})^2-1}}.\frac{1/3}{1/3}=\frac{1}{3}\sec^{-1}(\frac{\sin x}{3})+C$

7. $\int\frac{dx}{\sqrt{4x-x^2}}$

Rewrite $4x-x^2$ by completing the squares

$4x-x^2 \Rightarrow -(x^2-4x)=-(x^2-4x+4)+4=-(x-2)^2+4=4-(x-2)^2$

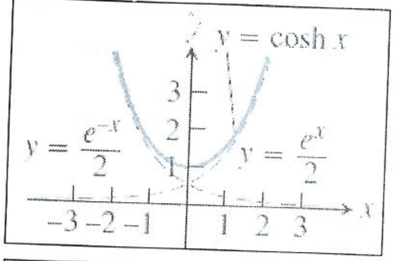
$\int\frac{dx}{\sqrt{4x-x^2}}=\int\frac{dx}{\sqrt{4-(x-2)^2}}=\int\frac{dx}{\sqrt{4[1-\frac{(x-2)^2}{4}]}}=\int\frac{dx}{2\sqrt{1-\left(\frac{x-2}{2}\right)^2}}$

$=\frac{1}{2}\int\frac{dx}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}}.\frac{1/2}{1/2}=\sin^{-1}(\frac{x-2}{2})+C.$

3. Hyperbolic functions:

The hyperbolic cosine and hyperbolic sine functions are defined by the following equations:

*Hyperbolic cosine of x: $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

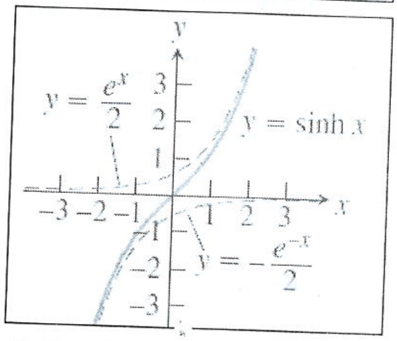


Note: when $x \to \infty, e^{-x} \to 0 \Rightarrow \cosh x \cong e^{x}/2$

$x \to -\infty, e^{x} \to 0 \Rightarrow \cosh x \cong e^{-x}/2$

$D_f = (-\infty, \infty)$ and $R_f = [1, \infty)$

*Hyperbolic sine of x: $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$



Note: when $x \to \infty, e^{-x} \to 0 \Rightarrow \sinh x \cong e^{x}/2$

$x \to -\infty, e^{x} \to 0 \Rightarrow \sinh x \cong -e^{-x}/2$

$D_f = (-\infty, \infty)$ and $R_f = (-\infty, \infty)$

The notation cosh x is often read "kosh x" and sinh x is pronounced as if spelled "cinch x" or "shine x".Four additional hyperbolic functions are defined in terms of cosh x and sinh x as shown below:

*Hyperbolic tangent of x: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

*Hyperbolic cotangent of x: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

*Hyperbolic secant of x: $\text{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$

*Hyperbolic cosecant of x: $\text{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$

Identities:

	Hyperbolic functions	Trigonometric Functions
1	$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
2	$\sinh 2x = 2\sinh x \cosh x$	$\sin 2x = 2\sin x \cos x$
3	$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cos 2x = \cos^2 x - \sin^2 x$
4	$\cosh^2 x = \frac{\cosh 2x + 1}{2}$	$\cos^2 x = \frac{1 + \cos 2x}{2}$
5	$\sinh^2 x = \frac{\cosh 2x - 1}{2}$	$\sin^2 x = \frac{1 - \cos 2x}{2}$
6	$\tanh^2 x = 1 - \text{sech}^2 x$	$\tan^2 x = \sec^2 x - 1$
7	$\coth^2 x = 1 + \text{csch}^2 x$	$\cot^2 x = \csc^2 x - 1$

Derivatives of hyperbolic function:

If u is any function of x, then:

	Derivative of hyperbolic functions	Derivative of trigonometric functions
1	$\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$	$\frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$
2	$\frac{d}{dx}\cosh u = \sinh u.\frac{du}{dx}$	$\frac{d}{dx}\cos u = -\sin u.\frac{du}{dx}$
3	$\frac{d}{dx}\tanh u = \sec h^2 u.\frac{du}{dx}$	$\frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$
4	$\frac{d}{dx}\coth u = -\csc h^2 u.\frac{du}{dx}$	$\frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$
5	$\frac{d}{dx}\sec hu = -\sec hu\tanh u.\frac{du}{dx}$	$\frac{d}{dx}\sec u = \sec u\tan u.\frac{du}{dx}$
6	$\frac{d}{dx}\csc hu = -\csc hu\coth u.\frac{du}{dx}$	$\frac{d}{dx}\csc u = -\csc u\cot u.\frac{du}{dx}$

Examples: Find $\frac{dy}{dx}$ for:

1. $y = \sinh 3x \quad \Rightarrow \quad \frac{dy}{dx} = \cosh 3x.(3) = 3\cosh 3x$

2. $y = \tanh(1+x^3) \quad \Rightarrow \quad \frac{dy}{dx} = \operatorname{sech}^2(1+x^3).(3x^2) = 3x^2\operatorname{sech}^2(1+x^3)$

3. $y = \ln\tanh 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\tanh 2x}.\operatorname{sech}^2 2x.(2) = 3x^2\operatorname{sech}(1+x^3) = \frac{2\cosh 2x}{\sinh 2x}.\frac{1}{\cosh^2 2x}$

$$= \frac{2}{\sinh 2x.\cosh 2x} = \frac{2}{\frac{\sinh 4x}{2}} = 4\operatorname{csch} 4x$$

4. $y = (\sinh x)^x \quad \Rightarrow \quad \ln y = \ln(\sinh x)^x = x\ln\sinh x$

$$\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\sinh x}.\cosh x + \ln\sinh x \quad \Rightarrow \quad \frac{dy}{dx} = (\sinh x)^x(x\coth x + \ln\sinh x)$$

5. $y = \operatorname{sech}^2 x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2.\operatorname{sech}x^2.(-\operatorname{sech}x^2.\tanh x^2).2x = -4x\operatorname{sech}^2 x^2.\tanh x^2$

$= (\operatorname{sech} x^2)^2$

$= \frac{1}{\tanh 2x}.\operatorname{sech}^2 2x(2)$

$= (2).\operatorname{sech}2x.\operatorname{csch}2x$

$= 2.\frac{1}{\cosh 2x}.\frac{1}{\sinh 2x}$

$= \frac{2}{\cosh 2x.\sinh 2x}$

$= \frac{2}{\frac{1}{2}\sinh 4x}$

$= 4\operatorname{csch}4x$

Integrals of hyperbolic function:

If u is any function of x, then:

1. $\int \sinh u.du = \cosh u + C$
2. $\int \cosh u.du = \sinh u + C$
3. $\int \sec h^2 u.du = \tanh u + C$
4. $\int \csc h^2 u.du = -\coth u + C$
5. $\int \sec hu.\tanh u.du = -\sec hu + C$
6. $\int \csc hu.\coth u.du = -\csc hu + C$

Examples: Evaluate the following integrals:

$\sinh^2 = \frac{\cosh^2 - 1}{2}$

1. $\int_0^1 \sinh^2 x dx \Rightarrow \int_0^1 \sinh^2 x dx = \int_0^1 (\frac{\cosh 2x - 1}{2}) dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$

$$= \frac{1}{2}\left[[\frac{\sinh 2}{2} - 1] - [\frac{\sinh 0}{2} - 0]\right] = \frac{\sinh 2}{4} - \frac{1}{2} = 040672$$

2. $\int \coth x.dx \Rightarrow \int \coth x.dx = \int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$

3. $\int \frac{\text{csch}^2 \sqrt{x}}{\sqrt{x}} dx \Rightarrow \int \frac{\text{csch}^2 \sqrt{x}}{\sqrt{x}} dx = -2\coth\sqrt{x} + C$

$u = \sqrt{x}$, $du = \frac{1}{2}\frac{1}{\sqrt{x}}$ $\qquad \frac{d}{dx}\coth = -\text{csch}^2$

4. $\int e^{\coth 2x}\,\text{csch}^2 2x dx \Rightarrow \int e^{\coth 2x}\,\text{csch}^2 2x dx = -\frac{e^{\coth 2x}}{2} + C$

$\coth = -\text{csch}$

5. $\int_0^{\ln 2} 4e^x \sinh x dx \Rightarrow \int_0^{\ln 2} 4e^x.\frac{1}{2}(e^x - e^{-x}) dx = \int_0^{\ln 2} (2e^{2x} - 2) dx = \left[e^{2x} - 2x\right]_0^{\ln 2}$

$$= \left|[e^{2\ln 2} - 2\ln 2] - [e^{2\ln 0} - 2\ln 0]\right| = 4 - 2\ln 2 - 1 = 3 - 2\ln 2$$

6. $\int \frac{dx}{\sinh x + \cosh x} \Rightarrow \int \frac{dx}{\frac{1}{2}(e^x - e^{-x}) + \frac{1}{2}(e^x + e^{-x})} = \int \frac{dx}{e^x} = \int e^{-x} dx = -e^{-x} + C$

$= \frac{e^x}{2} - \frac{e^{-x}}{2} + \frac{e^x}{2} + \frac{e^{-x}}{2}$

$= e^x$

4. Inverse of hyperbolic functions:

All hyperbolic functions have inverses, they are:

	Inverse of hyperbolic functions	Their domains
1	$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$	$(-\infty, \infty)$
2	$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$	$[1, \infty)$
3	$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$(-1,1)$
4	$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	$(-\infty, \infty) \setminus [-1,1]$
5	$\operatorname{sec}h^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$	$(0,1]$
6	$\operatorname{csc}h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, \infty) \setminus \{0\}$

Identities:

1. $\operatorname{sech}^{-1}x = \cosh^{-1}(\frac{1}{x})$
2. $\operatorname{csch}^{-1}x = \sinh^{-1}(\frac{1}{x})$
3. $\coth^{-1} x = \tanh^{-1}(\frac{1}{x})$

Differentiation and integration:

If u is a function of x, then

* $\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}.\frac{du}{dx} \qquad \Rightarrow \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + C$
* $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}, \quad u > 1 \qquad \Rightarrow \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$
* $\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}.\frac{du}{dx}, \quad |u| < 1 \qquad \Rightarrow$
* $\frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}.\frac{du}{dx}, \quad |u| > 1 \qquad \Rightarrow$

$$\int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + C \Leftrightarrow if\, |u| < 1 \\ \coth^{-1} u + C \Leftrightarrow if\, |u| > 1 \end{cases}$$

* $\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}}.\frac{du}{dx}, \quad 0 < u < 1 \Rightarrow \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C$
* $\frac{d}{dx}(\operatorname{csch}^{-1} u) = \frac{-1}{u\sqrt{1+u^2}}.\frac{du}{dx}, \quad u \neq 0 \qquad \Rightarrow \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$

Examples: Find $\frac{dy}{dx}$ for:

1. $y=\sinh^{-1}2x \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1+(2x)^2}}.(2)=\frac{2}{\sqrt{1+4x^2}}$

2. $y=\cosh^{-1}e^{x} \Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{(e^{x})^2-1}}.(e^{x})=\frac{e^{x}}{\sqrt{e^{2x}-1}}$

3. $y=\coth^{-1}(\frac{1}{x}) \Rightarrow \frac{dy}{dx}=\frac{1}{1-(1/x)^2}.(-1/x^2)=\frac{-1/x^2}{(x^2-1)/x^2}=\frac{-1}{x^2-1}=\frac{1}{1-x^2}$

4. $y=\operatorname{sech}^{-1}(\cos x) \Rightarrow \frac{dy}{dx}=\frac{-1}{\cos x\sqrt{1-(\cos x)^2}}.(-\sin x)=\frac{\sin x}{\cos x\sqrt{\sin^2 x}}=\frac{1}{\cos x}=\sec x$

$\sin^2+\cos^2=1$

Examples: Evaluate the following integrals:

1. $\int_0^1\frac{2.dx}{\sqrt{1+4x^2}} \Rightarrow \int_0^1\frac{2.dx}{\sqrt{1+4x^2}}=\int_0^1\frac{2.dx}{\sqrt{1+(2x)^2}}=\left[\sinh^{-1}2x\right]_0^1=\sinh^{-1}2-\sinh^{-1}0=1.4436$

2. $\int\frac{dx}{\sqrt{x^2-3}} \Rightarrow \int\frac{dx}{\sqrt{3(\frac{x^2}{3}-1)}}=\int\frac{dx}{\sqrt{3}\sqrt{(\frac{x}{\sqrt{3}})^2-1}}=\cosh^{-1}(\frac{x}{\sqrt{3}})+C$

3. $\int\frac{\sin x.dx}{\sqrt{1+\cos^2 x}} \Rightarrow \int\frac{\sin x.dx}{\sqrt{1+\cos^2 x}}=-\sinh^{-1}(\cos x)+C$ $u=\cos x$

4. $\int\frac{4\tanh^{-1}x}{1-x^2}dx \Rightarrow \int 4\tanh^{-1}x.\frac{1}{1-x^2}.dx=4.\frac{(\tanh^{-1}x)^2}{2}+C=2(\tanh^{-1}x)^2+C$

5. $\int\frac{\cos x.dx}{\sin x\sqrt{1+\sin^2 x}} \Rightarrow \int\frac{\cos x.dx}{\sin x\sqrt{1+\sin^2 x}}=-\operatorname{csch}^{-1}|\sin x|+C$

6. $\int\frac{dx}{9x^2-25} \Rightarrow \int\frac{dx}{25(\frac{9x^2}{25}-1)}=\frac{1}{25}\int\frac{dx}{(\frac{3x}{5})^2-1}=\frac{1}{25}\int\frac{dx}{(\frac{3x}{5})^2-1}=\frac{-1}{25}\int\frac{dx}{1-(\frac{3x}{5})^2}.\frac{3/5}{3/5}$

$$=\begin{cases}\frac{-1}{15}\tanh^{-1}(\frac{3x}{5})+C\\ \frac{-1}{15}\coth^{-1}(\frac{3x}{5})+C\end{cases}$$

$e^{-\infty} = 0 \quad e^{0} = 1$

$\frac{1}{0} = \infty \quad \frac{1}{\infty} = 0$

L'Hopital's Rule

Indeterminate forms 0/0, ∞/∞

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Examples:

(a) $\displaystyle \lim_{x\to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\bigg|_{x=0} = 2$

$\cos 0 = 1$

(b) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \frac{\frac{1}{2\sqrt{1+x}}}{1}\Bigg|_{x=0} = \frac{1}{2}$

Examples:

(a) $\displaystyle \lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\qquad \frac{0}{0}$

$\displaystyle = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$ $\qquad$ Still $\frac{0}{0}$: differentiate again.

$\displaystyle = \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$ $\qquad$ Not $\frac{0}{0}$: limit is found.

(b) $\displaystyle \lim_{x\to 0} \frac{x - \sin x}{x^3}$ $\qquad \frac{0}{0}$

$\displaystyle = \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$ $\qquad$ Still $\frac{0}{0}$

$\displaystyle = \lim_{x\to 0} \frac{\sin x}{6x}$ $\qquad$ Still $\frac{0}{0}$

$\displaystyle = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$ $\qquad$ Not $\frac{0}{0}$: limit is found.

$\frac{\sin x}{6x}$

Indeterminate forms 0.∞, ∞ - ∞

Examples:

(a) $\displaystyle \lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x}$ $\qquad \frac{\infty}{\infty}$ from the left

$\displaystyle = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$

$\frac{\sec \cdot \tan}{\sec^2} = \frac{\tan}{\sec} = \cos \cdot \frac{\sin}{\cos} = \sin$

(b) $\displaystyle \lim_{x\to\infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x\to\infty} \frac{1 - 4x}{6x + 5} = \lim_{x\to\infty} \frac{-4}{6} = -\frac{2}{3}$

$\frac{\infty}{\infty} \quad \frac{\infty}{\infty}$

$\frac{1}{\infty} = 0$

$\sin 0 = 0$

$\lim_{x \to \pi^+} \sqrt{x + \pi}$

(c)

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$$

$\infty \cdot 0$

هذا نستطيع ... نحول الى الصيغة الكسرية $\frac{0}{0}$

$\frac{1}{\infty} = 0$

$$= \lim_{h \to 0^+} \left(\frac{1}{h} \sin h \right)$$

let $h = 1/x$.

$= \lim_{h \to 0} \frac{\sin h}{h}$

∴ حولناها الى بسط ومقام فنطبق قاعدة لوبيتال

$h = \frac{1}{x}$

$\therefore x = \frac{1}{h}$

$$= 1$$

$= \frac{\text{مشتقة البسط}}{\text{مشتقة المقام}} = \frac{\cos h}{1}$

$\therefore \lim_{h \to 0} \frac{\cos h}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$

بما أن $x \to \infty$

$\therefore h = \frac{1}{\infty} = 0$

(d)

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

توحيد المقامات

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

حولناها الى بسط ومقام فنطبق قاعدة لوبيتال

Then apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

$\frac{0}{0}$

Still $\frac{0}{0}$

$\cos x \quad -x \sin x + \cos x$

$= \frac{}{2\cos x - x \sin x}$

Indeterminate forms 1^∞, ∞^0, 0^0

DEFINTION: If: $\lim_{x \to a} \ln f(x) = L$.

Then: $\lim_{x \to a} f(x) = e^L$

Example: Show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$. $\leftarrow 1^\infty$

Sol.: Let $f(x) = (1 + x)^{1/x} \Rightarrow \ln f(x) = \ln(1 + x)^{1/x} = \frac{1}{x} \ln(1 + x) = \frac{\ln(1 + x)}{x}$

مكان ما ادارة للوائيه

$\therefore \lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln(1 + x)}{x} = \frac{0}{0}$ (Indeterminate form)

$\ln$ رسم

∴ الاقتراب العدد من اليمين 0^+ ومن اليسار

By L'Hopital's Rule $\therefore \lim_{x \to 0^+} \frac{\ln(1 + x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{(1 + x)}}{1} = \frac{\frac{1}{(1 + 0)}}{1} = \frac{1}{1} = 1$

Therefore $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + x)^{1/x} = e^1 = e$ o.k.

<u>*Examples:*</u> Find the limits of the following:

1. $\lim_{x\to 0}(\sec^3 2x)^{\cot^2 3x} = (\sec^3(2*0))^{\cot^2 3*0} = 1^\infty$ (indeterminate form)

Sol.: Let $y = (\sec^3 2x)^{\cot^2 3x}$

$$\therefore \ln y = \ln(\sec^3 2x)^{\cot^2 3x} = \ln(\sec 2x)^{3\cot^2 3x} = 3\cot^2 3x \ln \sec 2x = \frac{3\ln \sec 2x}{\tan^2 3x}$$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{3\ln \sec 2x}{\tan^2 3x} = \frac{3\ln \sec 2*0}{\tan^2 3*0} = \frac{3\ln 1}{0} = \frac{0}{0} \quad \text{(also indeterminate form)}$$

$$\therefore \lim_{x\to 0} \frac{3\dfrac{\sec 2x \tan 2x * 2}{\sec 2x}}{2\tan 3x \sec^2 3x * 3} = \lim_{x\to 0} \frac{\tan 2x}{\tan 3x \sec^2 3x} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x}{\tan 3x * [2\sec 3x * \sec 3x \tan 3x * 3] + \sec^2 3x[\sec^2 3x * 3]}$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x}{6\sec^2 3x \tan^2 3x + 3\sec^4 3x} = \frac{2\sec^2 0}{6\sec^2 0 \tan^2 0 + 3\sec^4 0} = \frac{2*1}{6*0+3*1} = \frac{2}{3}$$

$$\therefore \lim_{x\to 0} \ln y = \frac{2}{3} \quad \Rightarrow \quad \lim_{x\to 0} y = \lim_{x\to 0}(\sec^3 2x)^{\cot^2 3x} = e^{2/3}$$

2. $\lim_{x\to 1} x^{\left(\frac{1}{x-1}\right)} = 1^{\left(\frac{1}{1-1}\right)} = 1^{\left(\frac{1}{0}\right)} = 1^\infty$ (indeterminate form)

Sol.: Let $y = x^{\left(\frac{1}{x-1}\right)} \quad \Rightarrow \quad \ln y = \ln x^{\left(\frac{1}{x-1}\right)} = \frac{1}{x-1}\ln x = \frac{\ln x}{x-1}$

$$\therefore \lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln x}{x-1} = \frac{\ln 1}{1-1} = \frac{0}{0} \quad \text{(also indeterminate form)}$$

$$= \lim_{x\to 1} \frac{1/x}{1} = \frac{1/1}{1} = 1$$

$$\therefore \lim_{x\to 1} y = \lim_{x\to 1} x^{\left(\frac{1}{x-1}\right)} = e^1 = e$$

3. $\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x} = \infty^0$ (indeterminate form)

Sol.: Let $y = (\tan x)^{\cos x} \quad \Rightarrow \quad \ln y = \ln(\tan x)^{\cos x} = \cos x \ln(\tan x) = \frac{\ln(\tan x)}{\sec x}$

$$\therefore \lim_{x\to \frac{\pi}{2}^-} \ln y = \lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\tan x)}{\sec x} = \frac{\ln \infty}{\infty} = \frac{\infty}{\infty} \quad \text{(also indeterminate form)}$$

9.

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\cos x}}{\left(\frac{\sin x}{\cos x}\right)^2}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

$$\therefore \lim_{x\to\frac{\pi}{2}^-} y = \lim_{x\to\frac{\pi}{2}^-} (\tan x)^{\cos x} = e^0 = 1$$

$\lim_{x\to\frac{\pi}{2}} \ln y = 0$

$\ln y = 0$

$e^{\ln y} = e^0$

$\therefore y = e^0$

4. $\lim_{x\to 0} x^{\sin x} = 0^0$ (indeterminate form)

Sol.: Let $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = \sin x \ln x = \frac{\ln x}{\csc x}$

$$\therefore \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln x}{\csc x} = \frac{\ln 0}{\infty} = \frac{-\infty}{\infty} \quad \text{(also indeterminate form)}$$

$$= \lim_{x\to 0} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = \lim_{x\to 0} \frac{-\sin^2 x}{x \cos x} = \frac{0}{0}$$

$\lim_{x\to 0^+} \ln x = -\infty$

$$= \lim_{x\to 0} \frac{-2\sin x \cos x}{-x\sin x + \cos x} = \lim_{x\to 0} \frac{-\sin 2x}{-x\sin x + \cos x} = \frac{0}{0+1} = 0$$

$$\therefore \lim_{x\to 0} y = \lim_{x\to 0} x^{\sin x} = e^0 = 1$$

Matrix

definition of a matrix

A matrix is defined as a rectangular array of quantities arranged in rows and columns.

A matrix with m rows and n columns can be expressed as follows.

$$A = [A] = \begin{array}{c} \left[\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ A_{31} & \cdots & \cdots & A_{ij} & A_{3n} \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{array}\right] \text{ i th row} \\ \text{j th column} \end{array}$$

Types of matrices

column matrix (vector)

$$B = \{B\} = \begin{bmatrix} 35 \\ 9 \\ 12 \\ 3 \end{bmatrix}$$

Row matrix

$$C = [9 \quad 35 \quad -12 \quad 7]$$

square matrix

$$A = \begin{bmatrix} 6 & 12 & 0 & 20 \\ 15 & -9 & -37 & 3 \\ -24 & 13 & 8 & 1 \\ 40 & 0 & 11 & -5 \end{bmatrix}$$

main diagonal

symmetric matrix

$$A = \begin{bmatrix} 6 & 15 & -24 & 40 \\ 15 & -9 & 13 & 0 \\ -24 & 13 & 8 & 11 \\ 40 & 0 & 11 & -5 \end{bmatrix}$$

lower triangular matrix

$$A = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 12 & -9 & 0 & 0 \\ 33 & 17 & 6 & 0 \\ -2 & 5 & 15 & 3 \end{bmatrix}$$

upper triangular matrix

$$A = \begin{bmatrix} -7 & 6 & 17 \\ 0 & 12 & 11 \\ 0 & 0 & 20 \end{bmatrix}$$

diagonal matrix

$$A = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 27 \end{bmatrix}$$

unit or Identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null matrix

$$O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplication of matrices

$$A = \begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}$$

3×2 $\qquad$ 2×2

$$A \cdot B = C$$

$$(l \times m) \; (m \times n) \qquad (l \times n)$$

equal

$$c_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

$$C = AB = \underset{(3\times 2)}{\begin{bmatrix} 1 & 8 \\ 4 & -2 \\ -5 & 3 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} 6 & -7 \\ -1 & 2 \end{bmatrix}} = \underset{(3\times 2)}{\begin{bmatrix} -2 & 9 \\ 26 & -32 \\ -33 & 41 \end{bmatrix}}$$

Transpose of matrix

$$B = \begin{bmatrix} 2 & -4 \\ -5 & 8 \\ 1 & 3 \end{bmatrix}_{3\times 2}$$

$$B^T = \begin{bmatrix} 2 & -5 & 1 \\ -4 & 8 & 3 \end{bmatrix}_{2\times 3}$$

~~Inverse of a square matrix~~

determinant

① if matrix 2×2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = (a_{11}\,a_{22}) - (a_{12}\,a_{21})$$

② if matrix 3×3

$$\cancel{A} \quad C_{ij} = (-1)^{i+j} M_{ij} \quad \{\text{co-factor of a matrix } [A]\}$$

where M_{ij} are the minors of matrix $[A]$.

$$\therefore |A| = \sum_{i=1}^{n} A_{ij}\, C_{ij} \quad (j = 1, 2, \ldots n)$$

or

$$|A| = \sum_{j=1}^{n} A_{ij}\, C_{ij} \quad (i = 1, 2, \ldots n)$$

example : Find the determinant of the matrix

$$[A] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 2 \\ 4 & 0 & 3 \end{bmatrix}$$

Solution : for column $\underline{\underline{2}}$ of matrix $[A]$

$$|A| = 6 * (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} + 5 * (-1)^{2+2} \begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix}$$

$$+ 0 (-1)^{3+2} \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix}$$

$$\therefore |A| = -6 (3*3 - 2*4) + 5 (5*3 - 2*4) + 0$$

$$\therefore |A| = 29$$

Inverse of square matrix

the inverse of a square matrix A is defined as a matrix A^{-1} with elements of such magnitudes that the product of the original matrix A and its inverse A^{-1} equals a unit matrix I ; that is

$$A A^{-1} = A^{-1} A = I$$

$$[A]^{-1} = \frac{adj\,[A]^T}{|A|}$$

Example : Find $[A]^{-1}$ for $[A]$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution :

for row 1

~~|A| =~~

$$|A| = (2) * (-1)^{1+1} \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} + (1) * (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$+ (0)$$

$$\therefore |A| = 12$$

Find co-factor

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = 10$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = -8$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 2$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = -3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = 6$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} = 1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = -2$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 5$$

$$\therefore C(A) = \begin{bmatrix} 10 & -8 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 5 \end{bmatrix}$$

$$\operatorname{adj} = (C(A))^T = \begin{bmatrix} 10 & -3 & 1 \\ -8 & 6 & -2 \\ 2 & -3 & 5 \end{bmatrix}$$

$$[A]^{-1} = \frac{1}{12}\begin{bmatrix} 10 & -3 & 1 \\ 8 & 6 & -2 \\ 2 & -3 & 5 \end{bmatrix}$$

$$\therefore [A]^{-1} = \begin{bmatrix} \frac{5}{6} & \frac{-1}{4} & \frac{1}{12} \\ \frac{-2}{3} & \frac{1}{2} & \frac{-1}{16} \\ \frac{1}{6} & \frac{-1}{4} & \frac{5}{12} \end{bmatrix}$$

Solution of simultaneous equations

① Cramer method

example: solve a system of linear equations.

$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 - x_2 - 2x_3 = 9$$
$$4x_1 + 3x_2 - 3x_3 = 3$$

Solution: $Ax = B$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 9 \\ 3 \end{bmatrix}$$

* for column 1

$$|A| = 1 * (-1)^{1+1} \begin{vmatrix} -1 & -2 \\ 3 & -3 \end{vmatrix} + 3(-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix} + 4(-1)^{3+1} \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$$\therefore |A| = (3+6) - 3[-6-3] + 4[-4+1]$$
$$= 9 - 3(-9) + 4(-3)$$
$$= 9 + 27 - 12$$
$$= 24$$

$$x_i = \frac{|A_i|}{|A|}$$

$$\therefore x_1 = \frac{|A_1|}{|A|}$$

$$\therefore x_1 = \frac{\begin{vmatrix} 0 & 2 & 1 \\ 9 & -1 & -2 \\ 3 & 3 & -3 \end{vmatrix}}{24}$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 9 & -1 & -2 \\ 3 & 3 & -3 \end{vmatrix} = 72$$

$$\therefore x_1 = \frac{72}{24} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 3 & 9 & -2 \\ 4 & 3 & -3 \end{vmatrix}}{24} = \frac{-48}{24} = -2$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{24}{24} = 1$$

② Matrix inverse method

example: solve a system of linear equations

$$x_1 + x_2 + 2x_3 = 9$$
$$2x_1 + 4x_2 - 3x_3 = 1$$
$$3x_1 + 6x_2 - 5x_3 = 0$$

Solution: write the system as $Ax = B$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$$A^{-1} = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \qquad \text{H.w}$$

$$\therefore x = A^{-1} B = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$