

<b>Subject :</b>	<b>Mathematics I</b>	<b>Units :</b>	<b>3</b>
<b>Code</b>	<b>E111</b>		
<b>Level</b>	<b>1<sup>st</sup> level</b>	<b>Theoretical :</b>	<b>3 Hr/wk</b>
<b>Pre-requisite:</b>	None	<b>Practical :</b>	

### **Algebraic Preliminaries**

Numbers, Sets, Inequalities, Absolute value.

### **Functions**

Domain, Range, graphs, Symmetry, Asymptotes.

### **Limits**

Definition of Limit, Theorems, Continuity, One-Sided Limits, Limits at Infinity, L Hopital's rule.

### **Derivatives**

Definition, Power and Sum Rules, Product and Quotient Rules, Chain rule, High-Order derivatives, Implicit differentiation.

### **Applications of Derivative**

Maximum and minimum, mean value theorem, Increasing and Decreasing Functions, Concavity and Points of inflection, Second Derivative Test.

### **Definite Integration**

Definition, Integral Theorems, Length of a Curve, Areas, Volume of Solids, Surface Area, Indefinite Integrals.

### **Transcendental Functions**

Trigonometric Functions, Graphs, Derivatives of trigonometric functions, Inverse trigonometric functions, Graphs, Derivatives of Inverse trigonometric functions, Natural Logarithm Functions, Exponential Functions, Functions  $a^u$  and  $\log_a u$ .

## PRELIMINARIES

### Sets and Intervals

Definitions:

**Set:** is a collection of objects, and these objects are the **elements** of the set. If  $S$  is a set, the notation  $a \in S$  means that  $a$  is an element of  $S$ , and  $a \notin S$  means that  $a$  is not an element of  $S$ . If  $S$  and  $T$  are sets, then  $S \cup T$  is their **union** and consists of all elements belonging either to  $S$  or  $T$  (or to both  $S$  and  $T$ ). The **intersection**  $S \cap T$  consists of all elements belonging to both  $S$  and  $T$ . The **empty set (null set)**  $\phi$  is the set that contains no elements.

Some sets can be described by listing their elements in braces. For instance, the set  $A$  consisting of positive integers less than 6 can be expressed as:

$$A = \{1, 2, 3, 4, 5\}.$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x: x \text{ is an integer and } 0 < x < 6\},$$

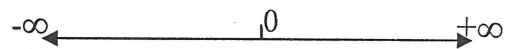
is the set of positive integers less than 6.

**Subset:** if each element of a set  $A$  is a member of a set  $B$ , then  $A$  is a subset of  $B$  and we write  $A \subseteq B$ . If the set  $B$  contains at least one element that is not a member of  $A$ , then  $A$  is a proper subset of  $B$  and this situation is indicated by  $A \subset B$ .

**Real numbers ( $R$ ):** are numbers that can be expressed as decimals, such as

$$-3/4 = -0.75000\dots, \quad 1/3 = 0.3333\dots, \quad \sqrt{2} = 1.4142\dots$$

The dots  $\dots$  in each case indicate that the sequence of decimal digits goes on forever. The real numbers can be represented geometrically as points on a number line called the **real number line**.  $R = \{-\infty, +\infty\}$



Real number line

These are four special subsets of real numbers:

1. The **natural numbers ( $N$ ):** consist of zero and positive integer numbers only.  $N = \{0, 1, 2, 3, \dots, +\infty\}$ .
2. The **integer numbers ( $I$ ):** consist of positive and negative integer numbers only.  $I = \{-\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, +\infty\}$ .
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction  $m/n$ , where  $m$  and  $n$  are integers and  $n \neq 0$ . Examples are:

$$1/3, \quad -4/9, \quad 200/13, \quad \text{and } 57 = 57/1.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

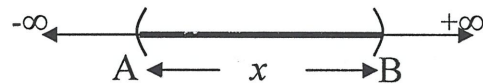


- (a) terminating (ending in an infinite string of zeros), for example,  $3/4 = 0.75000\dots$  or
- (b) eventually repeating (ending with a block of digits that repeats over and over), for example,  $23/11 = 2.090909\dots = 2.\overline{09}$ .

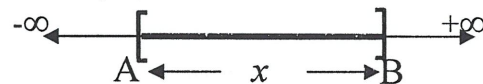
4. The **irrational numbers**, they are characterized by having nonterminating and nonrepeating decimal expansions. Examples are:  $\pi, \sqrt{2}, \sqrt[3]{5}, \log_{10} 3, \sin 41^\circ, 2^{\sqrt{3}}$ .

**Interval:** is a set of all real numbers between two points on the real number line (it is a subset of real numbers).

**1. Open interval:** is a set of all real numbers between A&B excluded (A&B are not elements in the set).  $(A, B) = \{x: A < x < B\}$ .

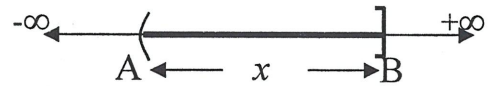


**2. Closed interval:** is a set of all real numbers between A&B included (A&B are elements in the set).  $[A, B] = \{x: A \leq x \leq B\}$ .

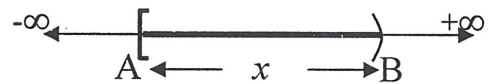


**3. Half-open interval:** is a set of all real numbers between A&B with one of the end-points as an element in the set.

a)  $(A, B] = \{x: A < x \leq B\}$



b)  $[A, B) = \{x: A \leq x < B\}$

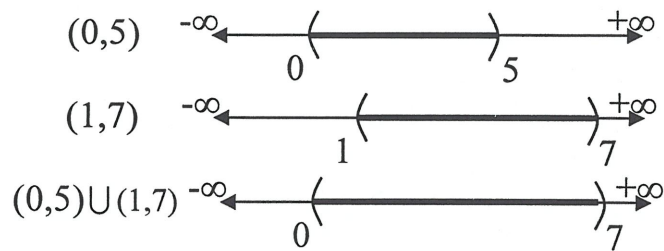


	Notation	Set description	Type	Picture
Finite:	$(a, b)$	$\{x a < x < b\}$	Open	
	$[a, b]$	$\{x a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x a < x \leq b\}$	Half-open	
Infinite:	$(a, \infty)$	$\{x x > a\}$	Open	
	$[a, \infty)$	$\{x x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x x < b\}$	Open	
	$(-\infty, b]$	$\{x x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Examples: Solve for  $x$  the following

1.  $\{x: 0 < x < 5\} \cup \{x: 1 < x < 7\}$

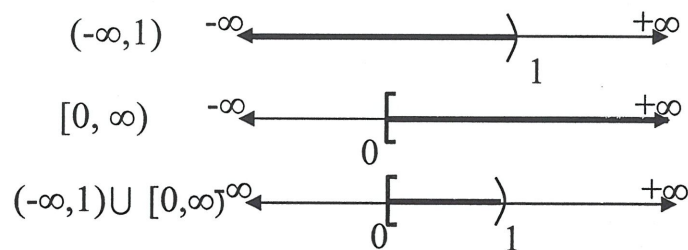
Sol: From the number line:



$\therefore$  The interval is  $\{0 < x < 7\}$  or  $(0, 5) \cup (1, 7) = (0, 7)$ .

2.  $\{x: x < 1\} \cap \{x: x \geq 0\}$

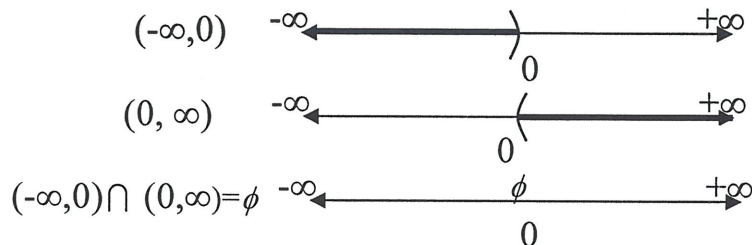
Sol: From the number line:



$\therefore$  The interval is  $\{x: 0 \leq x < 1\}$  or  $[0, 1)$ .

3.  $\{x: x < 0\} \cap \{x: x > 0\}$

Sol: From the number line:



**Inequalities:** At some times it is not sufficient to say that two numbers  $a$  and  $b$  are unequal. We can say something about unequal such as  $a$  is greater than  $b$  ( $a > b$ ) if and only  $(a - b)$  is positive and  $a$  is less than  $b$  ( $a < b$ ) if and only  $(a - b)$  is negative.

**Rules for inequalities:** If  $a$ ,  $b$ , and  $c$  are real numbers, then:

1.  $a < b \Rightarrow a + c < b + c$
  2.  $a < b \Rightarrow a - c < b - c$
  3.  $a < b$  and  $c > 0 \Rightarrow ac < bc$
  4.  $a < b$  and  $c < 0 \Rightarrow ac > bc$
- Special case  $a < b \Rightarrow -a > -b$

$$5. a > 0 \Rightarrow \frac{1}{a} > 0$$

$$6. \text{ If } a \text{ and } b \text{ are both positive or both negative, then } a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$$

**Solving inequalities:** The process of finding the interval or intervals of numbers that satisfy an inequality in  $x$  is called **solving** the inequality.

Examples: Solve the following inequalities and show their solution set

$$1. 3 + 7x \leq 2x - 9$$

$$\text{Sol: } 3 + 7x - 3 \leq 2x - 9 - 3 \Rightarrow 7x \leq 2x - 12 \Rightarrow 7x - 2x \leq 2x - 12 - 2x \\ \Rightarrow 5x \leq -12 \Rightarrow x \leq -12/5$$

The solution set is  $(-\infty, -12/5]$ .



$$2. 7 \leq 2 - 5x < 9$$

$$\text{Sol: } 7 - 2 \leq 2 - 5x - 2 < 9 - 2 \Rightarrow 5 \leq -5x < 7 \Rightarrow 5/(-5) \leq -5x/(-5) < 7/(-5) \\ \Rightarrow -1 \geq x > -7/5 \Rightarrow -1 \leq x \leq -7/5$$

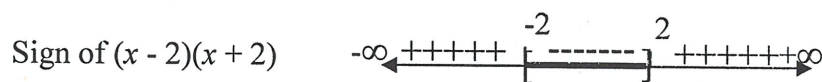
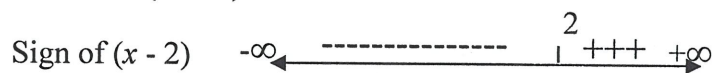
The solution set is  $(-7/5, -1]$ .



$$3. x^2 \leq 4$$

$$\text{Sol: } x^2 \leq 4 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x - 2)(x + 2) \leq 0$$

$$\text{Let } (x - 2)(x + 2) = 0 \Rightarrow \text{either } (x - 2) = 0 \Rightarrow x = 2 \\ \text{or } (x + 2) = 0 \Rightarrow x = -2$$

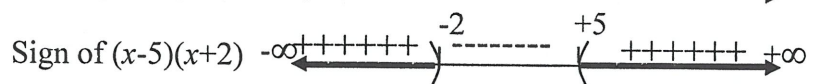
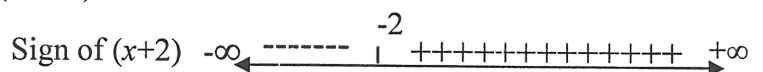


$\therefore$  The solution is  $\{x: -2 \leq x \leq 2\}$  or  $[-2, 2]$ .

$$4. x^2 - 3x > 10$$

$$\text{sol: } x^2 - 3x - 10 > 0 \Rightarrow (x - 5)(x + 2) > 0$$

$$\text{Let } (x - 5)(x + 2) = 0 \Rightarrow \text{either } (x - 5) = 0 \Rightarrow x = 5 \\ \text{or } (x + 2) = 0 \Rightarrow x = -2$$

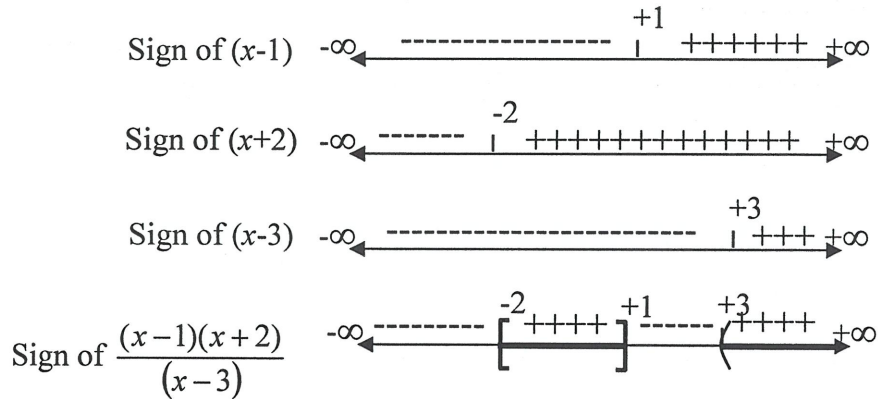


The solution set is  $(-\infty, -2) \cup (5, \infty)$ .



$$5. \frac{(x-1)(x+2)}{x-3} \geq 0$$

Sol:



$\therefore$  The solution set is  $\{x: -2 \leq x \leq +1\} \cup \{x: x > +3\}$   
or  $[-2,+1] \cup (+3, \infty)$ .

$$6. \frac{6}{x-1} \geq 5$$

Sol:

This inequality can hold only if  $x > 1$ , because otherwise  $6/(x - 1)$  is undefined or negative.

$$6 \geq 5x - 5 \Rightarrow 11 \geq 5x \Rightarrow 11/5 \geq x \text{ or } x \leq 11/5$$

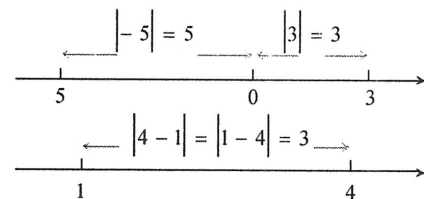
$\therefore$  The solution set is the half-open interval  $(1, 11/5]$ .

**Absolute value:** The absolute value of a number  $x$ , denoted by  $|x|$ , is defined by the formula:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example: Finding absolute values.

$$|3| = 3, |0| = 0, |-5| = -(-5) = 5.$$



Geometrically, the absolute value of  $x$  is the distance from  $x$  to 0 on the real number line. Since distances are always positive or 0, we see that  $|x| \geq 0$  for every real number  $x$ , and  $|x| = 0$  if and only if  $x = 0$ . Also  $|x - y|$  is the distance between  $x$  and  $y$  on the real line.

### Absolute Value Properties

1.  $|-a| = |a|$  A number and its additive inverse or negative have the same absolute value.
2.  $|ab| = |a||b|$  The absolute value of a product is the product of the absolute values.
3.  $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$  The absolute value of a quotient is the quotient of the absolute values.
4.  $|a + b| \leq |a| + |b|$  The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

### Absolute Values and Intervals

If  $a$  is any positive number, then

5.  $|x| = a$  if and only if  $x = \pm a$
6.  $|x| < a$  if and only if  $-a < x < a$
7.  $|x| > a$  if and only if  $x > a$  or  $x < -a$
8.  $|x| \leq a$  if and only if  $-a \leq x \leq a$
9.  $|x| \geq a$  if and only if  $x \geq a$  or  $x \leq -a$

Examples:

1. Solve the equation  $|2x - 3| = 1$

Sol:  $2x - 3 = \pm 1 \Rightarrow$  either  $2x - 3 = 1 \Rightarrow 2x = 4 \Rightarrow x = 2$   
or  $2x - 3 = -1 \Rightarrow 2x = 2 \Rightarrow x = 1$

$\therefore$  The solutions are  $x = 1$  and  $x = 2$ .

2. Solve the inequality  $|2x - 3| \leq 1$

Sol:  $|2x - 3| \leq 1 \Rightarrow -1 \leq 2x - 3 \leq 1 \Rightarrow 2 \leq 2x \leq 4 \Rightarrow 1 \leq x \leq 2$

$\therefore$  The solution set is the closed interval  $[1, 2]$ .

3. Solve the inequality  $|2x - 3| \geq 1$

Sol:  $|2x - 3| \geq 1 \Rightarrow$  either  $2x - 3 \geq 1 \Rightarrow 2x \geq 4 \Rightarrow x \geq 2$   
or  $2x - 3 \leq -1 \Rightarrow 2x \leq 2 \Rightarrow x \leq 1$

$\therefore$  The solution set is  $(-\infty, 1] \cup [2, \infty)$ .

4. Solve the inequality  $\left|5 - \frac{2}{x}\right| < 1$

Sol:  $\left|5 - \frac{2}{x}\right| < 1 \Rightarrow -1 < 5 - \frac{2}{x} < 1 \Rightarrow -6 < -\frac{2}{x} < -4$

$\Rightarrow 3 > \frac{1}{x} > 2 \Rightarrow \frac{1}{3} < x < \frac{1}{2}$

$\therefore$  The solution set is the open interval  $(\frac{1}{3}, \frac{1}{2})$ .

5. Solve the inequality  $|x-3|+|x+2|<11$ .

Sol. Recall the definition of absolute value:

$$y = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\text{It follows that: } |x-3| = \begin{cases} x-3 & \text{if } x-3 > 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \geq 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

$$\text{Similarly, } |x+2| = \begin{cases} x+2 & \text{if } x+2 \geq 0 \\ -(x+2) & \text{if } x+2 < 0 \end{cases} = \begin{cases} x+2 & \text{if } x \geq -2 \\ -x-2 & \text{if } x < -2 \end{cases}$$

These expressions show that we must consider three cases:

$$x < -2, \quad -2 \leq x < 3 \quad \text{and} \quad x \geq 3$$

Case I: if  $x < -2$  we have

$$\begin{aligned} |x-3|+|x+2| &< 11 \\ -x+3-x-2 &< 11 \\ -2x &< 10 \\ x &> -5 \end{aligned}$$

Case II: if  $-2 \leq x < 3$  we have

$$\begin{aligned} |x-3|+|x+2| &< 11 \\ -x+3+x+2 &< 11 \\ 5 &< 11 \quad (\text{always true}) \end{aligned}$$

Case III: if  $x \geq 3$  we have

$$\begin{aligned} |x-3|+|x+2| &< 11 \\ x-3+x+2 &< 11 \\ x &< 6 \end{aligned}$$

Combining cases I, II, and III, we see that the inequality is satisfied when

$$-5 < x < 6.$$

So the solution is the interval  $(-5, 6)$ .

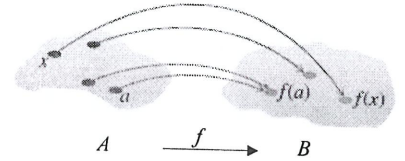


## Functions

**Definition:**

A **function**  $f$  is a rule that assigns to each element  $x$  in a set  $A$  exactly one element, called  $f(x)$ , in a set  $B$ .

- ❖ The set  $A$  is called the **domain** of the function  $f$ .
- ❖ The number  $f(x)$  or  $y$  is the value of  $f$  at  $x$ .
- ❖ The **range** of  $f$  is the set of all possible values of  $f(x)$  as  $x$  varies throughout the domain.
- ❖  $x$  &  $y$  are variables.
- ❖  $x$  is called the independent variable.
- ❖  $y$  is called the dependent variable.



Domain ( $D_f$ ): is the set of all possible inputs ( $x$ -values).

Range ( $R_f$ ): is the set of all possible outputs ( $y$ -values).

To find Domain ( $D_f$ ) and Range ( $R_f$ ) the following points must be noticed:

1. The denominator in a function must not equal zero (never divide by zero).
2. The values under even roots must be positive.

Examples: Find the domain ( $D_f$ ) and range ( $R_f$ ) of the following functions:

1.  $y = f(x) = \frac{1}{x}$

Sol: denominator must not equal zero  $\Rightarrow x \neq 0 \Rightarrow D_f = \{x: x \neq 0\}$ .

To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$\therefore x = \frac{1}{y} \Rightarrow R_f = \{y: y \neq 0\}, \text{ or } R_f = \mathbb{R} \setminus \{0\}, \text{ or } R_f = (-\infty, 0) \cup (0, \infty).$$

2.  $y = f(x) = \sqrt{4-x}$

Sol: The values under even roots must be positive

$$\Rightarrow 4 - x \geq 0 \Rightarrow 4 \geq x$$

$$\therefore D_f = \{x: x \leq 4\}.$$

To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$y = \sqrt{4-x} \Rightarrow y^2 = 4-x$$

$$\Rightarrow x = 4 - y^2 \Rightarrow$$

$$\therefore R_f = \mathbb{R}.$$

But the values of  $y$  must be always positive, we must exclude negative values,

$$\Rightarrow R_f = \{y: y \geq 0\}, \text{ or } R_f = [0, \infty).$$

3.  $y = f(x) = \sqrt{1-x^2}$

Sol: The values under even roots must be positive

$$\Rightarrow 1 - x^2 \geq 0 \Rightarrow -x^2 \geq -1 \Rightarrow x^2 \leq 1 \Rightarrow \text{either } x \leq 1, \text{ or } -x \leq 1 \Rightarrow x \geq -1$$

$$\therefore D_f = \{x: -1 \leq x \leq 1\}.$$

To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$$

$$\Rightarrow x^2 = 1-y^2 \Rightarrow x = \pm\sqrt{1-y^2}$$

So the values under even roots must be positive

$$1-y^2 \geq 0 \Rightarrow -y^2 \geq -1 \Rightarrow y^2 \leq 1 \Rightarrow \text{either } y \leq 1, \text{ or } -y \leq 1 \Rightarrow y \geq -1$$

$$\therefore R = \{y: -1 \leq y \leq 1\}.$$

But the values of  $y$  must be always positive, we must exclude negative values,

$$\Rightarrow R_f = \{y: 0 \leq y \leq 1\}.$$

$$4. y = f(x) = \frac{1}{\sqrt{9-x^2}}$$

Sol: The values under even roots must be positive and the denominator must not equal zero, so:

$$9-x^2 > 0 \Rightarrow -x^2 > -9 \Rightarrow x^2 < 9 \Rightarrow \text{either } x < 3, \text{ or } -x < 3 \Rightarrow x > -3$$

$$\therefore D_f = \{x: -3 < x < 3\}.$$

To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$y = \frac{1}{\sqrt{9-x^2}} \Rightarrow y^2 = \frac{1}{9-x^2} \Rightarrow 9-x^2 = \frac{1}{y^2} \Rightarrow x^2 = 9 - \frac{1}{y^2}$$

$$\Rightarrow x^2 = \frac{9y^2-1}{y^2} \Rightarrow x = \pm\sqrt{\frac{9y^2-1}{y^2}} \Rightarrow x = \pm\frac{\sqrt{9y^2-1}}{y}$$

The values under even roots must be positive, so:

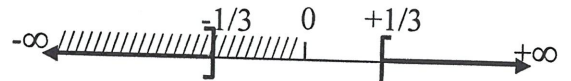
$$9y^2-1 \geq 0 \Rightarrow y^2 \geq 1/9 \Rightarrow \text{either } y \geq 1/3, \text{ or } -y \geq 1/3 \Rightarrow y \leq -1/3$$

$$\Rightarrow R_f = (-\infty, -1/3] \cup [1/3, \infty).$$

The denominator must not equal zero  $\Rightarrow y \neq 0$

But the values of  $y$  must be always positive; we must exclude negative values,

$$\Rightarrow R_f = [1/3, \infty), \text{ or } R_f = \{y: 1/3 \leq y \leq \infty\}.$$



$$5. y = f(x) = \frac{1}{x^2-9}$$

Sol: denominator must not equal zero  $\Rightarrow x^2-9 \neq 0 \Rightarrow x \neq \pm 3 \Rightarrow D_f = R \setminus \{-3, 3\}$ .

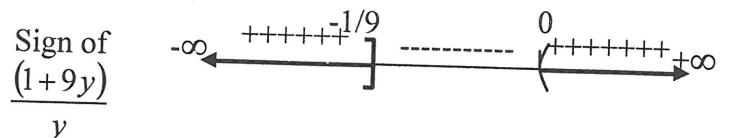
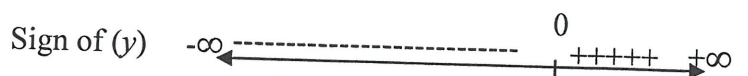
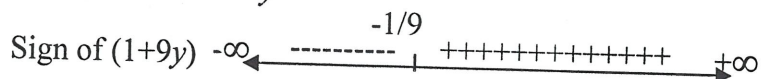
To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$y(x^2-9) = 1 \Rightarrow yx^2 - 9y = 1 \Rightarrow x = \pm\sqrt{\frac{1+9y}{y}}$$

The values under even roots must be positive, so  $\Rightarrow \frac{1+9y}{y} \geq 0$ .

$$\therefore R_f = (-\infty, -1/9] \cup (0, \infty).$$

$$\text{Or } R_f = R \setminus (-1/9, 0].$$



6.  $y = f(x) = -\sqrt{1-x^2}$

Sol: The values under even roots must be positive:

$$1 - x^2 \geq 0 \Rightarrow (1 - x)(1 + x) \geq 0$$

$$\therefore D_f = [-1, +1]$$

To find  $R_f$ : convert the function from  $y = f(x)$  into  $x = f(y)$ .

$$y = -\sqrt{1-x^2} \Rightarrow y^2 = 1-x^2$$

$$\Rightarrow x^2 = 1-y^2 \Rightarrow x = \mp\sqrt{1-y^2}$$

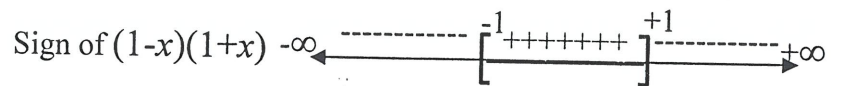
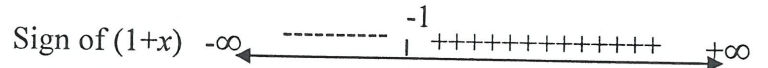
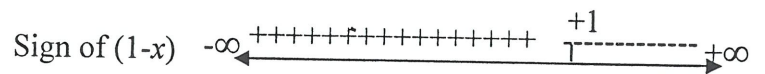
The values under even roots must be positive:

$$1 - y^2 \geq 0 \Rightarrow (1 - y)(1 + y) \geq 0$$

$$\Rightarrow R_f = [-1, +1]$$

But the values of  $y$  must be always negative; we must exclude positive values,

$$\Rightarrow R_f = [-1, 0] \therefore$$





## Chapter One

### Graph of Functions (Graph of Curves):

To graph the curve of a function, we can follow the following steps:

1. Find the domain and range of the function.
2. Check the symmetry of the function
3. Find (if any found) points of intersection with  $x$ -axis and  $y$ -axis.
4. Choose some another points on the curve.
5. Draw a smooth line through the above points.

#### Symmetry Tests for Graphs:

If  $f(x,y) = 0$  is any function then:

1. Symmetry about  $x$ -axis: If  $f(x,-y) = f(x,y)$
2. Symmetry about  $y$ -axis: If  $f(-x,y) = f(x,y)$  It is called an *even* function.
3. Symmetry about the origin: If  $f(-x,-y) = f(x,y)$  It is called an *odd* function

**Examples 1:** Check the symmetry of the graph of the following curves:

1.  $y = f(x) = x^2$

**sol.:**  $f(x,y) = x^2 - y = 0$

$\therefore$  (i)  $f(x,-y) = x^2 - (-y) = 0 \Rightarrow f(x,-y) = x^2 + y = 0 \neq f(x,y)$  not o.k.

(ii)  $f(-x,y) = (-x)^2 - y = 0 \Rightarrow f(-x,y) = x^2 - y = 0 = f(x,y)$  o.k.

(iii)  $f(-x,-y) = (-x)^2 - (-y) = 0 \Rightarrow f(-x,-y) = x^2 + y = 0 \neq f(x,y)$  not o.k.

So the function has symmetry only about  $y$ -axis. It is called an even function.

2.  $y = f(x) = x^3$

**Sol.**  $f(x,y) = x^3 - y = 0$

$\therefore$  (i)  $f(x,-y) = x^3 - (-y) = 0 \Rightarrow f(x,-y) = x^3 + y = 0 \neq f(x,y)$  not o.k.

(ii)  $f(-x,y) = (-x)^3 - y = 0 \Rightarrow f(-x,y) = -x^3 - y = 0 \neq f(x,y)$  not o.k.

(iii)  $f(-x,-y) = (-x)^3 - (-y) = 0 \Rightarrow f(-x,-y) = -x^3 + y = 0$  (multiply by -1)

$\Rightarrow f(-x,-y) = x^3 - y = 0 = f(x,y)$  o.k.

So the function has symmetry only about the origin. It is called an odd function.

**Chapter One**

3.  $x^2 = y^2 + 4$

**Sol.**  $f(x,y) = y^2 - x^2 + 4 = 0$

$\therefore$  (i)  $f(x,-y) = (-y)^2 - x^2 + 4 = 0 \Rightarrow f(x,-y) = y^2 - x^2 + 4 = 0 = f(x,y)$  o.k.

(ii)  $f(-x,y) = y^2 - (-x)^2 + 4 = 0 \Rightarrow f(-x,y) = y^2 - x^2 + 4 = 0 = f(x,y)$  o.k.

(iii)  $f(-x,-y) = (-y)^2 - (-x)^2 + 4 = 0 \Rightarrow f(-x,-y) = y^2 - x^2 + 4 = 0 = f(x,y)$  o.k.

So the function has symmetry about  $x$ -axis,  $y$ -axis and *the origin*.

**DEFINITIONS**                      **Even Function, Odd Function**

A function  $y = f(x)$  is an

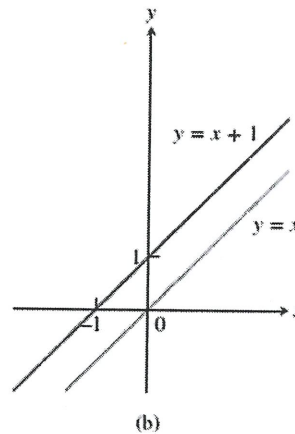
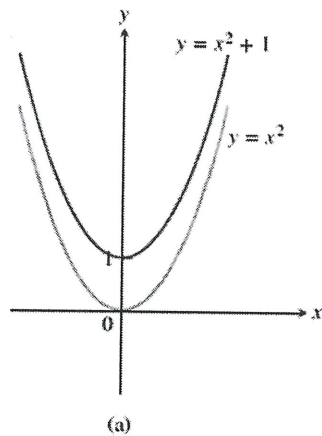
**even function of  $x$**  if  $f(-x) = f(x)$  symmetry about  $y$ -axis

**odd function of  $x$**  if  $f(-x) = -f(x)$  symmetry about origin

for every  $x$  in the function's domain.

**Examples 2:** Recognizing Even and Odd functions

- $f(x) = x^2$  Even function:  $(-x)^2 = x^2$  for all  $x$ ; symmetry about  $y$ -axis.
- $f(x) = x^2 + 1$  Even function:  $(-x)^2 + 1 = x^2 + 1$  for all  $x$ ; symmetry about  $y$ -axis.
- $f(x) = x$  Odd function:  $(-x) = -x$  for all  $x$ ; symmetry about *the origin*.
- $f(x) = x + 1$  Not odd:  $f(-x) = -x + 1$ , but  $-f(x) = -x - 1$ . The two are not equal.  
Not even:  $f(-x) = -x + 1$ , but  $f(x) = x + 1$ . for all  $x \neq 0$ .



**Example 3:** Sketch the graph of the curve  $y = f(x) = x^2 - 1$

**Chapter One**

**Sol.: Step 1:** Find  $D_f, R_f$  of the function?

$$D_f = (-\infty, \infty);$$

To find  $R_f$ : we must convert the function from  $y=f(x)$  into  $x=f(y)$ .

$$y = x^2 - 1 \Rightarrow x^2 = y + 1 \quad x = \pm\sqrt{y+1}$$

$$\text{So } y + 1 \geq 0 \Rightarrow y \geq -1 \Rightarrow R = [-1, \infty)$$

**Step 2:** Find  $x$  and  $y$  intercept?

$$\text{To find } x\text{-intercept put } y=0 \Rightarrow x^2 - 1 = 0 \Rightarrow x^2 = \pm 1$$

So  $x$ -intercept are  $(-1, 0)$  and  $(+1, 0)$ .

$$\text{To find } y\text{-intercept put } x=0 \Rightarrow y = 0 - 1 \Rightarrow y = -1$$

So  $y$ -intercept is  $(0, -1)$ .

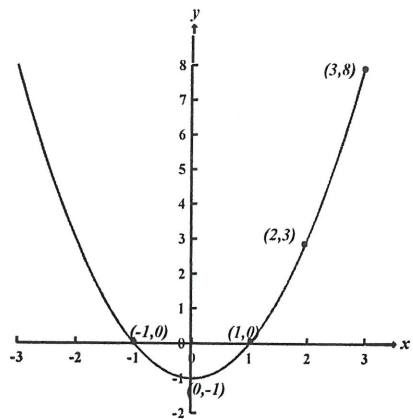
**Step 3:** check the symmetry:

$$f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$$

$$-f(x) = -(x^2 - 1) = -x^2 + 1 \neq f(x)$$

So it is an even function ( it is symmetric about  $y$ -axis).

**Step 4:** Choose some another point on the curve.



$x$	$y$
2	3
3	8

**Step 5:** Draw smooth line through the above points.

**Homework:** Draw the following functions:

1.  $y = f(x) = 3x^2 + 2$

2.  $x^2 + y^2 = 1$

3.  $y^2 = 4x - 1$

4.  $x = y^3$

5.  $y = [x]$ ; for  $-3 \leq x \leq 3$

6.  $y = x - [x]$ ; for  $-2 \leq x \leq 2$

7.  $y = \sqrt{4 - x}$



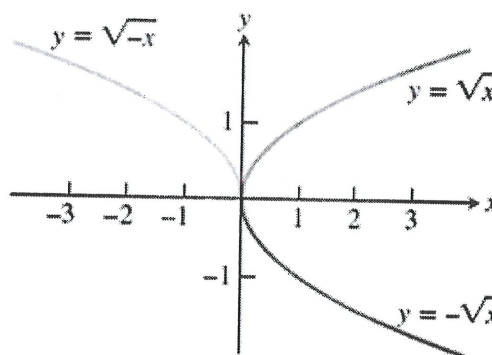
**Chapter One**

**Shifting, Shrinking and Stretching:**

<b>Shift formulas: (for <math>c &gt; 0</math>)</b>		
<b>Vertical shifts</b>		
$y = f(x) + c$	or	$y - c = f(x)$ shifts the graph of $f$ up by $c$ units.
$y = f(x) - c$	or	$y + c = f(x)$ shifts the graph of $f$ down by $c$ units.
<b>Horizontal shifts</b>		
$y = f(x + c)$		shifts the graph of $f$ left by $c$ units.
$y = f(x - c)$		shifts the graph of $f$ right by $c$ units.

<b>Shrinking, Stretching and Reflecting Formulas:</b>	
<b>(for <math>c &gt; 1</math>)</b>	
$y = c f(x)$	Stretches the graph of $f$ $c$ units along $y$ -axis.
$y = \frac{1}{c} f(x)$	Shrinks the graph of $f$ $c$ units along $y$ -axis.
$y = f(cx)$	Shrinks the graph of $f$ $c$ units along $x$ -axis.
$y = f\left(\frac{x}{c}\right)$	Stretches the graph of $f$ $c$ units along $x$ -axis.
<b>(for <math>c = -1</math>)</b>	
$y = -f(x)$	Reflects the graph of $f$ across the $x$ -axis.
$y = f(-x)$	Reflects the graph of $f$ across the $y$ -axis.

**Example 1:** The graph of  $y = -\sqrt{x}$  is a reflection of  $y = \sqrt{x}$  across the  $x$ -axis, and  $y = \sqrt{-x}$  is a reflection across the  $y$ -axis.



**Example 2:** Shift the graph of the function

$f(x) = x^2$ ; if  $D_f = \{x: -2 \leq x \leq 3\}$  and  $R_g = \{y: 0 \leq y \leq 9\}$ .

- (a) one unit right.
- (b) two units left.
- (c) one unit up.
- (d) two units down.

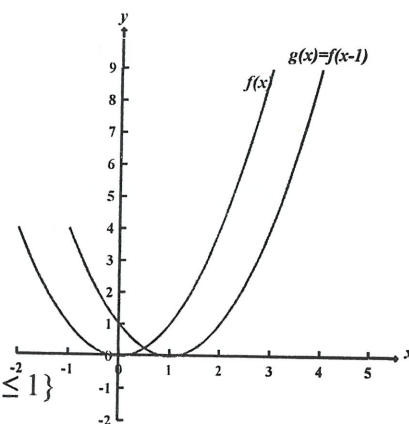
**Sol.:** (a) Shifting the function  $f(x)$  one unit right:

$g(x) = f(x-1) = (x-1)^2$  and  $D_g = \{x: -2 \leq x-1 \leq 3\} = \{x: -1 \leq x \leq 4\}$

**Note:** In case of horizontal shifts, the range of the function will not be changed.

**Chapter One**

$x$	$y=f(x)=x^2$	$x-1$	$y=g(x)=(x-1)^2$
-2	4	-	-
-1	1	-2	4
0	0	-1	1
1	1	0	0
2	4	1	1
3	9	2	4
4	-	3	9

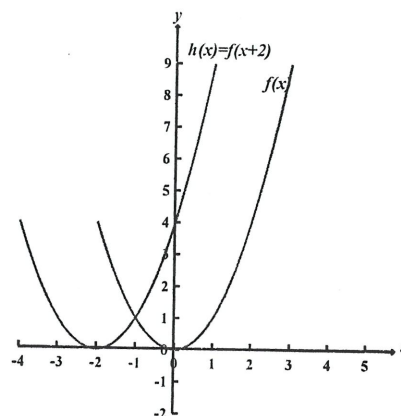


(b) Shifting the function  $f(x)$  two units left:

$$h(x) = f(x+2) = (x+2)^2 \text{ and } D_h = \{x: -2 \leq x+2 \leq 3\} = \{x: -4 \leq x \leq 1\}$$

**Note:** In case of horizontal shifts, the range of the function will not be changed.

$X$	$y=f(x)=x^2$	$x+2$	$y=h(x)=(x+2)^2$
-4	-	-2	4
-3	-	-1	1
-2	4	0	0
-1	1	1	1
0	0	2	4
1	1	3	9
2	4	-	-
3	9	-	-

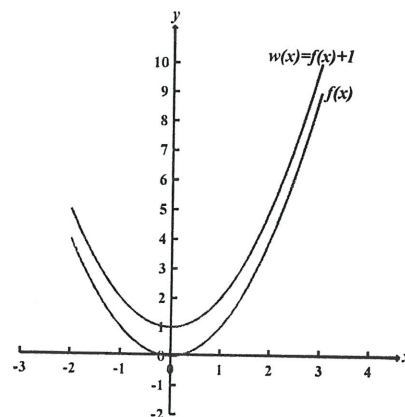


(c) Shifting the function  $f(x)$  one unit up:

$$w(x) = f(x)+1 = x^2 + 1 \text{ and } R_w = \{y: 0 \leq y-1 \leq 9\} = \{y: 1 \leq y \leq 10\}$$

**Note:** In case of vertical shifts, the domain of the function will not be changed.

$X$	$y=f(x)=x^2$	$y=w(x)=x^2+1$
-2	4	5
-1	1	2
0	0	1
1	1	2
2	4	5
3	9	10



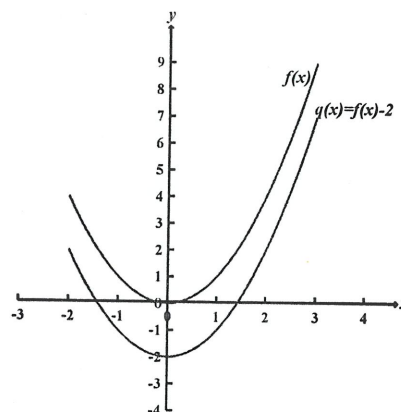
(d) Shifting the function  $f(x)$  two units down:

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$$q(x) = f(x) - 2 = x^2 - 2 \text{ and } R_q = \{y: 0 \leq y+2 \leq 9\} = \{y: -2 \leq y \leq 7\}$$

*Note:* In case of vertical shifts, the domain of the function will not be changed.

X	$y=f(x)=x^2$	$y=q(x)=x^2-2$
-2	4	2
-1	1	-1
0	0	-2
1	1	-1
2	4	2
3	9	7



**Example 3:** Sketch the graph of the curve  $y=f(x) = |x|$

**Sol.: Step1:** Find  $D_f, R_f$  of the function?

$$y = f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\Rightarrow D_f = (-\infty, \infty) \text{ and } R_f = [0, \infty);$$

**Step2:** Find  $x$  and  $y$  intercept?

To find  $x$ -intercept put  $y=0 \Rightarrow x=0$

To find  $y$ -intercept put  $x=0 \Rightarrow y=0$

So  $x$ - and  $y$ -intercept is  $(0,0)$ .

**Step 3:** check the symmetry:

$$f(-x) = |-x| = |x| = f(x)$$

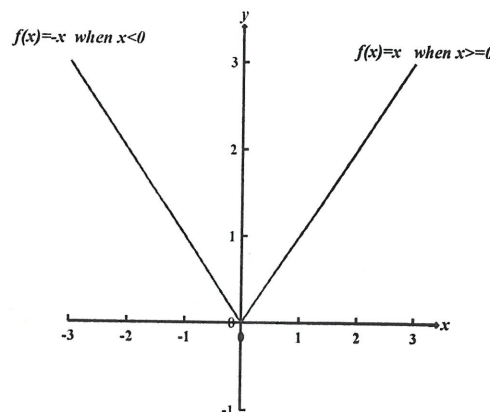
$$-f(x) = -|x| \neq f(x)$$

So it is an even function (it is symmetric about  $y$ -axis).

**Step 4:** Choose some another point on the curve.

x	y
1	1
2	2

**Step 5:** Draw smooth line through the above points.



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**Example 4:** Use graph of the function  $y=|x|$  to sketch the graph of the following functions, then show their domains and range

(a)  $y=|x+1|$

**Sol.**

$$y = |x+1| = \begin{cases} (x+1) & \text{if } (x+1) \geq 0 \\ -(x+1) & \text{if } (x+1) < 0 \end{cases}$$

$$= \begin{cases} (x+1) & \text{if } x \geq -1 \\ -x-1 & \text{if } x < -1 \end{cases}$$

Shifting the function  $y=|x|$  one unit left.

$D_f = (-\infty, \infty)$  and  $R_f = [0, \infty)$

(b)  $y=|x|+2$

**Sol.**  $y = |x| + 2 = \begin{cases} (x) + 2 & \text{if } (x) \geq 0 \\ (-x) + 2 & \text{if } (x) < 0 \end{cases}$

Shifting the function  $y=|x|$  two up.

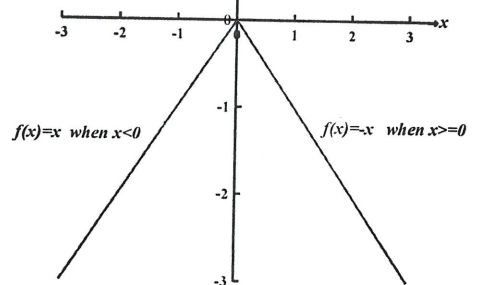
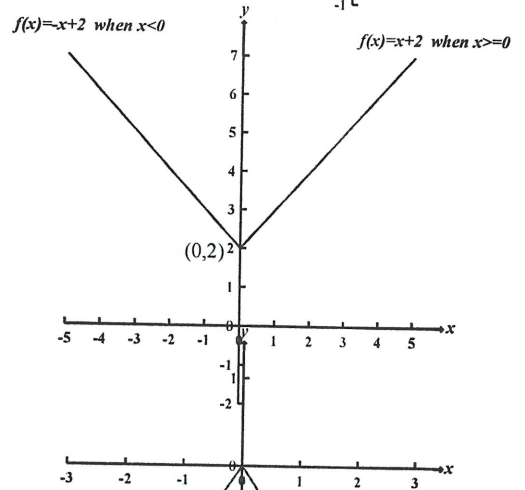
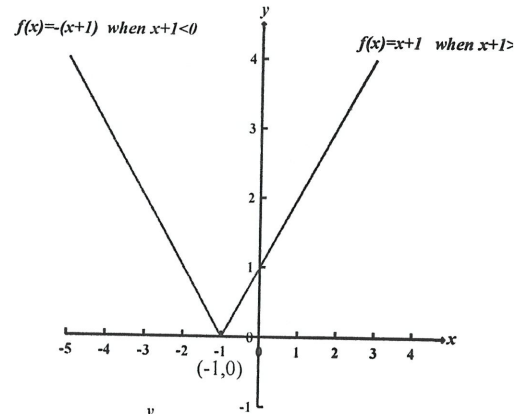
$D_f = (-\infty, \infty)$  and  $R_f = [2, \infty)$

(c)  $y=-|x|$

**Sol.**  $y = f(x) = -|x| = \begin{cases} -(x) = -x & \text{if } (x) \geq 0 \\ -(-x) = x & \text{if } (x) < 0 \end{cases}$

Reflecting the graph of the function  $y=|x|$  across  $x$ -axis.

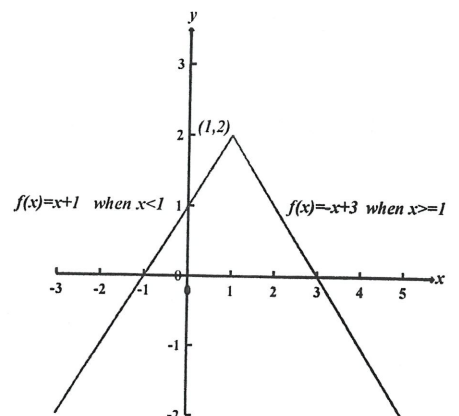
$D_f = (-\infty, \infty)$  and  $R_f = (-\infty, 0]$



(d)  $y=2-|1-x|$

**Sol.**  $y=2-|1-x| = -|1-x|+2 = -|x-1|+2$

$$= \begin{cases} -(x-1)+2 & \text{if } (x-1) \geq 0 \\ -(-(x-1))+2 & \text{if } x-1 < 0 \end{cases}$$





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$$= \begin{cases} -x+3 & \text{if } x \geq 1 \\ x+1 & \text{if } x < 1 \end{cases}$$

Reflecting the graph of the function  $y=|x|$  across  $x$ -axis, then shifting it one unit right and two units up.

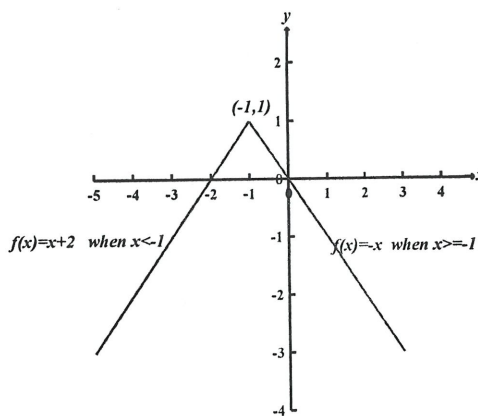
$$D_f = (-\infty, \infty) \text{ and } R_f = (-\infty, 2]$$

(e)  $y=1-|x+1|$

**Sol.**  $y=1-|x+1| = -|x+1|+1$

$$= \begin{cases} -(x+1)+1 & \text{if } (x+1) \geq 0 \\ -(-(x+1))+1 & \text{if } (x+1) < 0 \end{cases}$$

$$= \begin{cases} -x & \text{if } x \geq -1 \\ x+2 & \text{if } x < -1 \end{cases}$$



Reflecting the graph of the function  $y=|x|$  across  $x$ -axis, then shifting it one unit left and one unit up.

$$D_f = (-\infty, \infty) \text{ and } R_f = (-\infty, 1]$$

**Example 5:** If  $f(x) = \sqrt{4-x^2}$  which has  $D_f = [-2, 2]$  and  $R_f = [0, 2]$ , shrink and stretch it

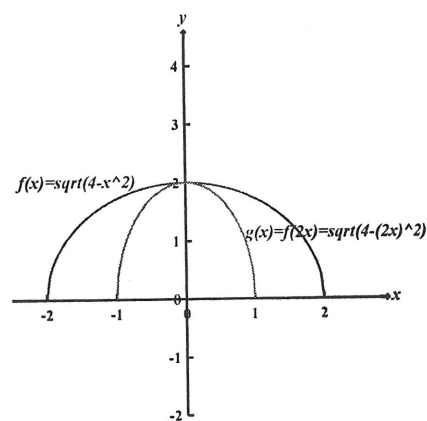
horizontally by two units and then sketch the original and resulting functions

**Sol.:** (a) shrinking:

$$g(x) = f(cx) = \sqrt{4-(2x)^2} = \sqrt{4-4x^2} = 2\sqrt{1-x^2}$$

$$D_g = \{x: -2 \leq 2x \leq 2\} = \{x: -1 \leq x \leq 1\}$$

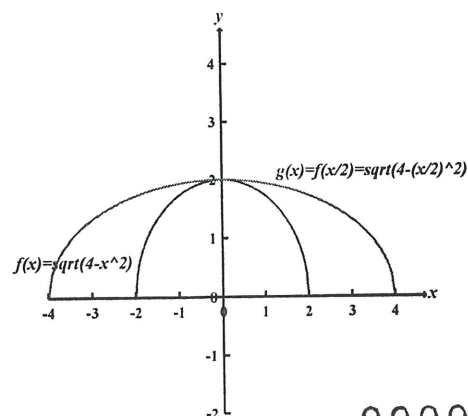
*Note:* In case of horizontal shrinks, the range of the function will not be changed.



(b) stretching:

$$g(x) = f\left(\frac{x}{c}\right) = \sqrt{4-\left(\frac{x}{2}\right)^2} = \sqrt{4-\frac{x^2}{4}} = \sqrt{\frac{16-x^2}{4}} = \frac{1}{2}\sqrt{16-x^2}$$

$$D_g = \{x: -2 \leq x/2 \leq 2\} = \{x: -4 \leq x \leq 4\}$$



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*Note:* In case of horizontal stretches, the range of the function will not be changed.

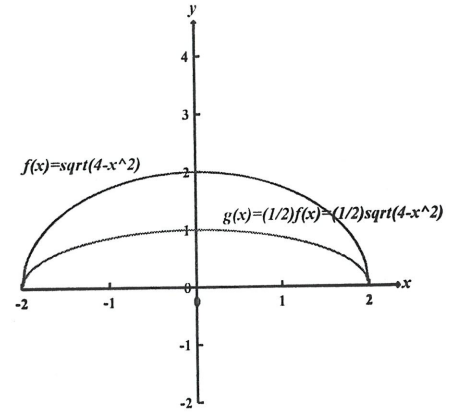
**Example 6:** Repeat the above example but here shrink and stretch the function vertically.

**Sol.:** (a) shrinking:

$$g(x) = \frac{1}{c} f(x) = \frac{1}{2} \sqrt{4-x^2}$$

$$R_g = \{y: 0 \leq 2y \leq 2\} = \{y: 0 \leq y \leq 1\}$$

*Note:* In case of vertical shrinks, the domain of the function will not be changed.

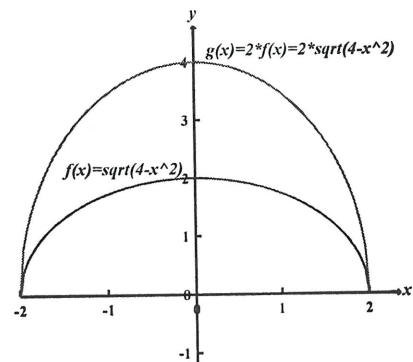


(b) stretching:

$$g(x) = cf(x) = 2\sqrt{4-x^2}$$

$$R_g = \{y: 0 \leq y/2 \leq 2\} = \{y: 0 \leq y \leq 4\}$$

*Note:* In case of vertical stretches, the domain of the function will not be changed.



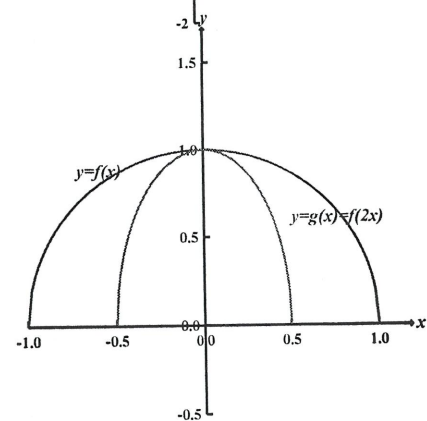
**Example 7:** Use the graph of the function  $y = f(x) = \sqrt{1-x^2}$  to sketch the graph of the following functions:

1.  $y = g(x) = \sqrt{1-4x^2}$

**Sol.:**  $y = \sqrt{1-4x^2} = \sqrt{1-(2x)^2}$

This function may be obtained by shrinking

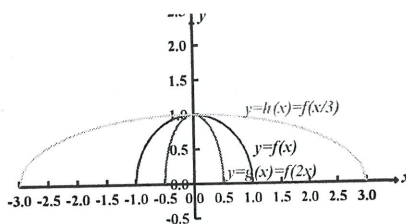
the function  $f(x) = \sqrt{1-x^2}$  by two units horizontally ( $g(x) = f(2x)$ ).



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**Sol.:**  $y = \sqrt{1 - \frac{x^2}{9}} = \sqrt{1 - \left(\frac{x}{3}\right)^2}$

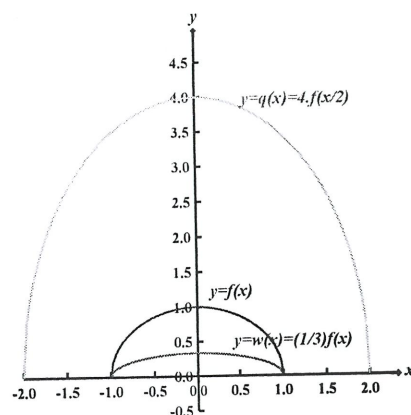
This function may be obtained by stretching the function  $f(x) = \sqrt{1 - x^2}$  by three units horizontally ( $h(x) = f\left(\frac{x}{3}\right)$ ).



3.  $y = w(x) = \frac{1}{3}\sqrt{1 - x^2}$

**Sol.:**  $y = w(x) = \frac{1}{3}\sqrt{1 - x^2}$

This function may be obtained by shrinking the function  $f(x) = \sqrt{1 - x^2}$  by three units vertically ( $h(x) = \frac{1}{3}f(x)$ ).



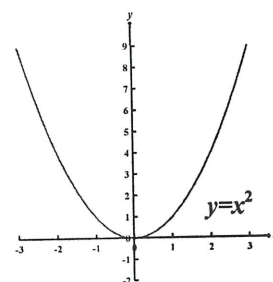
4.  $y = q(x) = 4\sqrt{1 - \frac{x^2}{4}}$

**Sol.:**  $y = q(x) = 4\sqrt{1 - \frac{x^2}{4}} = 4\sqrt{1 - \left(\frac{x}{2}\right)^2}$

This function may be obtained by stretching the function  $f(x) = \sqrt{1 - x^2}$  by two units horizontally and four units vertically ( $q(x) = 4.f\left(\frac{x}{2}\right)$ ).

**Homework:**

1. Sketch the graph of the following curves by shifting, reflecting, shrinking and stretching the graph of the given functions appropriately.



(a) The given function  $y = x^2$

(i)  $y = 1 + (x - 2)^2$

(ii)  $y = 2 - (x + 1)^2$

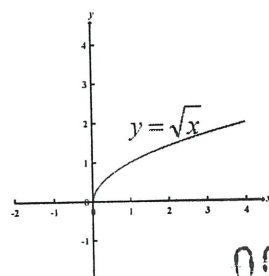
(iii)  $y = -2(x + 1)^2 - 3$

(iv)  $y = \frac{1}{2}(x - 3)^2 + 2$

(v)  $y = x^2 + 6x$

(vi)  $y = x^2 + 6x - 10$

(b) The given function  $y = \sqrt{x}$







## Asymptotes

Example: Find the asymptotes of  $xy - y - 1 = 0$ .

Sol. Solving for  $y$  in terms of  $x \Rightarrow y = \frac{1}{x-1}$

$$x-1=0 \Rightarrow x=1$$

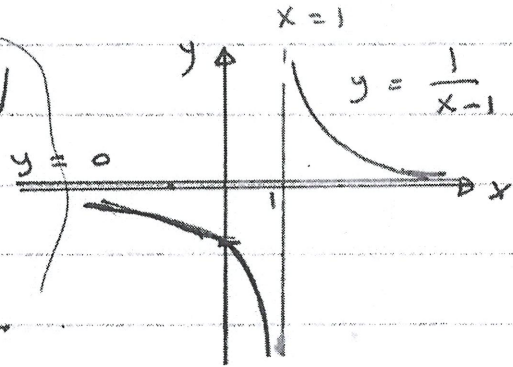
Thus the vertical asymptote is the vertical line through the point  $(1, 0)$ .

Solving for  $x$  in terms of  $y \Rightarrow x = \frac{y+1}{y}$

$$y=0$$

Thus the horizontal asymptote is the  $x$ -axis.

or to find the horizontal asymptote:



$$\lim_{x \rightarrow \infty} \frac{1}{x-1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$\therefore = \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty}} = \frac{0}{1-0}$$

$$= \frac{0}{1} = 0$$

$$\therefore y = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x-1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{x}{x} - \frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x}}$$

$$= \frac{\frac{1}{\infty}}{1 - \frac{1}{\infty}} = \frac{0}{1-0}$$

$$= \frac{0}{1} = 0$$

$\therefore$  horizontal asymptote is  $y=0$

$$\lim_{x \rightarrow \infty} \frac{1}{x-1} = \frac{1}{\infty-1} = \frac{1}{\infty} = 0$$

$$\therefore y = 0$$

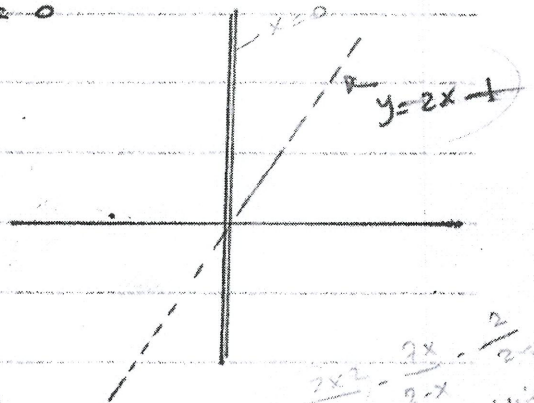
Example: find an equation of the linear oblique asymptote for curve?

①  $y = \frac{2x^3 - x^2 + 3}{x^2}$

sol.

$$y = 2x - 1 + \frac{3}{x^2}$$

if the absolute value of  $x$  is very large, the value of  $\frac{3}{x^2}$  approaches 0 and becomes insignificant in comparison to the ~~order~~ <sup>other</sup> two terms. Thus, the curve approaches the ~~line~~ <sup>line</sup>  $y = 2x - 1$ .  
vertical asymptote is at  $x = 0$



②  $y = \frac{2x^2 - 7x - 2}{2-x}$

thus

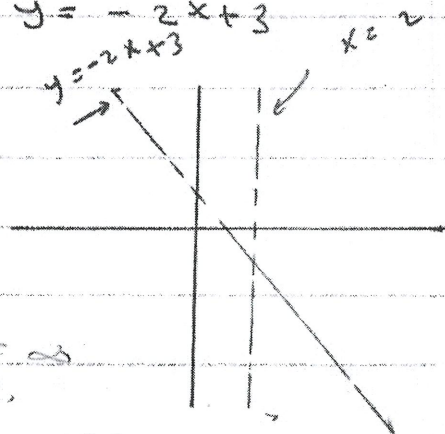
$$= -2x + 3 - \frac{8}{2-x}$$

$$\begin{array}{r} -2x+3 \overline{) 2x^2 - 7x - 2} \\ \underline{2x^2 - 4x} \phantom{- 2} \\ -3x - 2 \\ \underline{-3x + 6} \\ -8 \end{array}$$

$\frac{2x^2}{2-x} = \frac{2x}{2-x} = \frac{2}{2-x}$   
 oblique asymptote  $y = -2x + 3$   
 vertical asymptote  $x = 2$

the oblique asymptote is  $y = -2x + 3$

vertical asymptote at  $x = 2$



$$\lim_{x \rightarrow \infty} \frac{2x^2 - 7x - 2}{2-x}$$

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{7}{x} - \frac{2}{x^2}}{\frac{2}{x} - 1} = \frac{2-0-0}{0-1} = \frac{2}{-1} = -2$$

rest is ~~vertical~~ asymptote horizontal

H.W. Example: <sup>determine</sup> determining the asymptotes to a curve

$$y = \frac{x^2 - x - 2}{x - 1}$$

∴ sol.

$$y = \frac{(x+1)(x-2)}{x-1}$$

$x - 1 = 0 \Rightarrow x = 1 \Rightarrow$  vertical asymptote.

to find linear or oblique asymptote.

thus:

$$y = \frac{x^2 - x - 2}{x - 1}$$

$$\begin{array}{r} x \\ x-1 \overline{) x^2 - x - 2} \\ \underline{x^2 - x} \phantom{- 2} \\ -2 \end{array}$$

$$\therefore y = x - \frac{2}{x-1}$$

∴ the equation of oblique asymptote

is  $y = x$

~~Example: find a horizontal asymptotes for the following equation:~~

①  ~~$y = \frac{1}{10x^2}$~~

~~$$\lim_{x \rightarrow \infty} \frac{1}{10x^2} = \frac{1}{10\infty} = \frac{1}{\infty} = 0$$~~

~~$$\therefore y = 0$$~~

~~$$\text{or } y = \frac{1}{10x^2} \Rightarrow x^2 = \frac{1}{10y} \Rightarrow y = 0$$~~



H.W ②  $y = \frac{2x^2+1}{x^2+3}$

$$\lim_{x \rightarrow \infty} \frac{2x^2+1}{x^2+3} \Rightarrow$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{3}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 + \frac{1}{x^2}}{1 + \frac{3}{x^2}} = \frac{2 + \frac{1}{\infty}}{1 + \frac{3}{\infty}}$$

$$= \frac{2+0}{1+0} = 2 \quad \therefore y = 2 \leftarrow \text{vertical horizontal asymptote}$$

Example ③  $y = \frac{5x^4-4}{3x^2+2}$

Example: find a horizontal asymptote for the curve

Sol.  $\lim_{x \rightarrow \infty} \frac{\frac{5x^4}{x^4} - \frac{4}{x^4}}{\frac{3x^2}{x^4} + \frac{2}{x^4}} = \lim_{x \rightarrow \infty} \frac{5 - \frac{4}{x^4}}{\frac{3}{x^2} + \frac{2}{x^4}}$

$$= \frac{5 - \frac{4}{\infty}}{\frac{3}{\infty} + \frac{2}{\infty}} = \frac{5-0}{0} = \infty \quad \text{no horizontal asymptotes}$$

H.W Example

④ find a vertical and a horizontal asymptotes of the function  $y = \frac{(x+1)^2}{1+x^2}$

Sol.

- $1+x^2 \neq 0 \Rightarrow$  there is no vertical asymptote.
- to find horizontal asymptote.

$$\lim_{x \rightarrow \infty} \frac{(x+1)^2}{1+x^2} = \lim_{x \rightarrow \infty} \frac{(x+1)^2 \cdot \frac{1}{x^2}}{1+x^2 \cdot \frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{(\frac{x^2}{x^2} + \frac{1}{x^2})^2}{\frac{1}{x^2} + \frac{x^2}{x^2}}$$

$$= \lim_{x \rightarrow \infty} \frac{(1 + \frac{1}{x^2})^2}{\frac{1}{x^2} + 1} = \frac{(1 + \frac{1}{\infty})^2}{\frac{1}{\infty} + 1} = \frac{(1+0)^2}{0+1} = \frac{1}{1} = 1 \Rightarrow \boxed{y=1}$$

or  $y = \frac{(x+1)^2}{1+x^2} \Rightarrow y + yx^2 = x^2 + 2x + 1 \Rightarrow (yx^2 - x^2) + y - 2x - 1 = 0$

$$x^2(y-1) - 2x + (y-1) = 0$$



$$\frac{-b \pm \sqrt{b^2 - 4AC}}{2A}$$

$$\therefore \frac{+2 \pm \sqrt{4 - 4(y-1)(y-1)}}{2(y-1)}$$

$$\frac{2 \pm 2\sqrt{1 - (y-1)^2}}{2(y-1)}$$

$$= \frac{1 \pm \sqrt{1 - (y-1)^2}}{y-1}$$

$\Rightarrow y-1=0 \Rightarrow y=1$  is a horizontal asymptote.

## LIMITS AND CONTINUITY

**Limits**

Definition:

If the value of  $f(x)$  can be made as close as we like to  $L$  by taking the value of  $x$  sufficiently close to  $a$  (but not equal  $a$ ), then we write:

$$\lim_{x \rightarrow a} f(x) = L,$$

which is read "the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$ ".

**Properties of limits:**

1. If  $f(x) = k$ , then  $\lim_{x \rightarrow a} f(x) = k$ , where  $a$  and  $k$  are real numbers.
2. Sum rule:  $\lim_{x \rightarrow a} [f_1(x) + f_2(x)] = \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x)$ .
3. Difference rule:  $\lim_{x \rightarrow a} [f_1(x) - f_2(x)] = \lim_{x \rightarrow a} f_1(x) - \lim_{x \rightarrow a} f_2(x)$ .
4. Product rule:  $\lim_{x \rightarrow a} [f_1(x) \cdot f_2(x)] = \lim_{x \rightarrow a} f_1(x) \cdot \lim_{x \rightarrow a} f_2(x)$ .
5. Constant multiple rule:  $\lim_{x \rightarrow a} k \cdot f(x) = k \cdot \lim_{x \rightarrow a} f(x)$ , where  $k$  is a constant.
6. Quotient rule:  $\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} = \lim_{x \rightarrow a} f_1(x) / \lim_{x \rightarrow a} f_2(x)$ ,  $\lim_{x \rightarrow a} f_2(x) \neq 0$ .
7. Power rule:  $\lim_{x \rightarrow a} [f(x)]^{r/s} = [\lim_{x \rightarrow a} f(x)]^{r/s}$ , provided that  $\lim_{x \rightarrow a} f(x)$  is a real

number (if  $s$  is even, we assume  $\lim_{x \rightarrow a} f(x) \geq 0$ ).

\* Polynomials:  $\lim_{x \rightarrow a} (c_0 + c_1x + c_2x^2 + \dots + c_nx^n) = c_0 + c_1a + c_2a^2 + \dots + c_na^n$ .

\*  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

\* Sandwich theorem:

If  $g(x) \leq f(x) \leq h(x)$  are three functions such that:

$$\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} h(x) = L, \quad \text{then} \quad \lim_{x \rightarrow a} f(x) = L.$$

Note: Indeterminate quantities:  $(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 * \infty)$ .

Example: Find the limits of the following:

1.  $\lim_{x \rightarrow 2} x^2 - 4x = 2^2 - 4 * 2 = 4 - 8 = -4$ .

2.  $\lim_{x \rightarrow 1} x^3 + 2x^2 - 3x + 4 = 1^3 + 2 * 1^2 - 3 * 1 + 4 = 4$ .

3.  $\lim_{x \rightarrow 1} \frac{(3x-1)^2}{(x+1)^3} = \frac{(3*1-1)^2}{(1+1)^3} = \frac{2^2}{2^3} = \frac{1}{2}$ .

4.  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \frac{2^2 - 4}{2^2 - 5 * 2 + 6} = \frac{0}{0}$  (Indeterminate quantities)

$$\text{So } \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x \rightarrow 2} \frac{(x+2)}{(x-3)} = \frac{2+2}{2-3} = \frac{4}{-1} = -4.$$

$$\begin{aligned} 5. \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x^2-4}} &= \frac{2-2}{\sqrt{2^2-4}} = \frac{0}{0} && \text{(Indeterminate quantities)} \\ &= \lim_{x \rightarrow 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{(x-2)(x+2)}} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}\sqrt{x-2}}{\sqrt{x-2}\sqrt{x+2}} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{\sqrt{2-2}}{\sqrt{2+2}} = \frac{0}{\sqrt{4}} = 0 \end{aligned}$$

$$\begin{aligned} 6. \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{x^2-4} &= \frac{\sqrt{2-2}}{2^2-4} = \frac{0}{0} && \text{(Indeterminate quantities)} \\ &= \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{\sqrt{x-2}}{\sqrt{x-2}\sqrt{x-2}(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x-2}(x+2)} = \frac{1}{\sqrt{2-2}(2+2)} = \frac{1}{0 \cdot 4} = \frac{1}{0} = \infty \Rightarrow \text{the limit does not exist.} \end{aligned}$$

$$\begin{aligned} 7. \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} &= \frac{1-1}{\sqrt{1^2+3}-2} = \frac{0}{0} && \text{(Indeterminate quantities)} \\ &= \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2+3}-2} * \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} && \text{(Multiplying both the numerator and} \\ &&& \text{denominator by the conjugate factor)} \\ &= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2+3-4} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1} = \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{x^2+3}+2)}{(x+1)} = \frac{\sqrt{1^2+3}+2}{1+1} = \frac{4}{2} = 2. \end{aligned}$$

$$8. = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{x} * \frac{3}{3} = 3 * \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} = 3 * 1 = 3$$

$$9. = \lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x / \cos x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 * \frac{1}{1} = 1$$

$$10. = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x} \quad \text{Let } z = \frac{\pi}{2} - x, \quad \text{so as } x \rightarrow \frac{\pi}{2} \Rightarrow z \rightarrow 0$$

$$\therefore \lim_{z \rightarrow 0} \frac{\cos(\frac{\pi}{2} - z)}{z} = \lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

$$\begin{aligned} 11. &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} * \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{\sin x}{(1 + \cos x)} = 1 * \frac{0}{1+1} = 0 \end{aligned}$$

Example: Giving that  $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$  for all  $x \neq 0$ , find  $\lim_{x \rightarrow 0} u(x)$ .

Sol.: Since  $\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = 1$  and  $\lim_{x \rightarrow 0} 1 + \frac{x^2}{2} = 1$ , then

the Sandwich Theorem implies that  $\lim_{x \rightarrow 0} u(x) = 1$ .

### Right-hand limits and left-hand limits

#### One sided vs. two sided limits

Definition:

A function  $f(x)$  has a limit as  $x$  approaches  $c$  if and only if the right-hand and left-hand limits at  $c$  exist and are equal. In symbol:

$$\lim_{x \rightarrow c} f(x) = L \Leftrightarrow \lim_{x \rightarrow c^+} f(x) = L \text{ and } \lim_{x \rightarrow c^-} f(x) = L$$

Example: Discuss the limit properties of the function  $f(x)$  which shown in figure.

Sol.:

- At  $x = 0$   $\lim_{x \rightarrow 0^+} f(x) = 1$

$\lim_{x \rightarrow 0^-} f(x)$  does not exist (because the function is not defined to the left of  $x = 0$ )

- At  $x = 1$   $\lim_{x \rightarrow 1^-} f(x) = 0$  even though  $f(1) = 1$

$$\lim_{x \rightarrow 1^+} f(x) = 1$$

$\lim_{x \rightarrow 1} f(x)$  does not exist, because the right-hand and left-hand limits are not equal.

- At  $x = 2$   $\lim_{x \rightarrow 2^-} f(x) = 1$

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$\lim_{x \rightarrow 2} f(x) = 1 \text{ even though } f(2) = 2$$

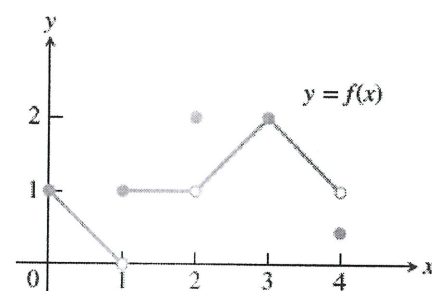
- At  $x = 3$   $\lim_{x \rightarrow 3^-} f(x) = 2$

$$\lim_{x \rightarrow 3^+} f(x) = 2$$

$$\lim_{x \rightarrow 3} f(x) = f(3) = 2$$

- At  $x = 4$   $\lim_{x \rightarrow 4^-} f(x) = 1$  even though  $f(4) = 0.5$

$\lim_{x \rightarrow 4^+} f(x)$  does not exist, because the function is not defined to the right of  $x = 4$ .



Example: Check the existence of the limit of the function  $f(x)$  at  $x = 1$ ,

$$f(x) = \begin{cases} 2x + 1 & -1 < x < 1 \\ x^2 / 2 - 3 & 1 < x < 4 \end{cases}$$

Sol.:  $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2x + 1 = 3.$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 / 2 - 3 = -2.5.$$

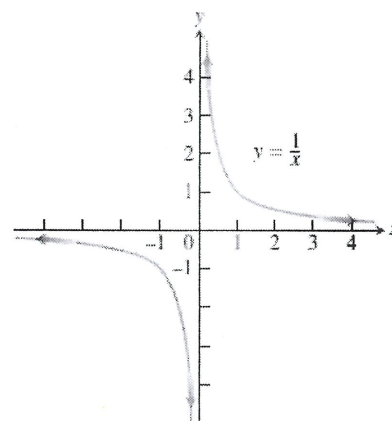
Since the right-hand and left-hand limits are not equal, thus the limit does not exist at  $x = 1$ .

### Limits Involving Infinity:

These are the limits that include  $x \rightarrow \infty$  or  $x \rightarrow -\infty$  and  $\lim f(x) = \infty$  or  $\lim f(x) = -\infty$ .

Let  $y = \frac{1}{x}$  then

1.  $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$
  2.  $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$
  3.  $\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$
  4.  $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$
- { One-sided limits  $\Rightarrow$  the limit does not exist



Example: Find the limits of the following:

$$1. \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x}\right) = \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} = 5 + 0 = 5$$

$$2. \lim_{x \rightarrow \infty} \frac{x}{7x + 4} = \lim_{x \rightarrow \infty} \frac{x/x}{7x/x + 4/x} = \lim_{x \rightarrow \infty} \frac{1}{7 + 4/x} = \frac{1}{7 + 0} = \frac{1}{7}$$

*Note:* In rational functions divide both the numerator and denominator by the largest power of  $x$  in the denominator.

$$3. \lim_{x \rightarrow \infty} \frac{2x^2 - x + 3}{3x^2 + 5} = \lim_{x \rightarrow \infty} \frac{2x^2/x^2 - x/x^2 + 3/x^2}{3x^2/x^2 + 5/x^2} = \lim_{x \rightarrow \infty} \frac{2 - 1/x + 3/x^2}{3 + 5/x^2} = \frac{2 - 0 + 0}{3 + 0} = \frac{2}{3}$$

$$4. \lim_{x \rightarrow \infty} \frac{4x^2 - 3}{3x} = \lim_{x \rightarrow \infty} \frac{4x^2/x - 3/x}{3x/x} = \lim_{x \rightarrow \infty} \frac{4x - 3/x}{3} = \frac{4 * \infty - 0}{3} = \infty \Rightarrow \text{the limit does not exist.}$$



$$5. \lim_{x \rightarrow \infty} \frac{5x+3}{2x^2-1} = \lim_{x \rightarrow \infty} \frac{5x/x^2 + 3/x^2}{2x^2/x^2 - 1/x^2} = \lim_{x \rightarrow \infty} \frac{5/x + 3/x^2}{2 - 1/x} = \frac{0-0}{2-0} = \frac{0}{2} = 0.$$

### Summary for Rational Functions

- a)  $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} = 0$  if  $\deg(f) < \deg(g)$   
 b)  $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$  is finite if  $\deg(f) = \deg(g)$   
 c)  $\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)}$  is infinite if  $\deg(f) > \deg(g)$

$$6. \text{ a } \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} \quad (\text{since } \sqrt{x^2} = |x|).$$

As  $x \rightarrow +\infty$ , the values of  $x$  under consideration are positive, so we can replace  $|x|$  by  $x$ :

$$\Rightarrow \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/x} = \lim_{x \rightarrow +\infty} \frac{\sqrt{1+2/x^2}}{(3-6/x)} = \frac{1}{3}.$$

$$\text{ b } \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}}{3x-6} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} \quad (\text{since } \sqrt{x^2} = |x|).$$

As  $x \rightarrow -\infty$ , the values of  $x$  under consideration are negative, so we can replace  $|x|$  by  $-x$ :

$$\Rightarrow \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/|x|}{(3x-6)/|x|} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}/\sqrt{x^2}}{(3x-6)/(-x)} = \lim_{x \rightarrow -\infty} \frac{\sqrt{1+2/x^2}}{(-3+6/x)} = -\frac{1}{3}.$$

$$7. \lim_{x \rightarrow \infty} \frac{\sin x}{x}$$

Remember that  $-1 \leq \sin x \leq 1$ . Dividing the inequality by  $x$  yields

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} -\frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0 \quad (\text{Sandwich theorem})$$

$$8. \lim_{x \rightarrow \infty} x \sin \frac{1}{x}$$

$$\text{Let } x = \frac{1}{z} \Rightarrow z = \frac{1}{x}$$

$$\text{When } x \rightarrow \infty \Rightarrow z \rightarrow 0$$

$$\therefore \lim_{z \rightarrow 0} \frac{1}{z} \sin z = 1$$

$$9. \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x} \quad (\infty - \infty)$$

$$\begin{aligned} \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x} &= \lim_{x \rightarrow \infty} \sqrt{x^2 + 6x + 1} - \sqrt{x^2 + x} * \frac{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{(x^2 + 6x + 1) - (x^2 + x)}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{x^2 + 6x + 1 - x^2 - x}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} = \lim_{x \rightarrow \infty} \frac{5x + 1}{\sqrt{x^2 + 6x + 1} + \sqrt{x^2 + x}} \\ &= \lim_{x \rightarrow \infty} \frac{5x/x + 1/x}{\sqrt{x^2/x^2 + 6x/x^2 + 1/x^2} + \sqrt{x^2/x^2 + x/x^2}} = \frac{5}{\sqrt{1+0+0} + \sqrt{1+0}} = \frac{5}{1+1} = \frac{5}{2} = 2.5. \end{aligned}$$

**Continuity:****Continuity at a point**

- A function  $y = f(x)$  is continuous at an interior point  $c$  of its domain if:

$$\lim_{x \rightarrow c} f(x) = f(c)$$



- A function  $y = f(x)$  is continuous at a left end-point  $a$  of its domain if:

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

- A function  $y = f(x)$  is continuous at a right end-point  $b$  of its domain if:

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

**Discontinuity at a point**

If a function  $f(x)$  is not continuous at a point  $c$ , we say that  $f(x)$  is discontinuous at  $c$  and call  $c$  a point of discontinuity of  $f(x)$ .

**Continuous Functions**

A function is continuous if it is continuous at each point of its domain.

**The Continuity Test**

A function  $y = f(x)$  is continuous at  $x = c$  if and only if it meets the following three conditions:

1.  $f(c)$  is defined ( $c$  lies in the domain of  $f$ ).
2.  $\lim_{x \rightarrow c} f(x)$  exists ( $f$  has a limit as  $x \rightarrow c$ ).
3.  $\lim_{x \rightarrow c} f(x) = f(c)$  (The limit equals the function value at  $c$ ).

Example: Discuss the continuity conditions of the function  $f(x)$  which shown in figure at  $x = 0$ ,  $x = 1$ ,  $x = 2$ ,  $x = 3$ ,  $x = 1.5$ , and  $x = 4$ .

Sol.:

-At  $x = 0$  (left end-point)

$$f(0) = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = f(0) = 1$$

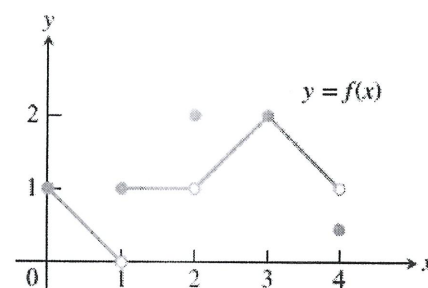
So it is continuous at the left end-point ( $x = 0$ ).

-At  $x = 1$  (interior point)

$$f(1) = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = 1$$



$\therefore \lim_{x \rightarrow 1} f(x)$  does not exist, because the right-hand and left-hand limits are not equal.

So it is discontinuous at  $x = 1$ .

-At  $x = 2$  (interior point)

$$f(2) = 2$$

$$\lim_{x \rightarrow 2^-} f(x) = 1, \quad \lim_{x \rightarrow 2^+} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 2} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 2} f(x) \neq f(2)$$

So it is discontinuous at  $x = 2$ .

-At  $x = 3$  (interior point)

$$f(3) = 2$$

$$\lim_{x \rightarrow 3^-} f(x) = 2, \quad \lim_{x \rightarrow 3^+} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 3} f(x) = 2$$

$$\therefore \lim_{x \rightarrow 3} f(x) = f(3) = 2$$

So it is continuous at  $x = 3$ .

-At  $x = 1.5$  (interior point)

$$f(1.5) = 1$$

$$\lim_{x \rightarrow 1.5^-} f(x) = 1, \quad \lim_{x \rightarrow 1.5^+} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 1.5} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 1.5} f(x) = f(1.5) = 1$$

So it is continuous at  $x = 1.5$ .

-At  $x = 4$  (right end-point)

$$f(4) = 0.5$$

$$\lim_{x \rightarrow 4^-} f(x) = 1$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) \neq f(4)$$

So it is discontinuous at right-end point ( $x = 4$ ).

Example: Determine whether the following functions are continuous at  $x = 2$ .

$$1. f(x) = \frac{x^2 - 4}{x - 2}$$

Sol.:  $f(2)$  is not found ( $2 \notin D_f$ )

So the function is discontinuous at  $x = 2$ .

$$2. f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x \neq 2 \\ 3 & x = 2 \end{cases}$$

Sol.:  $f(2) = 3$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} x + 2 = 2 + 2 = 4$$

$$\therefore f(2) \neq \lim_{x \rightarrow 2} f(x)$$

So the function is discontinuous at  $x = 2$ .

$$3. \quad f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

Sol.:  $f(2) = 4$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

$$\therefore f(2) = \lim_{x \rightarrow 2} f(x)$$

So the function is continuous at  $x = 2$ .

Example: Test the continuity of the following function at  $x = 1$ :

$$f(x) = \begin{cases} x^2 & x < 1 \\ \frac{x}{2} & x \geq 1 \end{cases}$$

Sol.:  $f(1) = \frac{1}{2}$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = (1)^2 = 1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x}{2} = \frac{1}{2}$$

$\lim_{x \rightarrow 1} f(x)$  is not found (the left-hand and right-hand limits are not equal).

So the function is discontinuous at  $x = 1$ .



## L'Hopital's Rule

Indeterminate forms  $0/0, \infty/\infty$

Suppose that  $f(a) = g(a) = 0$ , that  $f$  and  $g$  are differentiable on an open interval  $I$  containing  $a$ , and that  $g'(x) \neq 0$  on  $I$  if  $x \neq a$ . Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Examples:

$$(a) \lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$$

$$(b) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2\sqrt{1+x}} \Big|_{x=0} = \frac{1}{2}$$

Examples:

$$(a) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \quad \text{Still } \frac{0}{0}, \text{ differentiate again.}$$

$$= \lim_{x \rightarrow 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \quad \text{Not } \frac{0}{0}, \text{ limit is found.}$$

$$(b) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \quad \text{Still } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad \text{Still } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6} \quad \text{Not } \frac{0}{0}; \text{ limit is found.}$$

Indeterminate forms  $0 \cdot \infty, \infty - \infty$

Examples:

$$(a) \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} \quad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1$$

$$(b) \lim_{x \rightarrow \infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x \rightarrow \infty} \frac{1 - 4x}{6x + 5} = \lim_{x \rightarrow \infty} \frac{-4}{6} = -\frac{2}{3}$$

(c)

$$\begin{aligned}\lim_{x \rightarrow \infty} \left( x \sin \frac{1}{x} \right) & \quad \infty \cdot 0 \\ &= \lim_{h \rightarrow 0^+} \left( \frac{1}{h} \sin h \right) \quad \text{Let } h = 1/x. \\ &= 1\end{aligned}$$

(d)

$$\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right).$$

If  $x \rightarrow 0^+$ , then  $\sin x \rightarrow 0^+$  and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty.$$

Similarly, if  $x \rightarrow 0^-$ , then  $\sin x \rightarrow 0^-$  and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty - (-\infty) = -\infty + \infty.$$

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

Then apply l'Hôpital's Rule to the result:

$$\begin{aligned}\lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.\end{aligned}$$

Indeterminate forms  $1^\infty$ ,  $\infty^0$ ,  $0^0$

DEFINITION: If:  $\lim_{x \rightarrow a} \ln f(x) = L$ ,

Then:  $\lim_{x \rightarrow a} f(x) = e^L$

Example: Show that  $\lim_{x \rightarrow 0^+} (1+x)^{1/x} = e$ .

Sol.: Let  $f(x) = (1+x)^{1/x} \Rightarrow \ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x} \ln(1+x) = \frac{\ln(1+x)}{x}$

$$\therefore \lim_{x \rightarrow 0^+} \ln f(x) = \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} = \frac{0}{0} \text{ (Indeterminate form)}$$

$$\text{By L'Hopital's Rule } \therefore \lim_{x \rightarrow 0^+} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

Therefore  $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1+x)^{1/x} = e^1 = e$  o.k.

**Examples:** Find the limits of the following:

1.  $\lim_{x \rightarrow 0} (\sec^3 2x)^{\cot^2 3x} = (\sec^3(2 \cdot 0))^{\cot^2 3 \cdot 0} = 1^\infty$  (indeterminate form)

**Sol.:** Let  $y = (\sec^3 2x)^{\cot^2 3x}$

$$\therefore \ln y = \ln(\sec^3 2x)^{\cot^2 3x} = \ln(\sec 2x)^{3 \cot^2 3x} = 3 \cot^2 3x \ln \sec 2x = \frac{3 \ln \sec 2x}{\tan^2 3x}$$

$$\therefore \lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{3 \ln \sec 2x}{\tan^2 3x} = \frac{3 \ln \sec 2 \cdot 0}{\tan^2 3 \cdot 0} = \frac{3 \ln 1}{0} = \frac{0}{0} \quad (\text{also indeterminate form})$$

$$\therefore \lim_{x \rightarrow 0} \frac{3 \frac{\sec 2x \tan 2x \cdot 2}{\sec^2 2x}}{2 \tan 3x \sec^2 3x \cdot 3} = \lim_{x \rightarrow 0} \frac{\tan 2x}{\tan 3x \sec^2 3x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 2x}{\tan 3x \cdot [2 \sec 3x \cdot \sec 3x \tan 3x \cdot 3] + \sec^2 3x [\sec^2 3x \cdot 3]}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 2x}{6 \sec^2 3x \tan^2 3x + 3 \sec^4 3x} = \frac{2 \sec^2 0}{6 \sec^2 0 \tan^2 0 + 3 \sec^4 0} = \frac{2 \cdot 1}{6 \cdot 0 + 3 \cdot 1} = \frac{2}{3}$$

$$\therefore \lim_{x \rightarrow 0} \ln y = \frac{2}{3} \Rightarrow \lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} (\sec^3 2x)^{\cot^2 3x} = e^{2/3}$$

2.  $\lim_{x \rightarrow 1} x^{\left(\frac{1}{x-1}\right)} = 1^{\left(\frac{1}{1-1}\right)} = 1^{\left(\frac{1}{0}\right)} = 1^\infty$  (indeterminate form)

**Sol.:** Let  $y = x^{\left(\frac{1}{x-1}\right)} \Rightarrow \ln y = \ln x^{\left(\frac{1}{x-1}\right)} = \frac{1}{x-1} \ln x = \frac{\ln x}{x-1}$

$$\therefore \lim_{x \rightarrow 1} \ln y = \lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \frac{\ln 1}{1-1} = \frac{0}{0} \quad (\text{also indeterminate form})$$

$$= \lim_{x \rightarrow 1} \frac{1/x}{1} = \frac{1/1}{1} = 1$$

$$\therefore \lim_{x \rightarrow 1} y = \lim_{x \rightarrow 1} x^{\left(\frac{1}{x-1}\right)} = e^1 = e$$

3.  $\lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\cos x} = \infty^0$  (indeterminate form)

**Sol.:** Let  $y = (\tan x)^{\cos x} \Rightarrow \ln y = \ln(\tan x)^{\cos x} = \cos x \ln(\tan x) = \frac{\ln(\tan x)}{\sec x}$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} \ln y = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\tan x)}{\sec x} = \frac{\ln \infty}{\infty} = \frac{\infty}{\infty} \quad (\text{also indeterminate form})$$

$$\begin{aligned}
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec^2 x}{\sec x \tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\left(\frac{\sin x}{\cos x}\right)^2} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0 \\
\therefore \lim_{x \rightarrow \frac{\pi}{2}} y &= \lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{\cos x} = e^0 = 1
\end{aligned}$$

4.  $\lim_{x \rightarrow 0} x^{\sin x} = 0^0$  (indeterminate form)

Sol.: Let  $y = x^{\sin x} \Rightarrow \ln y = x^{\sin x} = \sin x \ln x = \frac{\ln x}{\csc x}$

$$\therefore \lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln x}{\csc x} = \frac{\ln 0}{\infty} = \frac{-\infty}{\infty} \text{ (also indeterminate form)}$$

$$= \lim_{x \rightarrow 0} \frac{1/x}{-\csc x \cot x} = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = \lim_{x \rightarrow 0} \frac{-\sin^2 x}{x \cos x} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin x \cos x}{-x \sin x + \cos x} = \lim_{x \rightarrow 0} \frac{-\sin 2x}{-x \sin x + \cos x} = \frac{0}{0+1} = 0$$

$$\therefore \lim_{x \rightarrow 0} y = \lim_{x \rightarrow 0} x^{\sin x} = e^0 = 1$$

## DIFFERENTIATION

### Derivatives:

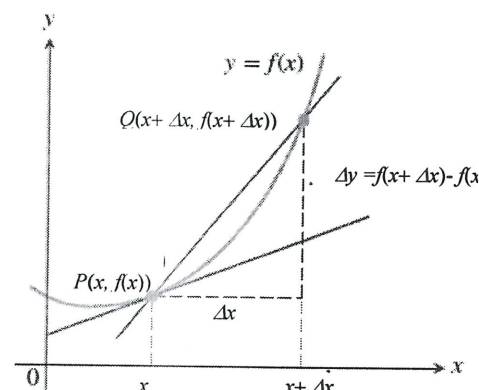
Derivatives are the functions which are used to measure rates at which things change. We define derivatives as limiting values of average change, just we define slope of curves as limiting values of slopes of secants.

If  $y=f(x)$

$$\therefore \Delta y = f(x + \Delta x) - f(x)$$

$$\text{So, slope of secant } PQ = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

As  $Q \rightarrow P$  then slope of secant  $PQ$  will equal to slope of tangent of the curve  $f(x)$  at  $P$  and  $\Delta x \rightarrow 0$



$\therefore \lim_{Q \rightarrow P} \text{slope of secant } PQ = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \text{slope of tangent of the curve } f(x) \text{ at } P.$  And this is called **the definition of derivative** of the function  $f(x)$  and this denoted by  $y'$ ,  $f'(x)$ ,  $\frac{dy}{dx}$ ,  $\frac{d}{dx} f(x)$ , and  $D_x f(x)$ .

$$\therefore f'(x) = \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The process of calculating a derivative is called **differentiation**. We consider that the derivative is found if the limit exists and finite at a certain point.

Example: Find the derivative of the function  $f(x) = x^2$  using the definition of derivative.

$$\begin{aligned} \text{Sol.: } \frac{dy}{dx} = f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x. \end{aligned}$$

Example: Using the definition of derivative, differentiate  $f(x) = \sqrt{x}$ .

$$\begin{aligned} \text{Sol.: } \frac{dy}{dx} = f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} = \lim_{\Delta x \rightarrow 0} \frac{1}{(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \frac{1}{(\sqrt{x} + \sqrt{x})} = \frac{1}{2\sqrt{x}}. \end{aligned}$$



**Laws of derivatives:**

1.  $\frac{d}{dx}c = 0$ , where  $c$  is a constant.

2.  $\frac{d}{dx}x^n = n \cdot x^{n-1}$ .

3. If  $U$  and  $V$  are two functions of  $x$  then:

(a)  $\frac{d}{dx}(c \cdot U) = c \cdot \frac{dU}{dx}$ , where  $c$  is a constant.

(b)  $\frac{d}{dx}(U \mp V) = \frac{dU}{dx} \mp \frac{dV}{dx}$ .

(c)  $\frac{d}{dx}(U \cdot V) = U \frac{dV}{dx} + V \frac{dU}{dx}$ .

(d)  $\frac{d}{dx}(U^n) = nU^{n-1} \cdot \frac{dU}{dx}$ .

(e)  $\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V \frac{dU}{dx} - U \frac{dV}{dx}}{V^2}$ .

Example: If  $y = x^3 + 7x^2 - 5x + 4$ , find  $\frac{dy}{dx}$ .

Sol.:  $\frac{dy}{dx} = \frac{d}{dx}(x^3) + \frac{d}{dx}(7x^2) - \frac{d}{dx}(5x) + \frac{d}{dx}(4)$   
 $= 3x^2 + 2 \cdot 7x - 5 + 0 = 3x^2 + 14x - 5$ .

**Higher order derivatives:**

The derivative:  $y' = \frac{dy}{dx}$  is the first derivative of  $y$  with respect to  $x$ . The first derivative may also be a differentiable function of  $x$ . If so its derivative:

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2},$$

is the second derivative of  $y$  with respect to  $x$ . If  $y''$  is also a differentiable function of  $x$ , its derivative:

$$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3},$$

is the third derivative of  $y$  with respect to  $x$ . The names continue as you imagine with

$$y^n = \frac{d}{dx}(y^{n-1}) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^n y}{dx^n},$$

is the  $n^{\text{th}}$  derivative of  $y$  with respect to  $x$  for any positive integer  $n$ .

Example: The first four derivatives of  $y = x^3 - 3x^2 + 2$  are:

First derivative:  $y' = 3x^2 - 6x$ .

Second derivative:  $y'' = 6x - 6$ .

Third derivative:  $y''' = 6$ .

Fourth derivative:  $y'''' = 0$ .

The function has derivatives of all orders, but the fifth and subsequent order derivatives are all zero.

### Implicit differentiation:

In some cases, it is difficult or impossible to solve  $y = f(x)$ , so to find  $\frac{dy}{dx}$  for such cases, implicit differentiation will be use.

Example: Find  $\frac{dy}{dx}$  of the following:

1.  $x^2 + y^2 = 1$ .

$$\text{Sol.: } 2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow 2y \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$$

2.  $2y = x^2 + 3xy^2$

$$\begin{aligned} \text{Sol.: } 2 \frac{dy}{dx} &= 2x + 3x(2y \frac{dy}{dx}) + 3y^2 \Rightarrow 2 \frac{dy}{dx} - 6xy \frac{dy}{dx} = 2x + 3y^2 \Rightarrow \frac{dy}{dx} (2 - 6xy) = 2x + 3y^2 \\ &\Rightarrow \frac{dy}{dx} = \frac{2x + 3y^2}{2 - 6xy}. \end{aligned}$$

Example: Find  $\frac{d^2y}{dx^2}$  if  $2x^3 - 3y^2 = 7$ .

Sol.: to find  $\frac{dy}{dx}$ :

$$\begin{aligned} 2x^3 - 3y^2 = 7 &\Rightarrow 6x^2 - 6y \frac{dy}{dx} = 0 \Rightarrow 6y \frac{dy}{dx} = 6x^2 \\ \Rightarrow 6y \frac{dy}{dx} &= 6x^2 \Rightarrow \frac{dy}{dx} = \frac{6x^2}{6y} = \frac{x^2}{y} = y' \quad \text{where } y \neq 0. \end{aligned}$$

We now apply the Quotient Rule to find  $\frac{d^2y}{dx^2}$  or  $y''$ .

$$\begin{aligned} \text{So } y'' &= \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{x^2}{y} \right) = \frac{y(2x) - x^2 \left( \frac{dy}{dx} \right)}{y^2} = \frac{2xy - x^2 \left( \frac{x^2}{y} \right)}{y^2} \\ &= \frac{2xy^2 - x^4}{y^3}. \end{aligned}$$

**The Chain Rule:**

If  $y = f(u)$ ;  $u = g(x)$ , and the derivatives  $\frac{dy}{du}$  and  $\frac{du}{dx}$  both exist then the composite function defined by  $f(g(x))$  has a derivative given by:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}}$$

Example: Let  $y = \sqrt{u^2 + 1}$ ;  $u = \frac{1}{x} + x^2$ , find  $\frac{dy}{dx}$ .

Sol.:  $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$

$$\frac{dy}{du} = \frac{2u}{2\sqrt{u^2 + 1}} = \frac{u}{\sqrt{u^2 + 1}}; \quad \frac{du}{dx} = -\frac{1}{x^2} + 2x$$

$$\therefore \frac{dy}{dx} = \frac{u}{\sqrt{u^2 + 1}} * \left(2x - \frac{1}{x^2}\right) = \frac{\left(\frac{1}{x} + x^2\right)}{\sqrt{\left(\frac{1}{x} + x^2\right)^2 + 1}} * \left(2x - \frac{1}{x^2}\right)$$

Another solution:

Find  $y \circ u = y(u(x)) = \sqrt{\left(\frac{1}{x} + x^2\right)^2 + 1}$

$$\therefore \frac{dy}{dx} = \frac{2\left(\frac{1}{x} + x^2\right) * \left(-\frac{1}{x^2} + 2x\right)}{2\sqrt{\left(\frac{1}{x} + x^2\right)^2 + 1}} = \frac{\left(\frac{1}{x} + x^2\right) * \left(2x - \frac{1}{x^2}\right)}{\sqrt{\left(\frac{1}{x} + x^2\right)^2 + 1}}$$

Example: If  $y = (3x^2 - 7x + 1)^5$ , use the chain rule to find  $\frac{dy}{dx}$ .

Sol.: We may express  $y$  as a composite function of  $x$  by letting:

$$y = u^5 \quad \text{and} \quad u = 3x^2 - 7x + 1$$

So,  $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = 5u^4 * (6x - 7) = 5(3x^2 - 7x + 1)^4 (6x - 7)$

**Derivative of Parametric Equations:**

If  $y = f(t)$  and  $x = g(t)$ , and the derivatives  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$  both exist, then:

$$\frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt}$$

and  $\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{d}{dx} y \right) = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$

Example: Find  $\frac{dy}{dx}$ , if  $y = t^2 - 1$  and  $x = 2t + 3$ .

Sol.:  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$\text{So } \frac{dy}{dt} = 2t, \quad \text{and } \frac{dx}{dt} = 2$$

$$\therefore \frac{dy}{dx} = \frac{2t}{2} = t = \frac{x-3}{2}$$

Another solution:

$$\text{From } x = 2t + 3 \quad \text{find } t = \frac{x-3}{2}$$

$$\text{Then: } y = \left(\frac{x-3}{2}\right)^2 - 1$$

$$\therefore \frac{dy}{dx} = 2\left(\frac{x-3}{2}\right) * \frac{1}{2} = \frac{x-3}{2}$$

Example: Find  $\frac{d^2y}{dx^2}$ , if  $x = t - t^2$  and  $y = t - t^3$ .

$$\text{Sol.: } \frac{dx}{dt} = 1 - 2t \quad \text{and } \frac{dy}{dt} = 1 - 3t^2$$

$$\therefore \frac{dy}{dx} = y' = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

$$\text{And } \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\frac{dy'}{dx} = \frac{(1 - 2t)(-6t) - (1 - 3t^2)(-2)}{(1 - 2t)^2} = \frac{6t^2 + 6t + 2}{(1 - 2t)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{6t^2 + 6t + 2}{(1 - 2t)^3}$$

### Tangent and normal lines:

Example: Does the curve  $y = x^4 - 2x^2 + 2$  has any horizontal tangent? If so, where?

Sol.: The horizontal tangents, if any, occur where the slope  $dy/dx$  is zero. To find these points, we should

$$1. \text{ Calculate } dy/dx: \frac{dy}{dx} = 4x^3 - 2(2x) = 4x^3 - 4x$$

$$2. \text{ Put } \frac{dy}{dx} = 0 \quad \Rightarrow 4x^3 - 4x = 0$$

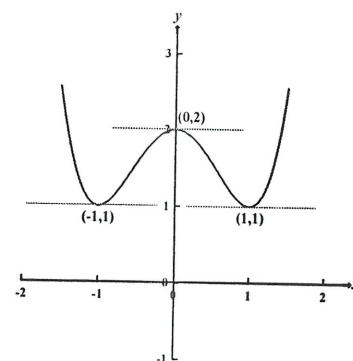
$$3. \text{ Solve the equation } \frac{dy}{dx} = 0 \text{ for } x.:$$

$$4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0$$

$$\text{either } 4x = 0 \quad \Rightarrow \quad x = 0$$

$$\text{or } x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$$

So the curve has horizontal tangents at  $x = 0, x = -1$  and  $x = 1$ .





The corresponding points on the curve (calculated from the equation  $y = x^4 - 2x^2 + 2$ ) are (0,2), (-1,1) and (1,1).

Example: Find the tangent and normal to the curve  $x^2 - xy + y^2 = 7$  at the point (-1,2).

Sol.: We first use the implicit differentiation to find  $\frac{dy}{dx}$ .

$$\begin{aligned}x^2 - xy + y^2 &= 7 \\2x - (x \frac{dy}{dx} + y) + 2y \frac{dy}{dx} &= 0 \Rightarrow \frac{dy}{dx}(2y - x) = y - 2x \\ \therefore \frac{dy}{dx} &= \frac{y - 2x}{2y - x}.\end{aligned}$$

We then evaluate the derivative (slope of the curve) at  $x = -1$  and  $y = 2$  to obtain:

$$\left. \frac{dy}{dx} \right|_{(-1,2)} = \frac{y - 2x}{2y - x} = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{2 + 2}{4 + 1} = \frac{4}{5}$$

So the tangent to the curve at the point (-1, 2) is:

$$\begin{aligned}y - y_1 &= m(x - x_1) \Rightarrow y - 2 = \frac{4}{5}(x - (-1)) \\ \Rightarrow y &= \frac{4}{5}x + \frac{4}{5} + 2 \Rightarrow \therefore y = \frac{4}{5}x + \frac{14}{5}.\end{aligned}$$

And the normal to the curve at the point (-1, 2) is (slope of normal is  $(-1/m)$ ):

$$\begin{aligned}y - y_1 &= m(x - x_1) \Rightarrow y - 2 = -\frac{5}{4}(x - (-1)) \\ \Rightarrow y &= -\frac{5}{4}x - \frac{5}{4} + 2 \Rightarrow \therefore y = -\frac{5}{4}x + \frac{3}{4}.\end{aligned}$$

### Derivative of Trigonometric Functions:

~~If  $u$  is function of  $x$ , then:~~

~~$$\begin{array}{ll}1. \frac{d}{dx} \sin u = \cos u * \frac{du}{dx} & 2. \frac{d}{dx} \cos u = -\sin u * \frac{du}{dx} \\ 3. \frac{d}{dx} \tan u = \sec^2 u * \frac{du}{dx} & 4. \frac{d}{dx} \cot u = -\csc^2 u * \frac{du}{dx} \\ 5. \frac{d}{dx} \sec u = \sec u \cdot \tan u * \frac{du}{dx} & 6. \frac{d}{dx} \csc u = -\csc u \cdot \cot u * \frac{du}{dx}\end{array}$$~~

Example: Find  $\frac{dy}{dx}$  of the following functions:

~~1.  $y = x^2 - \sin x$ .~~

~~Sol.:  $\frac{dy}{dx} = 2x - \cos x$ .~~

## APPLICATIONS OF DERIVATIVES

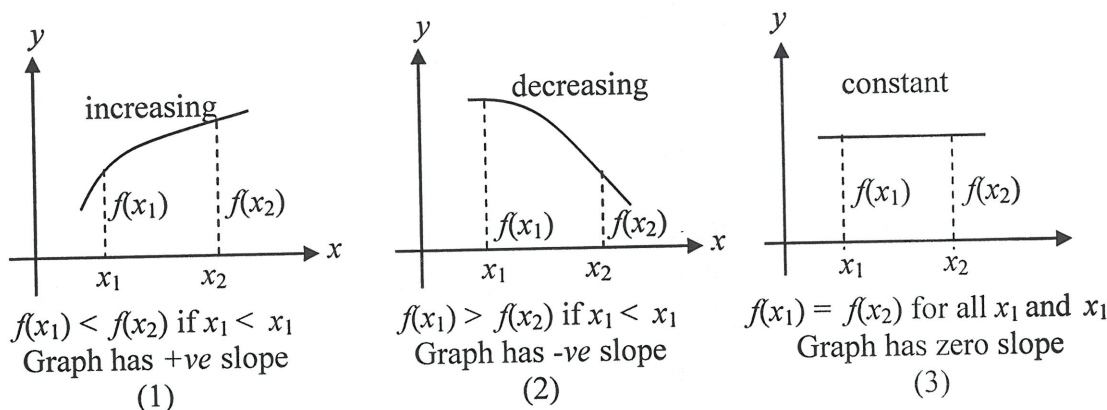
### 1. Analysis of Functions:

#### a. Increase, Decrease and Local (Relative) Extrema:

\* Definition: (Increasing, Decreasing Functions)

Let  $f$  be defined on an interval  $I$  and let  $x_1$  and  $x_2$  be any two points in  $I$ ;

1. If  $f(x_1) < f(x_2)$  whenever  $x_1 < x_2$ , then  $f$  is said to be increasing on  $I$ .
2. If  $f(x_1) > f(x_2)$  whenever  $x_1 < x_2$ , then  $f$  is said to be decreasing on  $I$ .
3. If  $f(x_1) = f(x_2)$  for all  $x_1$  and  $x_2$ , then  $f$  is said to be constant on  $I$ .



\* Theorem: (First derivative test for increasing and decreasing)

Let  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ ;

1. If  $f'(x) > 0$  at each point  $x \in (a, b)$ , then  $f$  is increasing on  $[a, b]$ .
2. If  $f'(x) < 0$  at each point  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$ .
3. If  $f'(x) = 0$  at each point  $x \in (a, b)$ , then  $f$  is constant on  $[a, b]$ .

\* Definition: (Local Minima, Local Maxima)

A function  $f$  has a local maximum value at interior point  $c$  of its domain if  $f(x) \leq f(c)$  for all  $x$  in some open interval containing  $c$ .

A function  $f$  has a local minimum value at interior point  $c$  of its domain if  $f(x) \geq f(c)$  for all  $x$  in some open interval containing  $c$ .

\* Definition: (Critical Point)

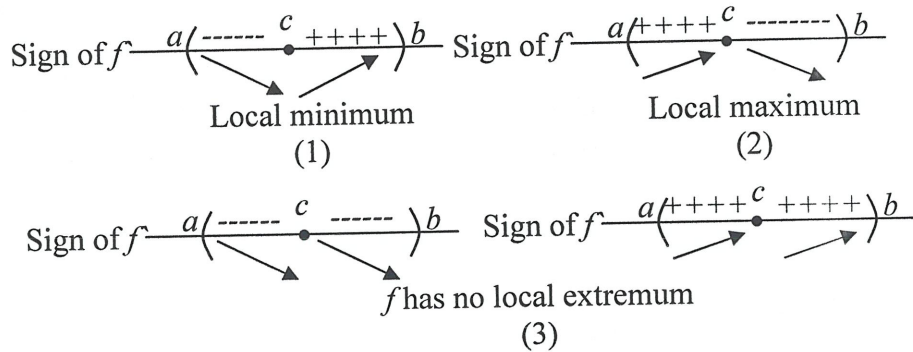
An interior point of the domain of a function  $f$  where  $f'$  is zero or undefined is a **critical point** of  $f$ .

\* Theorem: (First derivative test for local extrema)

Let  $c$  is a critical point of  $f$ . Moving across  $c$  from left to right;

1. If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ ;
2. If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ ;

3. If  $f'$  does not change sign at  $c$  (that is,  $f'$  is positive on both sides of  $c$  or  $f'$  is negative on both sides of  $c$ ), then  $f$  does not have a local extremum at  $c$ ;



\* Theorem: (Second derivative test for local extrema)

Let  $f$  is twice differentiable at  $c$ ;

1. If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $x = c$ .
2. If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $x = c$ .
3. If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test fails. The function  $f$  may have a local maximum, a local minimum, or neither at  $x = c$ .

Example: Find the intervals on which the following functions are increasing and the intervals on which decreasing, and also locate the maximum and the minimum.

(a)  $f(x) = x^2 - 4x + 3$

Sol.:  $f'(x) = 2x - 4 = 2(x - 2)$

Put  $f' = 0 \Rightarrow 2(x - 2) = 0 \Rightarrow x = 2$  (critical point)

Since  $f$  is continuous at  $x = 2$

$\therefore f$  is decreasing on  $(-\infty, 2]$ ,

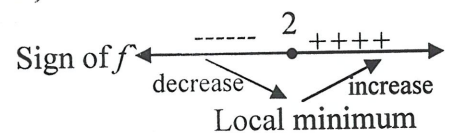
and  $f$  is increasing on  $[2, \infty)$ .

$f(2) = 2^2 - 4 * 2 + 3 = -1$

$\therefore (2, -1)$  is a minimum point.

Or by second derivative test:

$f''(x) = 2 > 0$ , so  $f$  has minimum point at  $x = 2$ .



(b)  $f(x) = x^3$

Sol.:  $f'(x) = 3x^2$

Put  $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$  point (critical point)

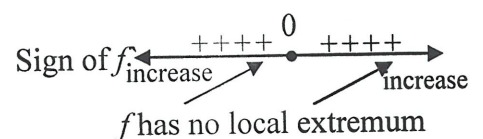
Since  $f$  is continuous at  $x = 0$

$\therefore f$  is increasing on  $(-\infty, 0]$ ,

and  $f$  is increasing on  $[0, \infty)$ .

So  $f$  is increasing over entire interval  $(-\infty, \infty)$

thus  $f$  has no local extremum at  $x = 0$ .



Or by second derivative test:

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0 \leftarrow \text{No indication.}$$

(c)  $f(x) = 3x^{5/3} - 15x^{2/3}$

Sol.:

$$\begin{aligned} f'(x) &= 3 * \frac{5}{3} x^{2/3} - 15 * \frac{2}{3} x^{-1/3} \\ &= 5x^{2/3} - 10x^{-1/3} \\ &= \frac{5(x-2)}{x^{1/3}} \end{aligned}$$

Put  $f' = 0 \Rightarrow \frac{5(x-2)}{x^{1/3}} = 0$

$$\Rightarrow x - 2 = 0 \Rightarrow x = 2$$

And  $f'$  does not exist (undefined) at  $x = 0$ .

Since the function is continuous at  $x = 0$  and  $x = 2$  then,

$\therefore f$  is increasing on  $(-\infty, 0]$  and  $[2, \infty)$ .

and  $f$  is decreasing on  $[0, 2]$ .

At  $x = 0 \Rightarrow f(0) = 3(0)^{5/3} - 15(0)^{2/3} = 0$

$\therefore (0, 0)$  is a local max.

At  $x = 2 \Rightarrow f(2) = 3(2)^{5/3} - 15(2)^{2/3} = -14.287$

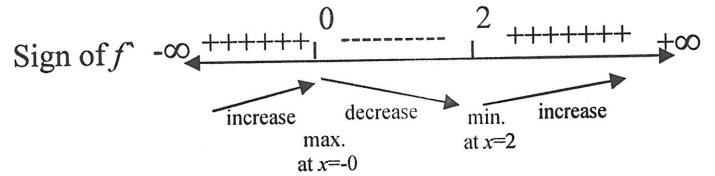
$\therefore (2, -14.287)$  is a local min.

Or by second derivative test:

$$f''(x) = \frac{10}{3} x^{-1/3} + \frac{10}{3} x^{-4/3} = \frac{10}{3x^{1/3}} + \frac{10}{3x^{4/3}} = \frac{10}{3} \left( \frac{x+1}{x^{4/3}} \right)$$

$$f''(0) = \frac{10}{3} \left( \frac{0+1}{0^{4/3}} \right) \Rightarrow f''(0) \text{ does not exist, so this test is failed.}$$

$$f''(2) = \frac{10}{3} \left( \frac{2+1}{2^{4/3}} \right) = 3.96 > 0, \text{ so the function has a min. point at } x = 2.$$



Example: If  $y = 2 + x^{2/3}$ , find the critical points and recognize them.

Sol.:

$$y' = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

The derivative does not exist (undefined) at  $x = 0$ .

So there is a critical point at  $x = 0$ .

At  $x = 0 \Rightarrow f(0) = 2 + (0)^{2/3} = 2$

$\therefore (0, 2)$  is a minimum point.



Example: Find the absolute maximum and minimum values of  $y = x^{2/3}$  on the interval  $-2 \leq x \leq 3$ .

Sol.:  $y = x^{2/3}$



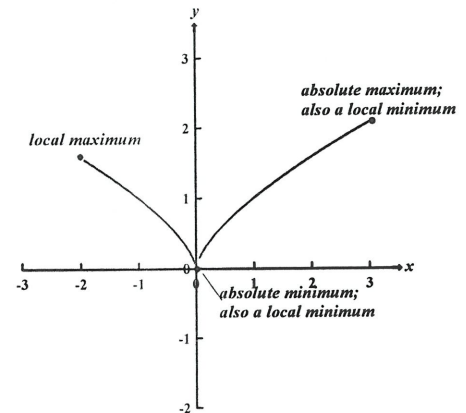
$$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}} \Rightarrow y' \neq 0,$$

But  $y'$  is undefined at  $x=0$ . The values of the function at this one critical point and the endpoints are:

Critical points value:  $f(0) = 0$   
 Endpoint values:  $f(-2) = (-2)^{2/3} = 4^{1/3}$ ,  
 $f(3) = (3)^{2/3} = 9^{1/3}$

We conclude that the function's maximum value is  $9^{1/3}$ , taken on at  $x=3$ .

The minimum value is 0, taken on at  $x=0$ .



**b) Concavity and Inflection Points (I.P.):**

\* Definition: (Concave up, Concave down)

If  $f$  is differentiable on an open interval  $I$ , then  $f$  is said to be;

- **Concave up** on  $I$  if  $f'$  is increasing on  $I$ .
- **Concave down** on  $I$  if  $f'$  is decreasing on  $I$ .

\* Theorem: (Second derivative test for concavity)

Let  $f$  be twice differentiable on an open interval  $I$ ;

1. If  $f''(x) > 0$  on  $I$ , then  $f$  is concave up on  $I$ .
2. If  $f''(x) < 0$  on  $I$ , then  $f$  is concave down on  $I$ .

\* Definition: (Inflection Point)

A point  $P$  on a curve  $y=f(x)$  is called an **inflection point (I.P.)** if  $f$  is continuous there and the curve changes from concave up to concave down or from concave down to concave up at  $P$ .

\* Let  $f$  be continuous and twice differentiable at  $x=c$ . If  $f''(c) = 0$  or undefined, then  $f$  may have an inflection point at  $x=c$ .

Example: Find the intervals on which the following functions are concave up and concave down, then, if any, locate the inflection points.

(a)  $f(x) = x^2 - 4x + 3$

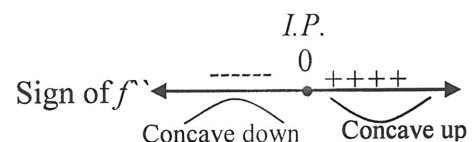
Sol.:  $f'(x) = 2x - 4$ , and  $f''(x) = 2 > 0$

Since  $f''(x) > 0$  for all  $x$ , the function  $f$  is concave up on the interval  $(-\infty, \infty)$  also  $f''(x) \neq 0$  for all  $x$ , the function  $f$  does not have inflection points.

(b)  $f(x) = x^3$

Sol.:  $f'(x) = 3x^2$ , and  $f''(x) = 6x$

Put  $f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$   
 $\Rightarrow y = 0^3 = 0$



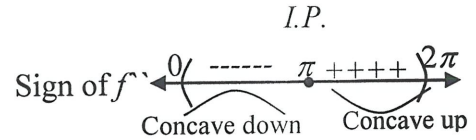
So,  $f$  is concave down on  $(-\infty, 0)$ ,  
and  $f$  is concave up on  $(0, \infty)$ .  
 $(0, 0)$  is the inflection point.

(c)  $f(x) = \sin x$  on  $[0, 2\pi]$

Sol.:  $f'(x) = \cos x$ , and  $f''(x) = -\sin x$

$$\begin{aligned} \text{Put } f''(x) = 0 &\Rightarrow -\sin x = 0 \Rightarrow x = \pi \\ &\Rightarrow y = \sin \pi = 0 \end{aligned}$$

So,  $f$  is concave down on  $(0, \pi)$ ,  
and  $f$  is concave up on  $(\pi, 2\pi)$ .  
 $(\pi, 0)$  is the inflection point.



(d)  $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

Sol.:  $f'(x) = 12x^3 + 12x^2 - 24x$

$$f''(x) = 36x^2 + 24x - 24$$

$$\begin{aligned} \text{Put } f''(x) = 0 &\Rightarrow 36x^2 + 24x - 24 = 0 \\ &\Rightarrow 3x^2 + 2x - 2 = 0 \end{aligned}$$

$$x = \frac{-B \mp \sqrt{B^2 - 4AC}}{2A},$$

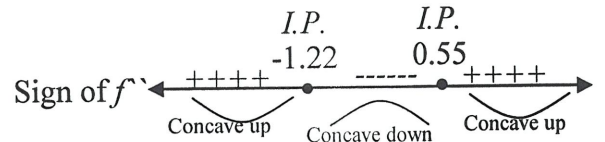
where  $A = 3$ ,  $B = 2$ ,  $C = -2$

$$\therefore x = \frac{-2 \mp \sqrt{2^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3} = \frac{-1 \mp \sqrt{7}}{3}$$

$$\text{Either } x = \frac{-1 - \sqrt{7}}{3} = -1.22 \Rightarrow y = -16.36$$

$$\text{Or } x = \frac{-1 + \sqrt{7}}{3} = 0.55 \Rightarrow y = -0.68$$

So,  $f$  is concave up on intervals  $(-\infty, -1.22)$  and  $(0.55, \infty)$ ,  
and  $f$  is concave down on interval  $(-1.22, 0.55)$ .  
It has *I.P.* at points  $(-1.22, -16.36)$  and  $(0.55, -0.68)$ .



Example: If  $y = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 8$ , find:

- Critical points.
- The intervals in which the function increases and decreases.
- Maximum and minimum values of  $y$ .
- The intervals in which the function is concave up or concave down.
- The inflection points.

Sol.:

(a) Critical points:

$$y = \frac{x^3}{3} + \frac{x^2}{2} - 6x + 8 \Rightarrow y' = \frac{3x^2}{3} + \frac{2x}{2} - 6 \Rightarrow y' = x^2 + x - 6$$

Put  $y' = 0 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$ ,  
 either  $x+3 = 0 \Rightarrow x = -3 \Rightarrow y = 43/2$   
 or  $x-2 = 0 \Rightarrow x = 2 \Rightarrow y = 2/3$

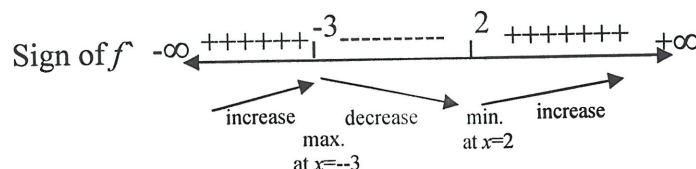
$\therefore$  The critical points are  $(-3, 43/2), (2, 2/3)$ .

(b) Increasing and decreasing intervals:

- The function increases on intervals:  $(-\infty, -3]$  and  $[2, \infty)$ .
- The function decreases on interval:  $[-3, 2]$

(c) Maximum and minimum values:

- The function has max. value of  $(y)$  at  $x = -3 \Rightarrow y = 43/2$
- The function has min. value of  $(y)$  at  $x = 2 \Rightarrow y = 2/3$ .



Or by second derivative test:

$y' = x^2 + x - 6 \Rightarrow y''(x) = 2x + 1$

$y''(-3) = 2(-3) + 1 = -5 < 0 \leftarrow$  There is a max. value at  $x = -3$ .

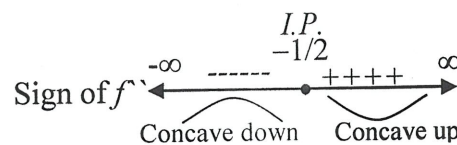
$y''(2) = 2(2) + 1 = 5 > 0 \leftarrow$  There is a min. value at  $x = 2$ .

(d) Concavity intervals:

$y'' = 2x + 1$

Put  $y'' = 0 \Rightarrow 2x + 1 = 0$

$\Rightarrow x = -\frac{1}{2} \Rightarrow y = \frac{133}{12}$



- The function is concave down on interval:  $(-\infty, -1/2)$ .
- The function is concave up on interval:  $(-1/2, \infty)$ .

(e) The inflection point is  $(-\frac{1}{2}, \frac{133}{12})$ .

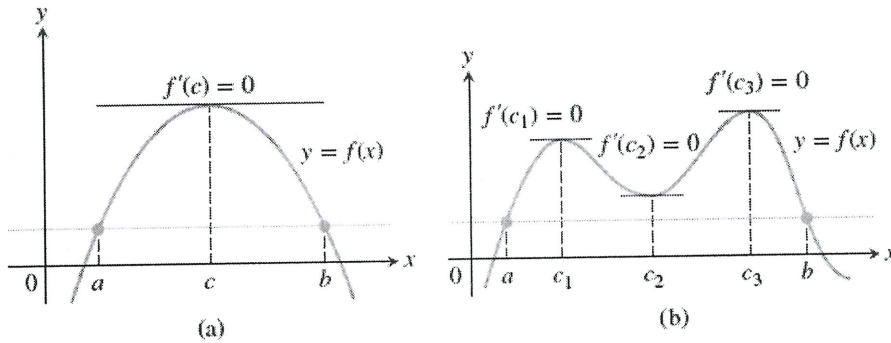
## 2. The Mean Value Theorem

**Rolle's Theorem:** Let  $y=f(x)$  is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ . If

$$f(a) = f(b) = 0,$$

then there is at least one number  $c$  in  $(a,b)$  at which

$$f'(c) = 0.$$



Example: Does Rolle's Theorem is applicable on the following functions. If so, find the value or values of  $c$ .

1.  $y = 2x - x^2$ ;  $[0,2]$

Sol.: 1. The function is continuous on  $[0,2]$ .

2.  $y' = 2 - 2x$  is differentiable on  $(0,2)$ .

3.  $f(0) = 2 \cdot 0 - 0^2 = 0$  and  $f(2) = 2 \cdot 2 - 2^2 = 0$  o.k.

$\therefore$  Rolle's Theorem is applicable on this function on  $[0,2]$ .

To find the value of  $c$ : Put  $y' = 0$

$$\therefore 2 - 2x = 0 \Rightarrow 2x = 2 \Rightarrow x = \frac{2}{2} = 1$$

$$\therefore c = 1$$

2.  $y = \frac{x^3}{3} - 3x$ ;  $[-3,3]$

Sol.: 1. The function is continuous on  $[-3, 3]$ .

2.  $y' = x^2 - 3$  is differentiable on  $(-3,3)$ .

3.  $f(-3) = \frac{(-3)^3}{3} - 3(-3) = \frac{-27}{3} + 9 = 0$  and  $f(3) = \frac{(3)^3}{3} - 3(3) = \frac{27}{3} - 9 = 0$  o.k.

$\therefore$  Rolle's Theorem is applicable on this function on  $[-3,3]$ .

To find the values of  $c$ : Put  $y' = 0$

$$\therefore x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

$$\therefore c_1 = -\sqrt{3} \text{ and } c_2 = \sqrt{3}.$$

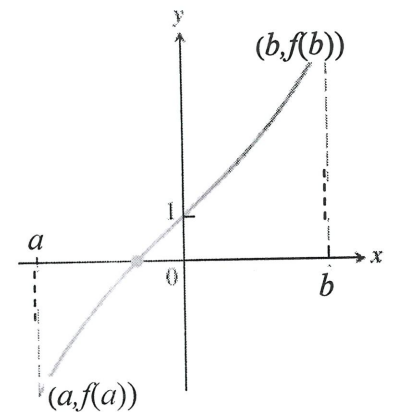


**Finding Solution of Equations:**

*Corollary*<sup>1</sup>: Suppose that:

1.  $f$  is continuous on  $[a, b]$ , and differentiable on  $(a, b)$ .
2.  $f(a)$  and  $f(b)$  have opposite signs.
3.  $f' \neq 0$  between  $a$  and  $b$ .

Then  $f$  has exactly one zero between  $a$  and  $b$ .



Example: Show that the equation  $x^3 + 3x + 1 = 0$  has exactly one real solution on the interval  $[-1, 1]$ .

Sol.: Let  $y = f(x) = x^3 + 3x + 1$

Then the derivative  $f'(x) = 3x^2 + 3$ , so

1.  $f$  is continuous on  $[-1, 1]$ , and differentiable on  $(-1, 1)$ .

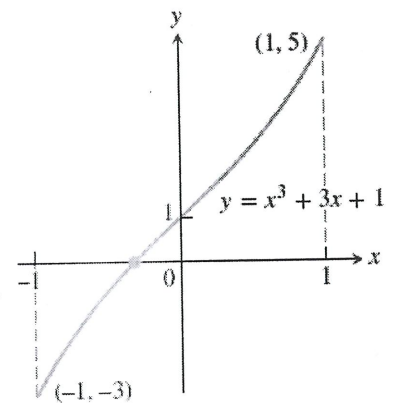
2.  $f(a) = f(-1) = (-1)^3 + 3(-1) + 1 = -3 < 0$  (negative)

$f(b) = f(1) = (1)^3 + 3(1) + 1 = 5 > 0$  (positive)

Thus  $f(-1)$  and  $f(1)$  have opposite signs.

3.  $f'(x) = 3x^2 + 3$  is never zero (because it is the sum of two positive numbers).

So corollary<sup>1</sup> is applicable on this function and the above equation has exactly one real solution on  $(-1, 1)$ .



Example: Show that the equation  $x^4 + 3x + 1 = 0$  has exactly one real root between  $a = -2$  and  $b = -1$ .

Sol.: Let  $y = f(x) = x^4 + 3x + 1$

Then the derivative  $f'(x) = 4x^3 + 3$ , so

3.  $f$  is continuous on  $[-2, -1]$ , and differentiable on  $(-2, -1)$ .

4.  $f(a) = f(-2) = (-2)^4 + 3(-2) + 1 = 11 > 0$  (positive)

$f(b) = f(-1) = (-1)^4 + 3(-1) + 1 = -1 < 0$  (negative)

Thus  $f(-2)$  and  $f(-1)$  have opposite signs.

3. When  $f'(x) = 4x^3 + 3 = 0 \Rightarrow x^3 = \frac{-3}{4} \Rightarrow x = \sqrt[3]{\frac{-3}{4}} \approx -0.91 \notin (-2, -1)$

Thus  $f'(x) \neq 0$  on  $(-1, -2)$ .

So corollary<sup>1</sup> is applicable on this function and the above equation has exactly one real root on  $(-2, -1)$ .



**The Mean Value Theorem (M. V. T.):**

Suppose  $y=f(x)$  is continuous on the closed interval  $[a,b]$  and differentiable on the open interval  $(a,b)$ . Then there is at least one point  $c$  in  $(a,b)$  at which

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Example: Is the M. V. T. is applicable on the following functions. If so find the value or values of  $c$ .

1.  $f(x) = x - 2\sin x$ ;  $0 \leq x \leq 2\pi$

Sol.: 1.  $f(x) = x - 2\sin x$  is continuous on  $[0, 2\pi]$ .

2.  $f'(x) = 1 - 2\cos x$  is differentiable on  $(0, 2\pi)$ .

$\therefore$  The M. V. T. is applicable on  $[0, 2\pi]$ .

To find  $c$ :

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

where  $f(b) = f(2\pi) = 2\pi - 2\sin 2\pi = 2\pi - 0 = 2\pi$

$$f(a) = f(0) = 0 - 2\sin 0 = 0 - 0 = 0$$

and  $f'(c) = 1 - 2\cos c$ , thus:

$$1 - 2\cos c = \frac{2\pi - 0}{2\pi - 0} \Rightarrow 1 - 2\cos c = 1 \Rightarrow 2\cos c = 0 \Rightarrow \cos c = 0$$

$$\therefore c = \mp \frac{n\pi}{2}; \quad n = 1, 3, 5, \dots$$

$$\therefore c_1 = \frac{\pi}{2} \quad \text{and} \quad c_2 = \frac{3\pi}{2} \quad \text{on the interval } [0, 2\pi].$$

2.  $f(x) = x^{2/3}$ ;  $[-8, 8]$

Sol.: 1.  $f(x) = x^{2/3} = \sqrt[3]{x^2}$  is continuous on  $[-8, 8]$ .

2.  $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$  is not differentiable  $x = 0 \in (-8, 8)$ .

$\therefore$  The M. V. T. is not applicable on  $[-8, 8]$ .

3.  $f(x) = x^{2/3}$ ;  $[0, 8]$

Sol.: 1.  $f(x) = x^{2/3} = \sqrt[3]{x^2}$  is continuous on  $[0, 8]$ .

2.  $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$  is not differentiable  $x = 0 \notin (0, 8)$

So it is differentiable on  $(0, 8)$

$\therefore$  The M. V. T. is applicable on  $[0, 8]$ .

To find  $c$ :

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

where  $f(b) = f(0) = 0^{2/3} = 0$

$$f(b) = f(8) = 8^{2/3} = 4$$

and  $f'(c) = \frac{2}{3\sqrt[3]{c}}$ , thus:

$$\frac{2}{3\sqrt[3]{c}} = \frac{4-0}{8-0} \Rightarrow \frac{2}{3\sqrt[3]{c}} = \frac{1}{2} \Rightarrow \sqrt[3]{c} = \frac{3}{4} \Rightarrow c = \left(\frac{3}{4}\right)^3 = \frac{27}{64} = 0.421875$$

Note: If  $f'(x)$  is continuous on  $[a, b]$ , the Max.-Min. Theorem for continuous functions tells us that  $f'$  has absolute maximum value ( $\max f'$ ) and absolute minimum value ( $\min f'$ ) on the interval, the equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

gives us the inequality:

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f'$$

Example: Estimate  $f(1)$  if  $f'(x) = \frac{1}{5-x^2}$  and  $f(0)=2$ .

Sol.:  $a = 0 \Rightarrow f(a) = f(0) = 2$

$b = 1 \Rightarrow f(b) = f(1) = ?$

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f'$$

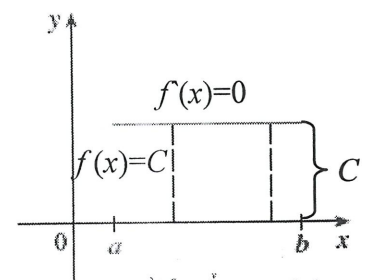
$$\frac{1}{5-0^2} \leq \frac{f(1) - 2}{1-0} \leq \frac{1}{5-1^2}$$

$$\frac{1}{5} \leq f(1) - 2 \leq \frac{1}{4}$$

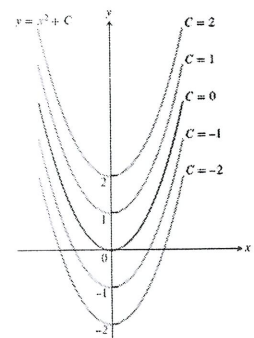
$$0.2 + 2 \leq f(1) \leq 0.25 + 2$$

$$2.2 \leq f(1) \leq 2.25$$

Corollary<sup>2</sup>: If  $f'(x) = 0$  for all  $x$  in an interval  $(a, b)$ , then  $f = C$ , for all  $x \in (a, b)$ , where  $C$  is a constant.



Corollary<sup>3</sup>: If  $f'(x) = g'(x)$  for all  $x$  in an interval  $(a, b)$ , then  $f-g$  is constant on  $(a, b)$ ; that is  $f(x) = g(x) + C$ , where  $C$  is a constant.



# Integration

1

There are two meanings of the integration:

1<sup>st</sup>: Integration means the process of finding a function  $F(x)$  given its derivative  $(f(x))$  (or the process of finding an antiderivative  $(F(x))$  of  $(f(x))$ ). "Indefinite Integral"

2<sup>nd</sup>: Integration means adding up (summation) infinitely many things. "Definite Integral"  
such as finding area and volume of <sup>any</sup> geometer, find area under a curve, finding area between curves, --- etc.

We must distinguish between Indefinite and Definite integrals, where the result of indefinite integral is a function, whereas the result of definite integral is a number.

## Antiderivative

Ex: Find an antiderivative  $(F(x))$  of  $f(x) = 2x$ .

Sol.: let  $\frac{dy}{dx} = f(x) = 2x \Rightarrow dy = 2x dx$

let  $y = F(x)$

~~is~~  $d(F(x)) = 2x dx$

So, antiderivative means finding  $F(x)$  when its derivative  $(f(x))$  equals  $(2x dx)$

By our experience in the <sup>derivative</sup> ~~differentiation~~:

$$F(x) = x^2$$

To check  $d(F(x)) = F'(x) = 2x dx$

We must note that the function  $F(x) = x^2$  is not the only

function whose derivative is  $2x$ . The functions  $(x^2+1)$ ,  $(x^2-\sqrt{2})$ ,  $(x^2+5\pi)$ , ... have the same derivative ( $2x$ ).

So, the functions  $(x^2+c)$ , where  $c$  is an arbitrary constant, ~~are~~ form all derivatives of  $f(x)=2x$

Question: Find an antiderivative of (1)  $f(x) = x^{-1/2}$   
(2)  $f(x) = x^2$

### Indefinite Integrals

The set of all antiderivatives of  ~~$f(x)$~~  is the indefinite integral of  $f$  with respect to  $x$  denoted by:

$$\int f(x) dx = \int d(F(x))$$

The symbol  $\int$  is an integral sign. The function  $f$  is the integrand of the integral, and  $x$  is the variable of integration.

### Rules of indefinite integral

1-  $\int dx = x + c$  ;  $c$ : an arbitrary constant

2-  $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

3-  $\int [g(x)]^n \cdot g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + c$

4-  $\int k f(x) dx = k \int f(x) dx$  ;  $k$ : constant

5-  $\int [f(x) \mp g(x)] dx = \int f(x) dx \mp \int g(x) dx$

Examples = Evaluate the following integrals:

1-  $\int x^5 dx$  ; according to rule no. (2)

$$\int x^5 dx = \frac{x^6}{6} + c$$

to check = By derivative;  $\frac{1}{6} \cdot (6x^5) dx = x^5 dx \therefore o.k$

2-  $\int (x+5)^5 dx$  ; according to rule no. (3)

let  $g(x) = x+5$  ;  $n=5$

$g'(x) = 1.0$

so;  $\int (x+5)^5 dx = \int [g(x)]^5 \cdot g'(x) dx = \frac{[g(x)]^{5+1}}{5+1} + c$

$$= \frac{(x+5)^6}{6} + c$$

check = By derivative;  $\frac{1}{6} \cdot 6(x+5)^5 dx = (x+5)^5 dx \therefore o.k$

3-  $\int \sqrt{2x+1} dx = \int (2x+1)^{1/2} dx$

according to rule no. (3) : let  $g(x) = 2x+1$  ;  $n=1/2$

$g'(x) = 2$

In the integration, we just have  $dx$ . So we need to multiply the integration by 2 and divided by 2:

$\therefore \int (2x+1)^{1/2} dx \cdot \frac{2}{2} = \int \frac{1}{2} (2x+1)^{1/2} \cdot 2 dx$

according to rule no. (3) :  $\int \frac{1}{2} (2x+1)^{1/2} \cdot 2 dx = \frac{1}{2} \int (2x+1)^{1/2} \cdot 2 dx$

$$= \frac{1}{2} \frac{(2x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{1}{2} \cdot \frac{(2x+1)^{3/2}}{3/2} + c$$

$$= \frac{1}{2} \cdot \frac{2}{3} (2x+1)^{3/2} + c$$

$$= \frac{1}{3} (2x+1)^{3/2} + c$$

check : By derivative;

$\frac{1}{3} \cdot \frac{3}{2} (2x+1)^{1/2} \cdot 2 dx = (2x+1)^{1/2} dx \therefore o.k$



$$4- \int (3x^2 - \sqrt{x}) dx = \int (3x^2 - x^{1/2}) dx = I$$

according to rule no. (5):

$$\therefore I = \int 3x^2 dx - \int x^{1/2} dx = I_1 - I_2$$

For  $I_1$ : According to rule no. (4) :  $I_1 = 3 \int x^2 dx = 3 \frac{x^3}{3} + c_1 = x^3 + c_1$

$$\therefore I_1 = x^3 + c_1 ; c_1 : \text{an arbitrary constant}$$

For  $I_2$ , According to rule no. (2):

$$I_2 = \int x^{1/2} dx = \frac{x^{3/2}}{3/2} + c_2 = \frac{2}{3} x^{3/2} + c_2 ; c_2 : \text{an arbitrary constant}$$

$$\therefore I = x^3 + c_1 - \left( \frac{2}{3} x^{3/2} + c_2 \right) = x^3 - \frac{2}{3} x^{3/2} + c_1 - c_2$$

$$\text{let } c_1 - c_2 = C$$

$$\therefore I = x^3 - \frac{2}{3} x^{3/2} + C$$

$$5- \int (x^2 + 2x - 3)^2 (x+1) dx$$

According to rule no. (3) : let  $g(x) = x^2 + 2x - 3 ; n=2$

$$\therefore g'(x) = 2x + 2 = 2(x+1)$$

So multiply the integration by  $\frac{3}{2}$

$$\begin{aligned} \therefore I &= \frac{1}{2} \int (x^2 + 2x - 3)^2 \cdot 2(x+1) dx = \frac{1}{2} \frac{(x^2 + 2x - 3)^3}{3} + C \\ &= \frac{1}{6} (x^2 + 2x - 3)^3 + C \end{aligned}$$

$$6- \int \frac{t^2 dt}{\sqrt[3]{(1+2t^3)^2}} = \int \frac{t^2 dt}{(1+2t^3)^{2/3}} = \int (1+2t^3)^{-2/3} t^2 dt$$

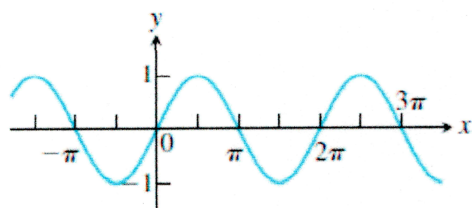
according to rule no. (3) : let  $g(x) = 1 + 2t^3 ; n = -\frac{2}{3}$

$$g'(x) = 6t^2$$

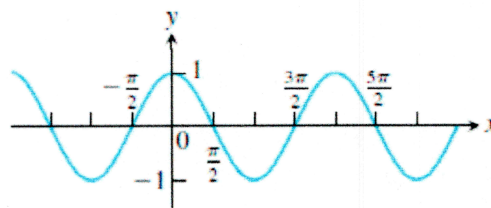
## Transcendental Functions

**Transcendental Functions** These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.

### Trigonometric Functions



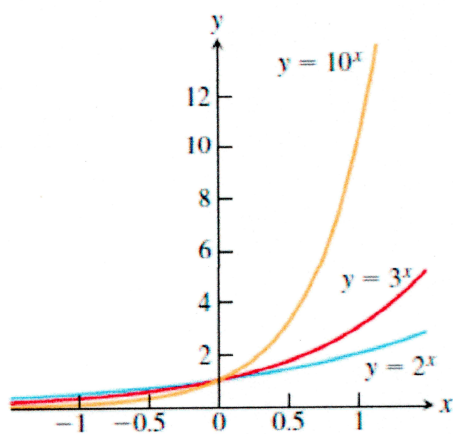
(a)  $f(x) = \sin x$



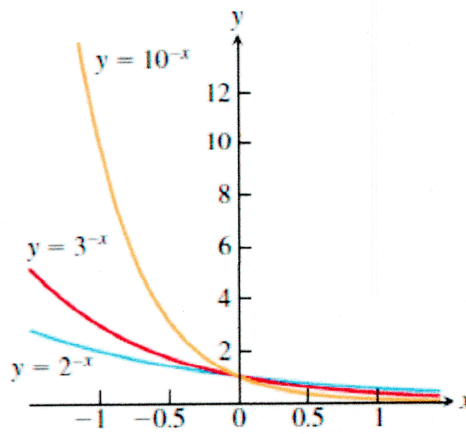
(b)  $f(x) = \cos x$

Graphs of the sine and cosine functions.

### Exponential Functions



(a)

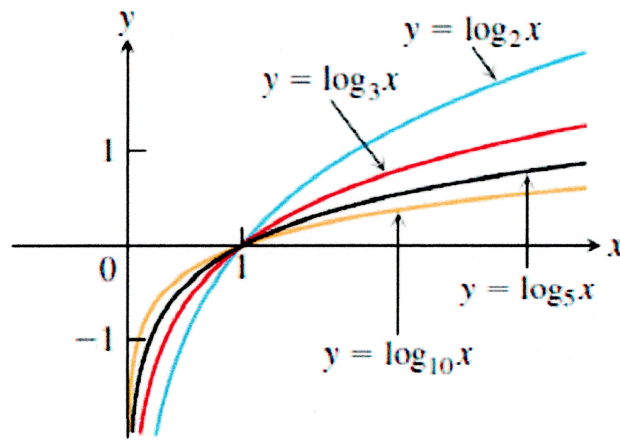


(b)

Graphs of exponential functions.

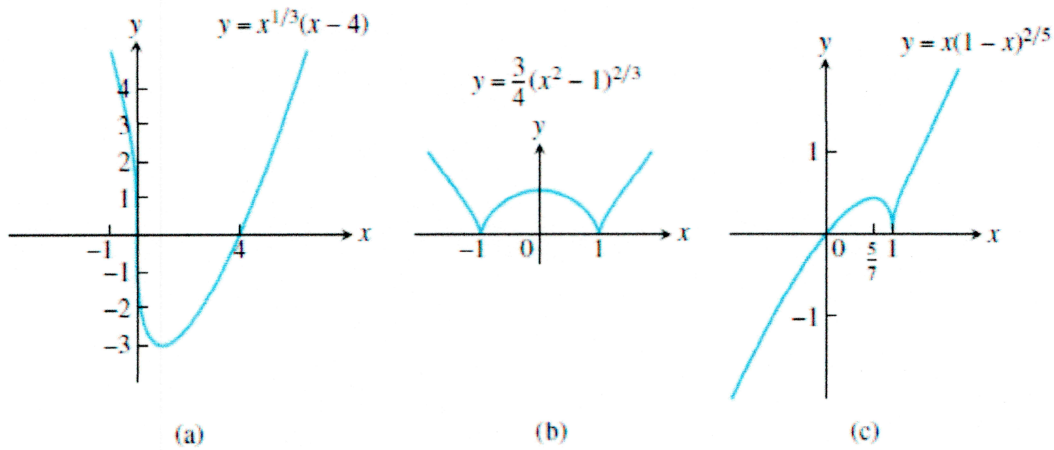
②

## Logarithmic Functions



Graphs of four logarithmic functions.

**Algebraic Functions** Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of algebraic functions.



Graphs of three algebraic functions.

## Inverse Functions

A function that undoes, or inverts, the effect of a function  $f$  is called the *inverse* of  $f$ . Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in applications.

The symbol  $f^{-1}$  for the inverse of  $f$  is read “ $f$  inverse.” The “ $-1$ ” in  $f^{-1}$  is *not* an exponent;  $f^{-1}(x)$  does not mean  $1/f(x)$ .

The process of passing from  $f$  to  $f^{-1}$  can be summarized as a two-step procedure.

1. Solve the equation  $y = f(x)$  for  $x$ . This gives a formula  $x = f^{-1}(y)$  where  $x$  is expressed as a function of  $y$ .
2. Interchange  $x$  and  $y$ , obtaining a formula  $y = f^{-1}(x)$  where  $f^{-1}$  is expressed in the conventional format with  $x$  as the independent variable and  $y$  as the dependent variable.

**EXAMPLE** Find the inverse of  $y = \frac{1}{2}x + 1$ , expressed as a function of  $x$ .

**Solution**

1. Solve for  $x$  in terms of  $y$ :  $y = \frac{1}{2}x + 1$

$$2y = x + 2$$

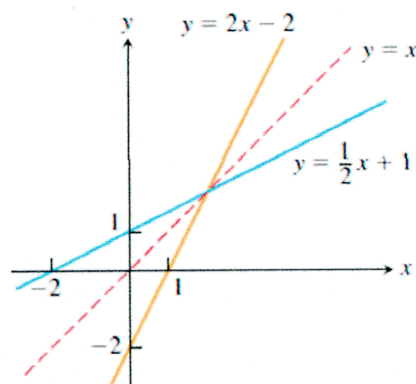
$$x = 2y - 2.$$

2. Interchange  $x$  and  $y$ :  $y = 2x - 2$ .

The inverse of the function  $f(x) = (1/2)x + 1$  is the function  $f^{-1}(x) = 2x - 2$ . (See Figure 1) To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$



**FIGURE 1** Graphing the functions  $f(x) = (1/2)x + 1$  and  $f^{-1}(x) = 2x - 2$  together shows the graphs' symmetry with respect to the line  $y = x$ .



(4)

## Logarithms

To understand the logarithm consider the expression:

$$16 = 2^4 \rightarrow \text{Power or Index}$$

$\rightarrow$  base

An alternative way of writing this expression is:

$$\text{Log}_2 16 = 4$$

$\rightarrow$  base

This is stated as 'Log to the base 2 of 16 equals 4'

The two statements:  $16 = 2^4$  and  $\text{Log}_2 16 = 4$

are equivalent. If we write either of them, we are automatically implying the other.

In general, if a number  $x$  can be written in the form of  $a^y$  ( $x = a^y$ ), then the index  $y$  is called the 'Logarithm of  $x$  to the base of  $a$ ',

i.e. if  $x = a^y$  then  $\text{Log}_a x = y$

Examples:

the equivalent statement of the expression

$$64 = 8^2 \text{ is } 2 = \text{Log}_8 64$$

and for the expression

$$\text{Log}_3 27 = 3 \text{ is } 3^3 = 27$$



⑤

From above, if we write down that:  $1000 = 10^3$ ,  
then  $3 = \text{Log}_{10} 1000$ . [check by your calculator]

Logarithms having a base of 10 are called **common Logarithms**  
and  $\text{Log}_{10}$  is usually abbreviated to **Log**.

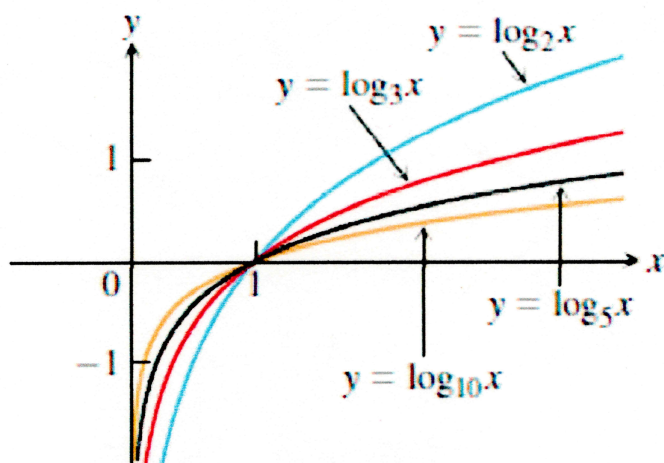
check the following values by using a calculator:

$$\text{Log } 27.5 = 1.4393\dots, \text{Log } 378.1 = 2.5776\dots$$

$$\text{Log } 0.0204 = -1.6903\dots$$

In the logarithm functions  $f(x) = \text{Log}_a x$ , the base  $a \neq 1$  is a positive constant. They are the inverse functions of the exponential functions.

Fig. 1 shows the graphs of four logarithmic functions with various bases. In each case the domain is  $(0, \infty)$  and the range is  $(-\infty, \infty)$



**FIGURE 1** Graphs of four logarithmic functions.

⑥

natural logarithms.

Logarithms having a base of  $e$  (where ' $e$ ' is a mathematical constant approximately equal to 2.7183) are called natural logarithms, and  $\text{Log}_e$  is usually abbreviated to  $\text{Ln}$ .

check the following values by your calculator:

$$\ln 3.65 = 1.2947\dots, \ln 417.3 = 6.0338\dots$$

$$\ln 0.182 = -1.7037\dots$$

problem 1. Evaluate:  $\log_3 9$  [log base 3 of 9]

Let  $\log_3 9 = y$  then  $3^y = 9$

i.e.  $3^y = 3^2$  from which,  $y = 2$  hence,  $\log_3 9 = 2$



problem 2. Evaluate:  $\log_{10} 10$

Let  $\log_{10} 10 = y$  then  $10^y = 10$

i.e.  $10^y = 10^1$  from which,  $y = 1$  hence,  $\log_{10} 10 = 1$

problem 3. Evaluate:  $\log_{16} 8$

Let  $\log_{16} 8 = y$  then  $16^y = 8$

i.e.  $(2^4)^y = 2^3$ , i.e.  $2^{4y} = 2^3$

from which,  $4y = 3$  and  $y = \frac{3}{4}$

Hence,  $\log_{16} 8 = \frac{3}{4}$

to calculate by calculator try  
 $\log_{16} 8 = \frac{\log 8}{\log 16} = \frac{3}{4}$   
or  $\frac{\ln 8}{\ln 16} = \frac{3}{4}$



(7)

problem 4. Evaluate:  $\log_{10} 0.001$

Let  $\log_{10} 0.001 = y$  then  $10^y = 0.001$

i.e.  $10^y = 10^{-3}$  from which  $y = -3$

problem 5. Evaluate:  $\ln e$

Let  $\ln e = y \Rightarrow \log_e e = y$  then  $e^y = e$

i.e.  $e^y = e^1 \Rightarrow y = 1 \Rightarrow \ln e = 1$

problem 6. Solve the equation:  $\log x = 3$

If  $\log x = 3$  then  $\log_{10} x = 3$

$10^3 = x$  i.e.  $x = 1000$

problem 7. solve the equation:  $\log_2 x = 5$

If  $\log_2 x = 5$  then  $2^5 = x = 32$

problem 8. solve the equation:  $\log_5 x = -2$

If  $\log_5 x = -2$  then  $5^{-2} = x = \frac{1}{5^2} = \frac{1}{25}$

problem 9. Evaluate:  $\log_2 \left(\frac{1}{8}\right)$

$\log_2 \left(\frac{1}{8}\right) = y$  then  $2^y = \frac{1}{8}$

$2^y = \frac{1}{2^3} \Rightarrow 2^y = 2^{-3} \Rightarrow y = -3$

$\log_2 \left(\frac{1}{8}\right) = -3$



⑧

problem 10. Evaluate:  $\log 1$

$$\log_{10} 1 = y \Rightarrow 10^y = 1 \Rightarrow y = 0$$

$$\log 1 = 0$$

$$\log_a 1 = 0$$

problem 11. Evaluate:  $\log 0$

$$\log_{10} 0 = y \Rightarrow 10^y = 0$$

{ there is no number of (y) that  
can make  $10^y = 0$

this log is [undefined]

problem 12. Evaluate:  $\log(-1)$

$$\log_{10}(-1) = y \Rightarrow 10^y = -1$$

{ there is no number of (y) that  
can make  $10^y = \text{negative number}$

this log is [undefined]

problem 13. Evaluate:  $\ln 1$

$$\ln 1 = \log_e 1 = y \Rightarrow e^y = 1 \Rightarrow y = 0 \Rightarrow \ln 1 = 0$$

problem 14. Evaluate:  $\ln(e^3)$

$$\log_e(e^3) = y \Rightarrow e^y = e^3 \Rightarrow y = 3$$

$$\ln(e^3) = 3$$

problem 15. solve  $\log_x 32 = 5$

$$x^5 = 32 = 2^5 \quad \text{then } \boxed{x=2}$$

problem 16. calculate  $\log_2 7$

$$2^y = 7, \quad \log_2 7 = \frac{\log 7}{\log 2} = 2.807..$$

## Properties of Logarithms

**THEOREM — Algebraic Properties of the Common Logarithm** For any numbers  $x > 0$  and  $y > 0$ , the common logarithm (base  $a$ ) satisfies the following rules:

1. *Product Rule:*  $\log_a xy = \log_a x + \log_a y$
2. *Quotient Rule:*  $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. *Reciprocal Rule:*  $\log_a \frac{1}{y} = -\log_a y$
4. *Power Rule:*  $\log_a x^y = y \log_a x$
5. *Equality Rule:* If  $[\log_a m = \log_a n]$  Then  $[m = n]$

**THEOREM — Algebraic Properties of the Natural Logarithm** For any numbers  $b > 0$  and  $x > 0$ , the natural logarithm satisfies the following rules:

1. *Product Rule:*  $\ln bx = \ln b + \ln x$
2. *Quotient Rule:*  $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule:*  $\ln \frac{1}{x} = -\ln x$       Rule 2 with  $b = 1$
4. *Power Rule:*  $\ln x^r = r \ln x$       For  $r$  rational

problem 17. solve for  $x$ . check your solution:

$$\log(12x - 7) = \log(3x + 11)$$

From the equality rule  $(12x - 7) = (3x + 11)$

$$\therefore 12x - 3x = 11 + 7 \Rightarrow 9x = 18 \Rightarrow x = 2$$

check:

$$\log(12 \cdot 2 - 7) \stackrel{?}{=} \log(3 \cdot 2 + 11)$$

$$\log(24 - 7) \stackrel{?}{=} \log(6 + 11)$$

$$\log(17) = \log(17) \quad \checkmark$$



(10)

problem 18. solve  $\log_2 x + \log_2 (x-2) = 3$

by using the product rule:

$$\log_2 x(x-2) = 3$$

$$\therefore 2^3 = x(x-2)$$

$$8 = x^2 - 2x$$

$$0 = x^2 - 2x - 8$$

$$0 = (x-4)(x+2)$$

$$x-4 = 0 \Rightarrow x=4$$

$$x+2 = 0 \Rightarrow x=-2$$

check:

$$\log_2 4 + \log_2 (4-2) \stackrel{?}{=} 3$$

$$\log_2 4 + \log_2 2 \stackrel{?}{=} 3$$

$$2 + 1 = 3 \quad \checkmark$$

$$\log_2 (-2) + \log_2 (-2-2) \stackrel{?}{=} 3$$

$$\log_2 (-2) + \log_2 (-4) \neq 3 \quad \times$$

$$\therefore x = 4$$

problem 19. solve. check your solutions.

$$\log_4 36 - \log_4 2x = 2 \log_4 3 + \log_4 2$$

$$\log_4 \left(\frac{36}{2x}\right) = \log_4 (3)^2 + \log_4 2$$

$$\log_4 \left(\frac{36}{2x}\right) = \log_4 (9 \cdot 2)$$

$$\log_4 \left(\frac{36}{2x}\right) = \log_4 (18)$$

$$\therefore \frac{36}{2x} = 18 \Rightarrow 36 = 36x \Rightarrow x = 1$$

$$\text{check} \Rightarrow \log_4 18 \stackrel{?}{=} \log_4 9 + \log_4 2 \Rightarrow \log_4 18 = \log_4 (9 \cdot 2) \quad \checkmark$$

①①

EX: Given that  $\log 2 \approx 0.3010$ ,  $\log 3 \approx 0.4771$ ,  
and  $\log 5 \approx 0.6990$ , use the laws of logarithms  
to find: a)  $\log 7.5$     b)  $\log 81$     c)  $\log 50$

Sol / a)  $\log 7.5 = \log\left(\frac{15}{2}\right)$   
 $= \log\left(\frac{3 \times 5}{2}\right)$   
 $= \log 3 + \log 5 - \log 2$   
 $= 0.4771 + 0.6990 - 0.3010$   
 $= 0.875$

b)  $\log 81 = \log 3^4$   
 $= 4 \log 3$   
 $\approx 4(0.4771) = 1.9084$

c)  $\log 50 = \log(5 \times 10)$   
 $= \log 5 + \log 10$   
 $= \log 5 + \log(5 \times 2)$   
 $= \log 5 + \log 5 + \log 2$   
 $= 0.6990 + 0.6990 + 0.3010$   
 $= 1.6990$

## The Derivative of $\ln x$

If  $u$  is a differentiable function of  $x$  whose values are positive, so that  $\ln u$  is defined, then applying the chain Rule we obtain

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}, \quad u > 0.$$

Ex. Find the derivatives of the following equations.

(a)  $\frac{d}{dx} \ln 2x$

$$= \frac{1}{2x} \frac{d}{dx} (2x) = \frac{1}{2x} (2) = \frac{1}{x}, \quad x > 0$$

(b)  $\frac{d}{dx} \ln(x^2+3) = \frac{2x}{x^2+3}$

(c)  $\frac{d}{dx} \ln x^r = \frac{r x^{r-1}}{x^r} = r \cdot \frac{1}{x}$

The Integral of  $\int \frac{1}{u} du$ :

If  $u$  is differentiable function that is never zero,

$$\int \frac{1}{u} du = \ln |u| + C$$

Ex. Find the integral of the following equations:

a)  $\int \frac{dx}{3+2x} = \frac{1}{2} \int \frac{2 dx}{3+2x} = \frac{1}{2} \ln |3+2x| + C$

b)  $\int \frac{dx}{x \ln x} = \int \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$$\begin{aligned} \text{let } u &= \ln x \\ du &= \frac{1}{x} dx \end{aligned}$$

$$= \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C$$

(13)

Ex. Find  $\frac{dy}{dx}$  if :

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \quad x > 1$$

$$\ln y = \ln \frac{(x^2+1)(x+3)^{1/2}}{(x-1)}$$

$$\ln y = \ln [(x^2+1)(x+3)^{1/2}] - \ln(x-1) \quad \{\text{Quotient Rule}\}$$

$$\ln y = \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) \quad \{\text{product Rule}\}$$

$$\ln y = \ln(x^2+1) + \frac{1}{2} \ln(x+3) - \ln(x-1) \quad \{\text{Power Rule}\}$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2} \frac{1}{x+3} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y \left[ \frac{2x}{x^2+1} + \frac{1}{2} \frac{1}{x+3} - \frac{1}{x-1} \right]$$

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/2}}{x-1} \left[ \frac{2x}{x^2+1} + \frac{1}{2} \frac{1}{x+3} - \frac{1}{x-1} \right]$$



$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} && u = \cos x > 0 \text{ on } (-\pi/2, \pi/2), \\ & && du = -\sin x \, dx \\ &= -\ln |u| + C = -\ln |\cos x| + C \\ &= \ln \frac{1}{|\cos x|} + C = \ln |\sec x| + C.\end{aligned}$$

$$\begin{aligned}\int \cot x \, dx &= \int \frac{\cos x \, dx}{\sin x} = \int \frac{du}{u} && u = \sin x, \\ & && du = \cos x \, dx \\ &= \ln |u| + C = \ln |\sin x| + C = -\ln |\csc x| + C.\end{aligned}$$

$$\int \sec x \, dx =$$

To integrate  $\sec x$ , we multiply and divide by  $(\sec x + \tan x)$

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C && u = \sec x + \tan x, \\ & && du = (\sec x \tan x + \sec^2 x) dx\end{aligned}$$

$$\int \csc x \, dx =$$

For  $\csc x$ , we multiply and divide by  $(\csc x + \cot x)$

$$\begin{aligned}\int \csc x \, dx &= \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx \\ &= \int \frac{-du}{u} = -\ln |u| + C = -\ln |\csc x + \cot x| + C && u = \csc x + \cot x, \\ & && du = (-\csc x \cot x - \csc^2 x) dx\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/6} \tan 2x \, dx &= \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2} \int_0^{\pi/3} \tan u \, du \\ &= \frac{1}{2} \ln |\sec u| \Big|_0^{\pi/3} = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2\end{aligned}$$

Substitute  $u = 2x$ ,  
 $dx = du/2$ ,  
 $u(0) = 0$ ,  
 $u(\pi/6) = \pi/3$



## Derivatives and Integrals involving $\log_a x$ .

To find derivatives or integrals involving base  $a$  logarithms, we convert them to natural logarithms. If  $u$  is a positive differentiable function of  $x$ , then:

$$\frac{d}{dx} (\log_a u) = \frac{d}{dx} \left( \frac{\ln u}{\ln a} \right) = \frac{1}{\ln a} \frac{d}{dx} (\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}$$

$$\boxed{\frac{d}{dx} (\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \cdot \frac{du}{dx}}$$

Ex. Find the derivatives of the following equations:

$$\begin{aligned} \text{a) } y = \log_{10} (3x+1) &\Rightarrow \frac{dy}{dx} = \frac{1}{\ln 10} \cdot \frac{1}{3x+1} \cdot \frac{d}{dx} (3x+1) \\ &= \frac{3}{(\ln 10)(3x+1)} \end{aligned}$$

$$\text{b) } y = \log e^x$$

$$y = \frac{\ln e^x}{\ln 10} = \frac{x}{\ln 10} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 10}$$

$$\text{c) } y = \log_5 (x^2+1)^2$$

$$y = \frac{\ln (x^2+1)^2}{\ln 5} = \frac{2 \ln (x^2+1)}{\ln 5}$$

$$\frac{dy}{dx} = \frac{2}{\ln 5} \cdot \frac{2x}{(x^2+1)} = \frac{4x}{\ln 5 (x^2+1)}$$

(16)

Ex. Find the integrals:

$$a) \int \frac{\log_2 X}{X} dx = \frac{1}{\ln 2} \int \frac{\ln X}{X} dx$$

$$\log_2 X = \frac{\ln X}{\ln 2}$$

$$= \frac{1}{\ln 2} \int \frac{u}{\cancel{X}} \cancel{X} du$$

$$= \frac{1}{\ln 2} \int u du = \frac{1}{\ln 2} \frac{u^2}{2} + C$$

$$= \frac{1}{\ln 2} \frac{(\ln X)^2}{2} + C = \frac{(\ln X)^2}{2 \ln 2} + C$$

$$\begin{aligned} \text{Let } u &= \ln X \\ du &= \frac{1}{X} dx \\ \therefore dx &= X du \end{aligned}$$

$$b) \int_1^4 \frac{\ln 2 \log_2 X}{X} dx$$

$$= \frac{\ln 2}{\ln 2} \int_1^4 \frac{\ln X}{X} dx$$

$$= \int_0^{\ln 4} \frac{u}{\cancel{X}} \cancel{X} du$$

$$= \frac{u^2}{2} \Big|_0^{\ln 4}$$

$$= \frac{(\ln 4)^2}{2} = \frac{(\ln 2 + \ln 2)^2}{2} = \frac{(\ln 2)^2 + 2(\ln 2)(\ln 2) + (\ln 2)^2}{2}$$

$$= \frac{4(\ln 2)^2}{2} = 2(\ln 2)^2$$

$$\begin{aligned} \text{Let } u &= \ln X \\ du &= \frac{dx}{X} \\ dx &= X du \end{aligned}$$

$$\begin{aligned} u(1) &= \ln 1 = 0 \\ u(4) &= \ln 4 \end{aligned}$$

(17)

$$c) \int_0^2 \frac{\log_2(x+2)}{(x+2)} dx$$

$$\int_0^2 \frac{\ln(x+2)}{\ln 2 \cdot (x+2)} dx$$

$$= \frac{1}{\ln 2} \int_0^2 \frac{\ln(x+2)}{(x+2)} dx$$

$$= \frac{1}{\ln 2} \int_{\ln 2}^{\ln 4} \frac{u}{\cancel{(x+2)} \cancel{(x+2)}} du$$

$$= \frac{1}{\ln 2} \left[ \frac{u^2}{2} \right]_{\ln 2}^{\ln 4}$$

$$= \frac{1}{2 \ln 2} \left[ (\ln 4)^2 - (\ln 2)^2 \right]$$

$$= \frac{1}{2 \ln 2} \left[ (\ln 2 \times 2)^2 - (\ln 2)^2 \right]$$

$$= \frac{1}{2 \ln 2} \left[ (\ln 2 + \ln 2)^2 - (\ln 2)^2 \right]$$

$$= \frac{1}{2 \ln 2} \left[ (\ln 2)^2 + 2(\ln 2)^2 + (\ln 2)^2 - (\ln 2)^2 \right]$$

$$= \frac{1}{2 \ln 2} \left[ 3(\ln 2)^2 \right] = \frac{3}{2} \ln 2$$

Let  $u = \ln(x+2)$   
 $du = \frac{dx}{(x+2)}$   
 $dx = (x+2) du$   
 $u(0) = \ln 2$   
 $u(2) = \ln 4$

## Exponential Functions

**Exponential Functions** Functions of the form  $f(x) = a^x$ , where the base  $a > 0$  is a positive constant and  $a \neq 1$ , are called exponential functions. All exponential functions have domain  $(-\infty, \infty)$  and range  $(0, \infty)$ , so an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 1

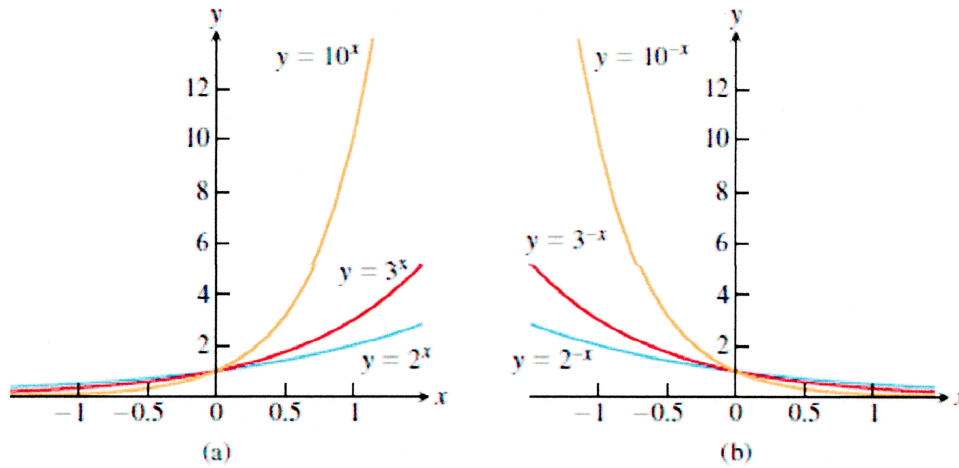


Figure 1 Graphs of exponential functions.

If  $a$  is any positive number other than 1, the function  $a^x$  is one-to-one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the logarithm of  $x$  with base  $a$  and denote it by  $\log_a x$ .

**DEFINITION** For any positive number  $a \neq 1$ ,

$\log_a x$  is the inverse function of  $a^x$ .

The graph of  $y = \log_a x$  can be obtained by reflecting the graph of  $y = a^x$  across the  $45^\circ$  line  $y = x$  (Figure 2). When  $a = e$ , we have  $\log_e x =$  inverse of  $e^x = \ln x$ .

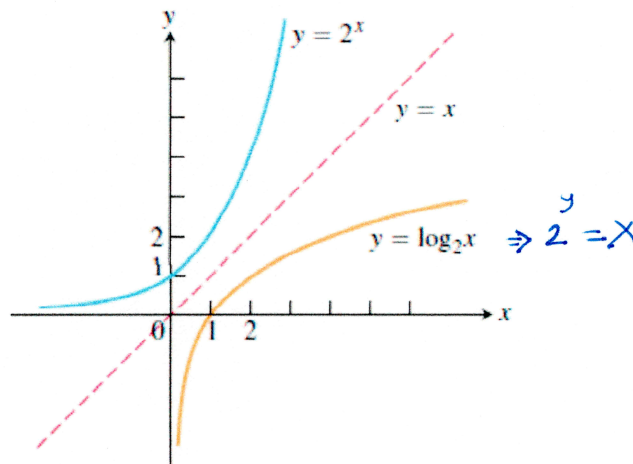


Figure 2 The graph of  $2^x$  and its inverse,  $\log_2 x$ .

## The Inverse of $\ln x$ and the Number $e$

The function  $\ln x$ , being an increasing function of  $x$  with domain  $(0, \infty)$  and range  $(-\infty, \infty)$ , has an inverse  $\ln^{-1} x$  with domain  $(-\infty, \infty)$  and range  $(0, \infty)$ . The graph of  $\ln^{-1} x$  is the graph of  $\ln x$  reflected across the line  $y = x$ . As you can see in Figure 3.

$$\lim_{x \rightarrow \infty} \ln^{-1} x = \infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} \ln^{-1} x = 0.$$

The function  $\ln^{-1} x$  is usually denoted as  $\exp x$ . We now show that  $\exp x$  is an exponential function with base  $e$ . The number  $e$  was defined to satisfy the equation  $\ln(e) = 1$ , so  $e = \exp(1)$ .

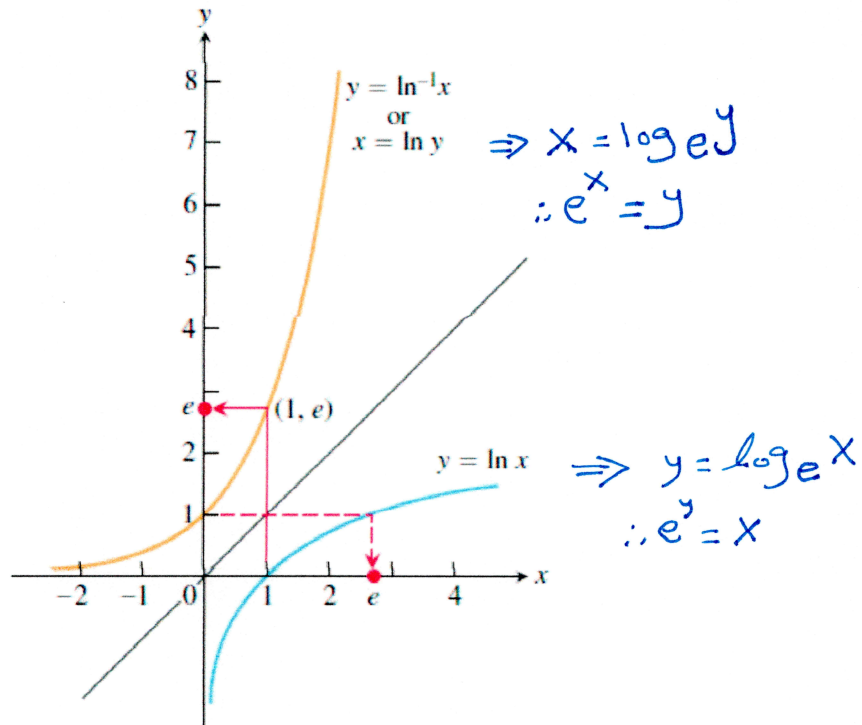


Figure 3. The graphs of  $y = \ln x$  and  $y = \ln^{-1} x = \exp x$ . The number  $e$  is  $\ln^{-1} 1 = \exp(1)$ .

**DEFINITION** For every real number  $x$ , we define the **natural exponential function** to be  $e^x = \exp x$ .

### Inverse Equations for $e^x$ and $\ln x$

$$e^{\ln x} = x \quad (\text{all } x > 0)$$

$$\ln(e^x) = x \quad (\text{all } x)$$



## Laws of Exponents

Even though  $e^x$  is defined in a seemingly roundabout way as  $\ln^{-1} x$ , it obeys the familiar laws of exponents from algebra. The following Theorem 3 shows us that these laws are consequences of the definitions of  $\ln x$  and  $e^x$ .

**THEOREM 3** For all numbers  $x, x_1$ , and  $x_2$ , the natural exponential  $e^x$  obeys the following laws:

1.  $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2.  $e^{-x} = \frac{1}{e^x}$
3.  $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4.  $(e^{x_1})^r = e^{rx_1}$ , if  $r$  is rational

### The Derivative and Integral of $e^x$

If  $u$  is any differentiable function of  $x$ , then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

The general antiderivative of the exponential function

$$\int e^u du = e^u + C$$

### The Derivative and Integral of $a^u$

If  $a > 0$  and  $u$  is a differentiable function of  $x$ , then  $a^u$  is a differentiable function of  $x$  and

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}. \quad (3)$$

$$\int a^u du = \frac{a^u}{\ln a} + C. \quad (4)$$

(21)

Solve for  $x$  :  $\ln(2x+1)=2$

Taking  $e$  of both sides gives:

$$e^{\ln(2x+1)} = e^2 \Rightarrow 2x+1 = e^2$$

$$\Rightarrow x = \frac{e^2 - 1}{2} = 3.19$$

Solve  $4^{x^2} = 5$

Taking  $\ln$  of both sides gives:

$$\ln 4^{x^2} = \ln 5$$

$$x^2 \ln 4 = \ln 5 \Rightarrow x^2 = \frac{\ln 5}{\ln 4}$$

$$\therefore x = \pm \sqrt{\frac{\ln 5}{\ln 4}} = \pm 1.077$$

Solve the equation  $e^{2x-6} = 4$  for  $x$ .

$$\ln e^{2x-6} = \ln 4$$

{inverse relationship}

$$2x-6 = \ln 4$$

$$2x = \ln 4 + 6$$

$$x = \frac{1}{2} \ln 4 + 3$$

$$x = \ln 4^{1/2} + 3$$

$$x = \ln 2 + 3$$

(22)

Ex. Find the derivatives for the following equations.

$$a) y = 5e^x \Rightarrow \frac{dy}{dx} = 5e^x$$

$$b) y = e^{-x} \Rightarrow \frac{dy}{dx} = e^{-x} (-1) = -e^{-x}$$

$$c) y = e^{\sin x} \Rightarrow \frac{dy}{dx} = e^{\sin x} \cdot \cos x$$

$$d) y = e^{\sqrt{3x+1}} \Rightarrow \frac{dy}{dx} = e^{\sqrt{3x+1}} \cdot \frac{3}{2\sqrt{3x+1}}$$

$$e) y = 3^x \Rightarrow 3^x \ln 3 = \frac{dy}{dx}$$

$$f) y = 3^{\sin x} \Rightarrow \frac{dy}{dx} = 3^{\sin x} (\ln 3) \cos x$$

$$g) y = e^{3x} \sin 2x \Rightarrow \frac{dy}{dx} = e^{3x} \cdot \cos 2x \cdot (2) + e^{3x} \cdot (3) \cdot \sin 2x \\ = 2e^{3x} \cdot \cos 2x + 3e^{3x} \cdot \sin 2x$$

$$h) y = x^2 e^{-x^2} \Rightarrow \frac{dy}{dx} = x^2 [e^{-x^2} (-2x)] + e^{-x^2} (2x)$$

$$i) y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot (1) \Rightarrow \frac{dy}{dx} = y \left[ x \cdot \frac{1}{x} + \ln x \right] = x^x (1 + \ln x)$$

$$j) y = (\sin x)^{\tan x} \Rightarrow \ln y = \ln (\sin x)^{\tan x}$$

$$\therefore \ln y = \tan x \ln (\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = \tan x \cdot \frac{\frac{\sin x}{\cos x}}{\sin x} + \ln (\sin x) \cdot \sec^2 x$$

$$* y = (\sin x)^{\tan x}$$

$$\frac{dy}{dx} = (\sin x)^{\tan x} [1 + \sec^2 x \cdot \ln (\sin x)]$$



(23)

Ex. Find the integrals for the following Equations:

$$a) \int_0^{\ln 2} e^{3x} dx$$

$$u = 3x \Rightarrow du = 3 dx \Rightarrow dx = \frac{du}{3}$$
$$u(0) = 0, u(\ln 2) = 3 \ln 2 = \ln 2^3 = \ln 8$$

$$= \int_0^{\ln 8} e^u \cdot \frac{1}{3} du = \frac{1}{3} \int_0^{\ln 8} e^u du = \frac{1}{3} e^u \Big|_0^{\ln 8}$$

$$= \frac{1}{3} (8 - 1) = \frac{7}{3}$$

$$b) \int_0^{\pi/2} e^{\sin x} \cdot \cos x dx$$

$$u = \sin x \Rightarrow du = \cos x dx$$
$$\therefore dx = \frac{du}{\cos x}$$
$$u(0) = 0, u(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$$

$$= \int_0^1 e^u \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int_0^1 e^u du = e^u \Big|_0^1 = e^1 - e^0 = e - 1$$

$$c) \int 2^x dx = \frac{2^x}{\ln 2} + C$$

$$d) \int 2^{\sin x} \cos x dx$$

$$\text{Let } u = \sin x \Rightarrow du = \cos x dx$$
$$\therefore dx = \frac{du}{\cos x}$$

$$= \int 2^u \cdot \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int 2^u du = \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{\sin x}}{\ln 2} + C$$

(24)

EX. solve the following integrals:

$$a) \int \frac{e^{\sin x} \cdot \cos x}{e^{\sin x} + 1} dx$$

$$= \int \frac{\cancel{e^{\sin x}} \cdot \cancel{\cos x}}{e^{\sin x} + 1} \frac{du}{\cancel{e^{\sin x}} \cdot \cancel{\cos x}}$$

$$= \int \frac{du}{u+1} = \ln|u+1| + C = \ln|e^{\sin x} + 1| + C$$

Let  $u = e^{\sin x}$   
 $\therefore du = e^{\sin x} \cdot \cos x dx$   
 $dx = \frac{du}{e^{\sin x} \cdot \cos x}$

$$b) \int \frac{3x+1}{\sqrt{x}(\sqrt{x^3} + \sqrt{x} + 4)} dx$$

$$= \int \frac{3x+1}{x^{1/2}(x^{3/2} + x^{1/2} + 4)} dx$$

$$= \int \frac{3u^2+1}{u(u^3+u+4)} \cdot 2u du$$

$$= 2 \int \frac{3u^2+1}{u^3+u+4} du$$

$$= 2 \ln|u^3+u+4| + C = 2 \ln|x^{3/2} + x^{1/2} + 4| + C$$

$$= 2 \ln|\sqrt{x^3} + \sqrt{x} + 4| + C$$

Let  $u = x^{1/2}$   
 $u^2 = x$   
 $2u du = dx$

$$c) \int \frac{\sqrt{x}}{\sqrt{x^3} + 1} dx -$$

$$= \int \frac{x^{1/2}}{x^{3/2} + 1} dx$$

$$= \int \frac{u}{u^3+1} \cdot 2u du = 2 \int \frac{u^2}{u^3+1} du$$

$$= \frac{2}{3} \int \frac{3u^2}{u^3+1} du = \frac{2}{3} \ln|u^3+1| + C$$

$$= \frac{2}{3} \ln|x^{3/2} + 1| + C = \frac{2}{3} \ln|\sqrt{x^3} + 1| + C$$

Let  $u = x^{1/2} \Rightarrow u^2 = x$   
 $2u du = dx$



(25)

$$e) \int_1^{\sqrt{2}} x \cdot 2^{(x^2)} dx$$

$$= \int_1^2 2^u \cdot \frac{du}{2x}$$

$$= \frac{1}{2} \int_1^2 2^u du$$

$$= \frac{1}{2} \left[ \frac{2^u}{\ln 2} \right]_1^2$$

$$= \frac{1}{2 \ln 2} [4 - 2] = \frac{2}{2 \ln 2} = \frac{1}{\ln 2}$$

$$f) \int_0^{\pi/2} 7^{\cos t} \sin t dt$$

$$= \int_1^0 -7^u \frac{du}{\sin t}$$

$$= - \int_1^0 7^u du$$

$$= - \left[ \frac{7^u}{\ln 7} \right]_1^0$$

$$= \frac{-1}{\ln 7} [1 - 7] = \frac{6}{\ln 7}$$

$$\text{Let } u = x^2$$

$$du = 2x dx$$

$$u(1) = 1$$

$$u(\sqrt{2}) = (\sqrt{2})^2 = 2$$

$$\text{Let } u = \cos t$$

$$du = -\sin t dt$$

$$\therefore dt = -\frac{du}{\sin t}$$

$$u(0) = 1$$

$$u(\pi/2) = 0$$

(26)

Solve  $32 = 70(1 - e^{-t/2})$  to find  $t$ .

$$\frac{32}{70} = 1 - e^{-t/2} \Rightarrow 1 - \frac{32}{70} = e^{-t/2}$$

$$\therefore e^{-t/2} = \frac{38}{70} \Rightarrow e^{t/2} = \frac{70}{38}$$

Taking  $\ln$  of both sides gives

$$\ln e^{t/2} = \ln\left(\frac{70}{38}\right)$$

$$\frac{t}{2} = \ln\left(\frac{70}{38}\right) \Rightarrow t = 2 \ln\left(\frac{70}{38}\right) = \boxed{1.22}$$

Solve  $e^{x-1} = 2e^{3x-4}$  to find  $x$ .

Taking  $\ln$  of both sides gives

$$\ln(e^{x-1}) = \ln(2e^{3x-4})$$

$$\ln(e^{x-1}) = \ln 2 + \ln(e^{3x-4})$$

$$x-1 = \ln 2 + 3x-4$$

$$4-1-\ln 2 = 3x-x$$

$$3-\ln 2 = 2x$$

$$\therefore x = \frac{3-\ln 2}{2} = 1.153$$

(27)

Solve,  $\ln(x-2)^2 = \ln(x-2) - \ln(x+3) + 1.6$  to find  $x$ .

$$\ln(x-2)^2 - \ln(x-2) + \ln(x+3) = 1.6$$

$$\ln \left\{ \frac{(x-2)^2 (x+3)}{\cancel{(x-2)}} \right\} = 1.6$$

$$\ln\{(x-2)(x+3)\} = 1.6$$

Taking  $e$  of both sides gives:

$$e^{\ln\{(x-2)(x+3)\}} = e^{1.6}$$

$$(x-2)(x+3) = e^{1.6}$$

$$x^2 + x - 6 = e^{1.6} \Rightarrow x^2 + x - 6 - e^{1.6} = 0$$

$$x^2 + x - 10.953 = 0$$

using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-10.953)}}{2} = \frac{-1 \pm \sqrt{44.812}}{2}$$

$$x = \frac{-1 \pm 6.6942}{2}$$

i.e.  $x = 2.847$  or  $-3.8471$

$x = -3.8471$  is not valid since the logarithm of a negative number has no real root.

Hence, the solution of the equation is  $x = 2.847$

(28)

Ex. population records of city (A) for three censuses are as follows:

<u>year</u>	<u>population</u>
1980	100 000
1990	108 000
2000	116 100

Estimate the population of (A) in the year 2035.

Sol/ The rate of population growth is assumed to be proportional to population.

$$\frac{dp_t}{dt} \propto p_t \Rightarrow \frac{dp_t}{p_t} = K dt \Rightarrow \text{by integrating both sides after rearrangement}$$

$$\frac{dp_t}{p_t} = K dt \Rightarrow \ln p_t = Kt + C \quad \text{--- (1)}$$

B.C at time = (t<sub>0</sub>) ⇒ p = p<sub>0</sub>

∴ ln p<sub>0</sub> = K(0) + C ⇒ C = ln p<sub>0</sub> → sub in (1)

$$\ln p_t = Kt + \ln p_0 \Rightarrow p_t = e^{(\ln p_0 + Kt)} \quad \text{where } t \text{ is } \Delta t = (t - t_0)$$

$$K_1 = \frac{\ln 108000 - \ln 100000}{10} = 0.0076$$

$$K_2 = \frac{\ln 116100 - \ln 108000}{10} = 0.0072$$

$$K_{av.} = \frac{K_1 + K_2}{2} = 0.0074$$

$$p_{2035} = e^{(\ln 116100 + 0.0074 * 35)} = 150423$$



### Logarithmic scale

Log scale is a way of displaying numerical data over a very wide range of values in a compact way. Often exponential growth curves are displayed on a Log scale.

To understanding the different between a linear scale and Log scale look the figures below. Figure 1 shows a population growth in an arbitrary city and figure 2 shows the worldwide spread (total cases and total deaths) of COVID-19 CORONAVIRUS.

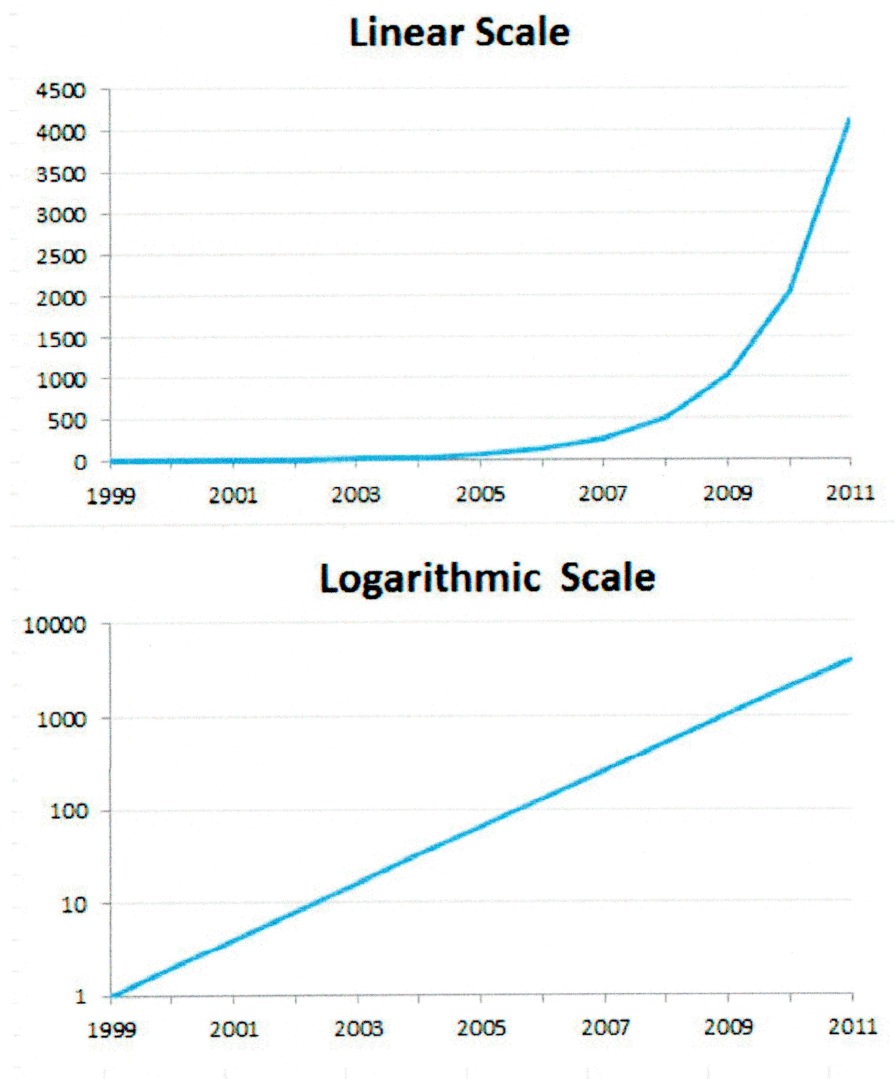


Figure 1

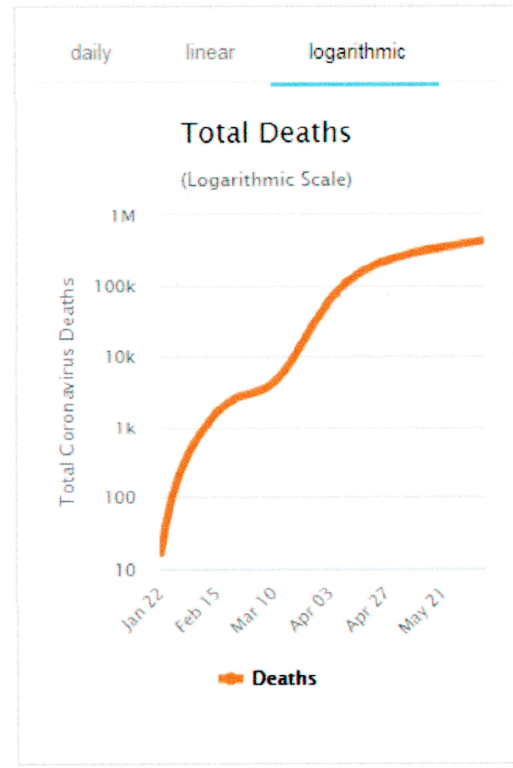
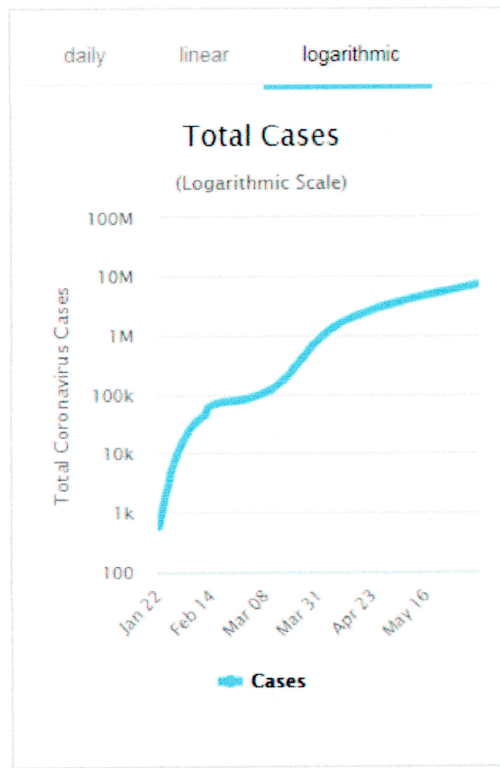
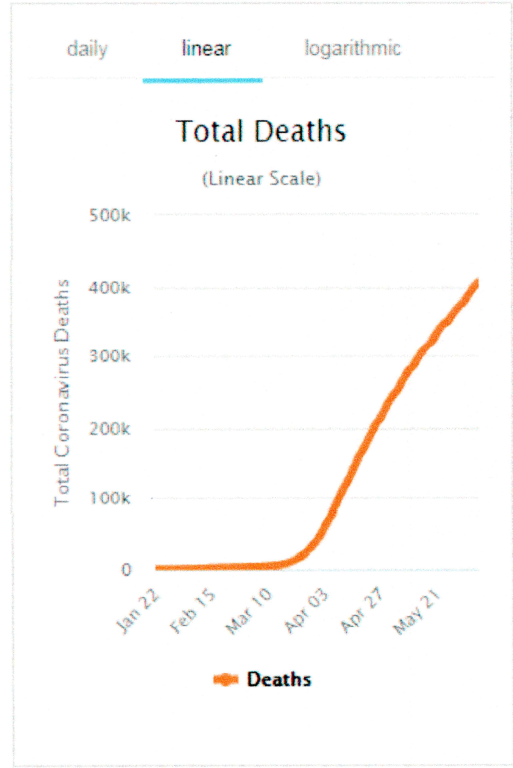
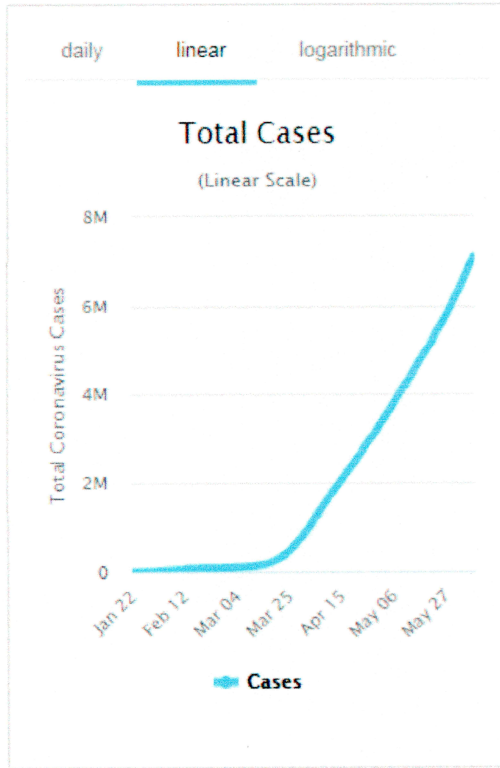


Figure 2