Chapter 6
Root-Locus Analysis

plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.(See Chapter 7.) Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros. Therefore, it is desired that the designer have a good understanding of the method for generating the root loci of the closed-loop system, both by hand and by use of a computer software like MATLAB.

Root-Locus Method. The basic idea behind the root-locus method is that the values of s that make the transfer function around the loop equal -1 must satisfy the characteristic equation of the system.

The root locus is the locus of roots of the characteristic equation of the closed-loop system as a specific parameter (usually, gain K) is varied from zero to infinity, giving the method its name. Such a plot clearly shows the contributions of each open-loop pole or zero to the locations of the closed-loop poles.

In designing a linear control system, we find that the root-locus method proves quite useful since it indicates the manner in which the open-loop poles and zeros should be modified so that the response meets system performance specifications. This method is particularly suited to obtaining approximate results very quickly.

Because generating the root loci by use of MATLAB is very simple, one may think sketching the root loci by hand is a waste of time and effort. However, experience in sketching the root loci by hand is invaluable for interpreting computer-generated root loci, as well as for getting a rough idea of the root loci very quickly.

By using the root-locus method, it is possible to determine the value of the loop gain K that will make the damping ratio of the dominant closed-loop poles as prescribed. If the location of an open-loop pole or zero is a system variable, then the root-locus method suggests the way to choose the location of an open-loop pole or zero.

Outline of the Chapter. This chapter introduces the basic concept of the root-locus method and presents useful rules for graphically constructing the root loci, as well as the generation of root loci with MATLAB.

The outline of the chapter is as follows: Section 6–1 has presented an introduction to the root-locus method. Section 6–2 details the concepts underlying the root-locus method and presents the general procedure for sketching root loci using illustrative examples. Section 6–3 summarizes general rules for constructing root loci. (If the designer follows the general rules for constructing the root loci, sketching the root loci for a given system will become a simple matter.) Section 6–4 discusses generating root-locus plots with MATLAB. Section 6–5 treats a special case when the closed-loop system has positive feedback. Section 6–6 treats conditionally stable systems. Finally, Section 6–7 extends the root-locus method to treat closed-loop systems with transport lag.

6-2 ROOT-LOCUS PLOTS

Angle and Magnitude Conditions. Consider the system shown in Figure 6-1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{6-1}$$

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (6-1) equal to zero. That is,

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1 \tag{6-2}$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in s. [Later, in Section 6-7, we extend the analysis to the case when $G(s)H(s)$ involves the transport lag e^{-Ts}.] Since $G(s)H(s)$ is a complex quantity, Equation (6-2) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\underline{/G(s)H(s)} = \pm 180°(2k + 1) \qquad (k = 0, 1, 2, \dots) \tag{6-3}$$

Magnitude condition:

$$|G(s)H(s)| = 1 \tag{6-4}$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition. The details of applying the angle and magnitude conditions to obtain the closed-loop poles are presented later in this section.

In many cases, $G(s)H(s)$ involves a gain parameter K, and the characteristic equation may be written as

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0$$

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of $G(s)H(s)$. Remember that the angles

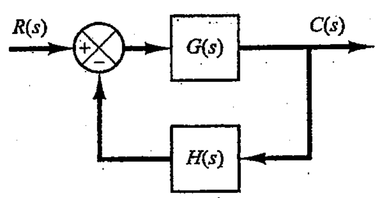

Figure 6-1
Control system.

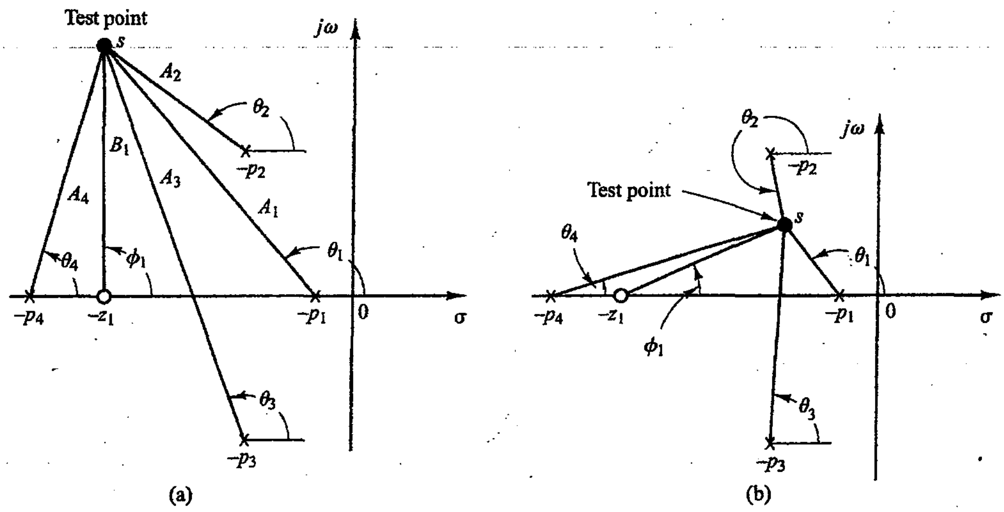

Figure 6–2
(a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point s.

of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counterclockwise direction. For example, if $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$

where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where $\phi_1, \theta_1, \theta_2, \theta_3$, and θ_4 are measured counterclockwise as shown in Figures 6–2(a) and (b). The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

where A_1, A_2, A_3, A_4, and B_1 are the magnitudes of the complex quantities $s + p_1$, $s + p_2$, $s + p_3$, $s + p_4$, and $s + z_1$, respectively, as shown in Figure 6–2(a).

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s plane.

Illustrative Examples. In what follows, two illustrative examples for constructing root-locus plots will be presented. Although computer approaches to the construction of the root loci are easily available, here we shall use graphical computation, combined with inspection, to determine the root loci upon which the roots of the characteristic equation of the closed-loop system must lie. Such a graphical approach will enhance understanding of how the closed-loop poles move in the complex plane as the open-loop poles and zeros are moved. Although we employ only simple systems for illustrative purposes, the procedure for finding the root loci is no more complicated for higher-order systems.

The first step in the procedure for constructing a root-locus plot is to seek out the loci of possible roots using the angle condition. Then, if necessary, the loci can be scaled, or graduated, in gain using the magnitude condition.

Because graphical measurements of angles and magnitudes are involved in the analysis, we find it necessary to use the same divisions on the abscissa as on the ordinate axis when sketching the root locus on graph paper.

EXAMPLE 6–1 Consider the system shown in Figure 6–3. (We assume that the value of gain K is nonnegative.) For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)}, \qquad H(s) = 1$$

Let us sketch the root-locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

For the given system, the angle condition becomes

$$\begin{aligned} \underline{/G(s)} &= \Bigg/\frac{K}{s(s+1)(s+2)} \\ &= -\underline{/s} - \underline{/s+1} - \underline{/s+2} \\ &= \pm 180^\circ(2k+1) \qquad (k = 0, 1, 2, \ldots) \end{aligned}$$

The magnitude condition is

$$|G(s)| = \left|\frac{K}{s(s+1)(s+2)}\right| = 1$$

A typical procedure for sketching the root-locus plot is as follows:

1. *Determine the root loci on the real axis.* The first step in constructing a root-locus plot is to locate the open-loop poles, $s = 0$, $s = -1$, and $s = -2$, in the complex plane. (There are no open-loop zeros in this system.) The locations of the open-loop poles are indicated by crosses. (The locations of the open-loop zeros in this book will be indicated by small circles.) Note that the starting points of the root loci (the points corresponding to $K = 0$) are open-loop poles. The number of individual root loci for this system is three, which is the same as the number of open-loop poles.

To determine the root loci on the real axis, we select a test point, s. If the test point is on the positive real axis, then

$$\underline{/s} = \underline{/s+1} = \underline{/s+2} = 0^\circ$$

This shows that the angle condition cannot be satisfied. Hence, there is no root locus on the positive real axis. Next, select a test point on the negative real axis between 0 and −1. Then

$$\underline{/s} = 180^\circ, \qquad \underline{/s+1} = \underline{/s+2} = 0^\circ$$

Thus

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -180^\circ$$

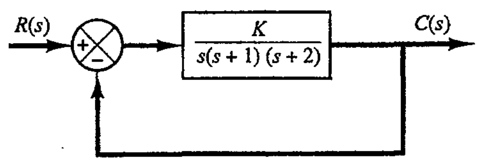

Figure 6–3
Control system.

and the angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and −1 forms a portion of the root locus. If a test point is selected between −1 and −2, then

$$\angle s = \angle s+1 = 180°, \qquad \angle s+2 = 0°$$

and

$$-\angle s - \angle s+1 - \angle s+2 = -360°$$

It can be seen that the angle condition is not satisfied. Therefore, the negative real axis from −1 to −2 is not a part of the root locus. Similarly, if a test point is located on the negative real axis from −2 to −∞, the angle condition is satisfied. Thus, root loci exist on the negative real axis between 0 and −1 and between −2 and −∞.

2. *Determine the asymptotes of the root loci.* The asymptotes of the root loci as s approaches infinity can be determined as follows: If a test point s is selected very far from the origin, then

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

and the angle condition becomes

$$-3\angle s = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \ldots)$$

or

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} \qquad (k = 0, 1, 2, \ldots)$$

Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60°, −60°, and 180°. Thus, there are three asymptotes. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis. Since

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

if a test point is located very far from the origin, then $G(s)$ may be written as

$$G(s) = \frac{K}{s^3 + 3s^2 + \cdots}$$

For large values of s, this last equation may be approximated by

$$G(s) \doteqdot \frac{K}{(s+1)^3} \tag{6–5}$$

A root-locus diagram of $G(s)$ given by Equation (6–5) consists of three straight lines. This can be seen as follows: The equation of the root locus is

$$\angle \frac{K}{(s+1)^3} = \pm 180°(2k+1)$$

or

$$-3\angle s+1 = \pm 180°(2k+1)$$

which can be written as

$$\angle s+1 = \pm 60°(2k+1)$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\underline{/\sigma + j\omega + 1} = \pm 60°(2k + 1)$$

or

$$\tan^{-1} \frac{\omega}{\sigma + 1} = 60°, \quad -60°, \quad 0°$$

Taking the tangent of both sides of this last equation,

$$\frac{\omega}{\sigma + 1} = \sqrt{3}, \quad -\sqrt{3}, \quad 0$$

which can be written as

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0, \quad \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0, \quad \omega = 0$$

These three equations represent three straight lines, as shown in Figure 6–4. The three straight lines shown are the asymptotes. They meet at point $s = -1$. Thus, the abscissa of the intersection of the asymptotes and the real axis is obtained by setting the denominator of the right-hand side of Equation (6–5) equal to zero and solving for s. The asymptotes are almost parts of the root loci in regions very far from the origin.

3. *Determine the breakaway point.* To plot root loci accurately, we must find the breakaway point, where the root-locus branches originating from the poles at 0 and −1 break away (as K is increased) from the real axis and move into the complex plane. The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.

A simple method for finding the breakaway point is available. We shall present this method in the following: Let us write the characteristic equation as

$$f(s) = B(s) + KA(s) = 0 \tag{6–6}$$

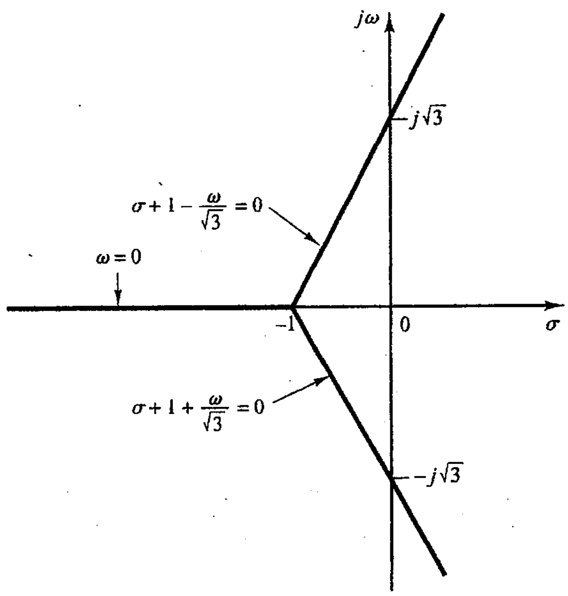

Figure 6–4
Three asymptotes.

where $A(s)$ and $B(s)$ do not contain K. Note that $f(s) = 0$ has multiple roots at points where

$$\frac{df(s)}{ds} = 0$$

This can be seen as follows: Suppose that $f(s)$ has multiple roots of order r. Then $f(s)$ may be written as

$$f(s) = (s - s_1)^r(s - s_2)\cdots(s - s_n)$$

If we differentiate this equation with respect to s and set $s = s_1$, then we get

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0 \tag{6–7}$$

This means that multiple roots of $f(s)$ will satisfy Equation (6–7). From Equation (6–6), we obtain

$$\frac{df(s)}{ds} = B'(s) + KA'(s) = 0 \tag{6–8}$$

where

$$A'(s) = \frac{dA(s)}{ds}, \qquad B'(s) = \frac{dB(s)}{ds}$$

The particular value of K that will yield multiple roots of the characteristic equation is obtained from Equation (6–8) as

$$K = -\frac{B'(s)}{A'(s)}$$

If we substitute this value of K into Equation (6–6), we get

$$f(s) = B(s) - \frac{B'(s)}{A'(s)}A(s) = 0$$

or

$$B(s)A'(s) - B'(s)A(s) = 0 \tag{6–9}$$

If Equation (6–9) is solved for s, the points where multiple roots occur can be obtained. On the other hand, from Equation (6–6) we obtain

$$K = -\frac{B(s)}{A(s)}$$

and

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

If dK/ds is set equal to zero, we get the same equation as Equation (6–9). Therefore, the breakaway points can be simply determined from the roots of

$$\frac{dK}{ds} = 0$$

It should be noted that not all the solutions of Equation (6–9) or of $dK/ds = 0$ correspond to actual breakaway points. If a point at which $dK/ds = 0$ is on a root locus, it is an actual breakaway or break-in point. Stated differently, if at a point at which $dK/ds = 0$ the value of K takes a real positive value then that point is an actual breakaway or break-in point.

For the present example, the characteristic equation $G(s) + 1 = 0$ is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0$$

or

$$K = -(s^3 + 3s^2 + 2s)$$

By setting $dK/ds = 0$, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

or

$$s = -0.4226, \qquad s = -1.5774$$

Since the breakaway point must lie on a root locus between 0 and -1, it is clear that $s = -0.4226$ corresponds to the actual breakaway point. Point $s = -1.5774$ is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of K corresponding to $s = -0.4226$ and $s = -1.5774$ yields

$$K = 0.3849, \qquad \text{for } s = -0.4226$$

$$K = -0.3849, \qquad \text{for } s = -1.5774$$

4. *Determine the points where the root loci cross the imaginary axis.* These points can be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

the Routh array becomes

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

The value of K that makes the s^1 term in the first column equal zero is $K = 6$. The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

which yields

$$s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$. The gain value corresponding to the crossing points is $K = 6$.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K. For the present system, the characteristic equation, with $s = j\omega$, is

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

or

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

Equating both the real and imaginary parts of this last equation to zero, we obtain

$$K - 3\omega^2 = 0, \qquad 2\omega - \omega^3 = 0$$

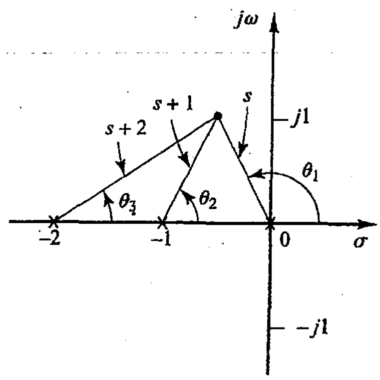

Figure 6–5
Construction of root locus.

from which

$$\omega = \pm\sqrt{2}, \quad K = 6 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Thus, root loci cross the imaginary axis at $\omega = \pm\sqrt{2}$, and the value of K at the crossing points is 6. Also, a root-locus branch on the real axis touches the imaginary axis at $\omega = 0$.

5. *Choose a test point in the broad neighborhood of the jω axis and the origin,* as shown in Figure 6–5, and apply the angle condition. If a test point is on the root loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must be 180°. If the test point does not satisfy the angle condition, select another test point until it satisfies the condition. (The sum of the angles at the test point will indicate which direction the test point should be moved.) Continue this process and locate a sufficient number of points satisfying the angle condition.

6. *Draw the root loci,* based on the information obtained in the foregoing steps, as shown in Figure 6–6.

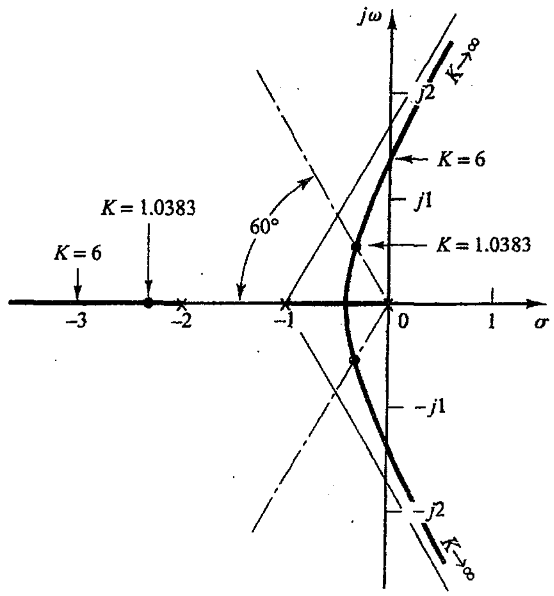

Figure 6–6
Root-locus plot.

7. *Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio ζ is 0.5.* Closed-loop poles with $\zeta = 0.5$ lie on lines passing through the origin and making the angles $\pm\cos^{-1}\zeta = \pm\cos^{-1}0.5 = \pm 60°$ with the negative real axis. From Figure 6–6, such closed-loop poles having $\zeta = 0.5$ are obtained as follows:

$$s_1 = -0.3337 + j0.5780, \qquad s_2 = -0.3337 - j0.5780$$

The value of K that yields such poles is found from the magnitude condition as follows:

$$K = |s(s+1)(s+2)|_{s=-0.3337+j0.5780}$$
$$= 1.0383$$

Using this value of K, the third pole is found at $s = -2.3326$.

Note that, from step 4, it can be seen that for $K = 6$ the dominant closed-loop poles lie on the imaginary axis at $s = \pm j\sqrt{2}$. With this value of K, the system will exhibit sustained oscillations. For $K > 6$, the dominant closed-loop poles lie in the right-half s plane, resulting in an unstable system.

Finally, note that, if necessary, the root loci can be easily graduated in terms of K by use of the magnitude condition. We simply pick out a point on a root locus, measure the magnitudes of the three complex quantities s, $s + 1$, and $s + 2$, and multiply these magnitudes; the product is equal to the gain value K at that point, or

$$|s| \cdot |s+1| \cdot |s+2| = K$$

Graduation of the root loci can be done easily by use of MATLAB. (See Section 6–4.)

EXAMPLE 6–2 In this example, we shall sketch the root-locus plot of a system with complex-conjugate open-loop poles. Consider the system shown in Figure 6–7. For this system,

$$G(s) = \frac{K(s+2)}{s^2 + 2s + 3}, \qquad H(s) = 1$$

where $K \geq 0$. It is seen that $G(s)$ has a pair of complex conjugate poles at

$$s = -1 + j\sqrt{2}, \qquad s = -1 - j\sqrt{2}$$

A typical procedure for sketching the root-locus plot is as follows:

1. *Determine the root loci on the real axis.* For any test point s on the real axis, the sum of the angular contributions of the complex-conjugate poles is 360°, as shown in Figure 6–8. Thus the net effect of the complex-conjugate poles is zero on the real axis. The location of the root locus on the real axis is determined from the open-loop zero on the negative real axis. A simple test reveals that a section of the negative real axis, that between −2 and −∞, is a part of the root locus. It is noted that, since this locus lies between two zeros (at $s = -2$ and $s = -\infty$), it is actually a part of two root loci, each of which starts from one of the two complex-conjugate poles. In other words, two root loci break in the part of the negative real axis between −2 and −∞.

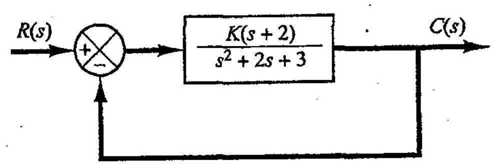

Figure 6–7
Control system.

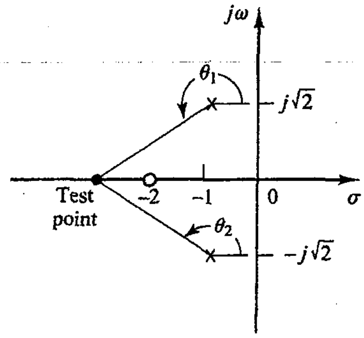

Figure 6–8
Determination of the root locus on the real axis.

Since there are two open-loop poles and one zero, there is one asymptote, which coincides with the negative real axis.

2. *Determine the angle of departure from the complex-conjugate open-loop poles.* The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles. Knowledge of this angle is important since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Referring to Figure 6–9, if we choose a test point and move it in the very vicinity of the complex open-loop pole at $s = -p_1$, we find that the sum of the angular contributions from the pole at $s = p_2$ and zero at $s = -z_1$ to the test point can be considered remaining the same. If the test point is to be on the root locus, then the sum of ϕ_1', $-\theta_1$, and $-\theta_2'$ must be $\pm 180°(2k + 1)$, where $k = 0, 1, 2, \ldots$. Thus, in the example,

$$\phi_1' - (\theta_1 + \theta_2') = \pm 180°(2k + 1)$$

or

$$\theta_1 = 180° - \theta_2' + \phi_1' = 180° - \theta_2 + \phi_1$$

The angle of departure is then

$$\theta_1 = 180° - \theta_2 + \phi_1 = 180° - 90° + 55° = 145°$$

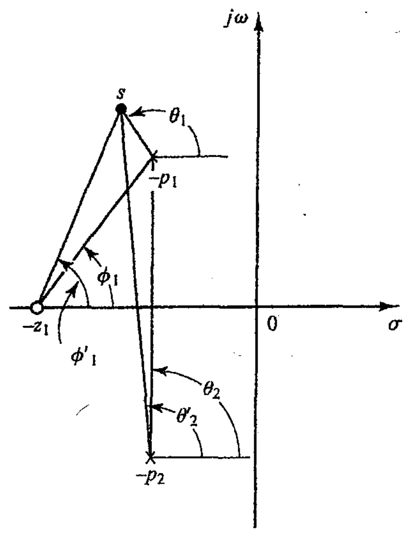

Figure 6–9
Determination of the angle of departure.

Since the root locus is symmetric about the real axis, the angle of departure from the pole at $s = -p_2$ is $-145°$.

3. *Determine the break-in point.* A break-in point exists where a pair of root-locus branches coalesces as K is increased. For this problem, the break-in point can be found as follows: Since

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

we have

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

which gives

$$s^2 + 4s + 1 = 0$$

or

$$s = -3.7320 \qquad \text{or} \qquad s = -0.2680$$

Notice that point $s = -3.7320$ is on the root locus. Hence this point is an actual break-in point. (Note that at point $s = -3.7320$ the corresponding gain value is $K = 5.4641$.) Since point $s = -0.2680$ is not on the root locus, it cannot be a break-in point. (For point $s = -0.2680$, the corresponding gain value is $K = -1.4641$.)

4. *Sketch a root-locus plot, based on the information obtained in the foregoing steps.* To determine accurate root loci, several points must be found by trial and error between the break-in point and the complex open-loop poles. (To facilitate sketching the root-locus plot, we should find the direction in which the test point should be moved by mentally summing up the changes on the angles of the poles and zeros.) Figure 6–10 shows a complete root-locus plot for the system considered.

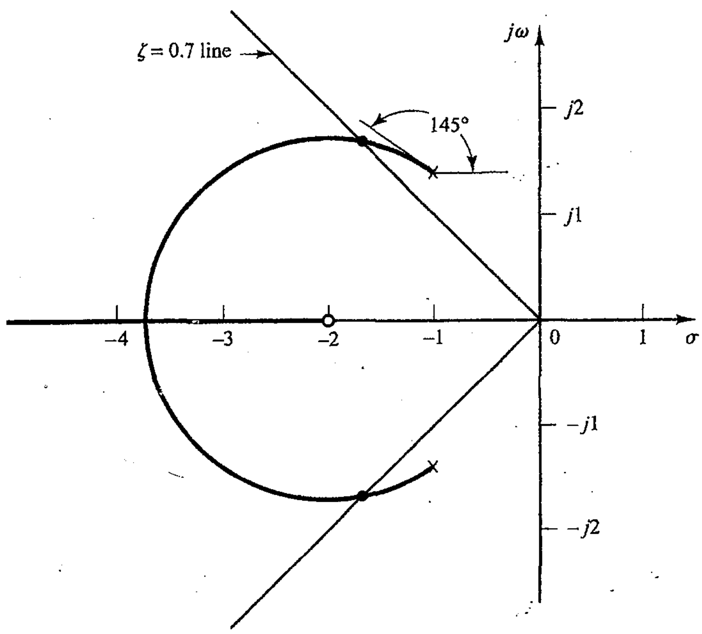

Figure 6–10
Root-locus plot.

The value of the gain K at any point on root locus can be found by applying the magnitude condition or by use of MATLAB (see Section 6–4). For example, the value of K at which the complex-conjugate closed-loop poles have the damping ratio $\zeta = 0.7$ can be found by locating the roots, as shown in Figure 6–10, and computing the value of K as follows:

$$K = \left| \frac{(s + 1 - j\sqrt{2})(s + 1 + j\sqrt{2})}{s + 2} \right|_{s=-1.67+j1.70} = 1.34$$

Or use MATLAB to find the value of K. (See Section 6–4.)

It is noted that in this system the root locus in the complex plane is a part of a circle. Such a circular root locus will not occur in most systems. Circular root loci may occur in systems that involve two poles and one zero, two poles and two zeros, or one pole and two zeros. Even in such systems, whether circular root loci occur depends on the locations of poles and zeros involved.

To show the occurrence of a circular root locus in the present system, we need to derive the equation for the root locus. For the present system, the angle condition is

$$\angle s + 2 - \angle s + 1 - j\sqrt{2} - \angle s + 1 + j\sqrt{2} = \pm 180^\circ(2k + 1)$$

If $s = \sigma + j\omega$ is substituted into this last equation, we obtain

$$\angle \sigma + 2 + j\omega - \angle \sigma + 1 + j\omega - j\sqrt{2} - \angle \sigma + 1 + j\omega + j\sqrt{2} = \pm 180^\circ(2k + 1)$$

which can be written as

$$\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) - \tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) - \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \pm 180^\circ(2k + 1)$$

or

$$\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ(2k + 1)$$

Taking tangents of both sides of this last equation using the relationship

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \tag{6–10}$$

we obtain

$$\tan\left[\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right)\right] = \tan\left[\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ(2k + 1)\right]$$

or

$$\frac{\dfrac{\omega - \sqrt{2}}{\sigma + 1} + \dfrac{\omega + \sqrt{2}}{\sigma + 1}}{1 - \left(\dfrac{\omega - \sqrt{2}}{\sigma + 1}\right)\left(\dfrac{\omega + \sqrt{2}}{\sigma + 1}\right)} = \frac{\dfrac{\omega}{\sigma + 2} \pm 0}{1 \mp \dfrac{\omega}{\sigma + 2} \times 0}$$

which can be simplified to

$$\frac{2\omega(\sigma + 1)}{(\sigma + 1)^2 - (\omega^2 - 2)} = \frac{\omega}{\sigma + 2}$$

or

$$\omega\left[(\sigma + 2)^2 + \omega^2 - 3\right] = 0$$

This last equation is equivalent to

$$\omega = 0 \qquad \text{or} \qquad (\sigma + 2)^2 + \omega^2 = (\sqrt{3})^2$$

These two equations are the equations for the root loci for the present system. Notice that the first equation, $\omega = 0$, is the equation for the real axis. The real axis from $s = -2$ to $s = -\infty$ corresponds to a root locus for $K \geq 0$. The remaining part of the real axis corresponds to a root locus when K is negative. (In the present system, K is nonnegative.) The second equation for the root locus is an equation of a circle with center at $\sigma = -2$, $\omega = 0$ and the radius equal to $\sqrt{3}$. That part of the circle to the left of the complex-conjugate poles corresponds to a root locus for $K \geq 0$. The remaining part of the circle corresponds to a root locus when K is negative.

It is important to note that easily interpretable equations for the root locus can be derived for simple systems only. For complicated systems having many poles and zeros, any attempt to derive equations for the root loci is discouraged. Such derived equations are very complicated and their configuration in the complex plane is difficult to visualize.

6–3 SUMMARY OF GENERAL RULES FOR CONSTRUCTING ROOT LOCI

For a complicated system with many open-loop poles and zeros, constructing a root-locus plot may seem complicated, but actually it is not difficult if the rules for constructing the root loci are applied. By locating particular points and asymptotes and by computing angles of departure from complex poles and angles of arrival at complex zeros, we can construct the general form of the root loci without difficulty.

Some of the rules for constructing root loci were given in Section 6–2. The purpose of this section is to summarize the general rules for constructing root loci of the system shown in Figure 6–11. While the root-locus method is essentially based on a trial-and-error technique, the number of trials required can be greatly reduced if we use these rules.

General Rules for Constructing Root Loci. We shall now summarize the general rules and procedure for constructing the root loci of the system shown in Figure 6–11.

First, obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form

$$1 + \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 0 \qquad (6\text{–}11)$$

In the present discussions, we assume that the parameter of interest is the gain K, where $K > 0$. (If $K < 0$, which corresponds to the positive-feedback case, the angle condition must be modified. See Section 6–5.) Note, however, that the method is still applicable to systems with parameters of interest other than gain. (See Section 7–6.)

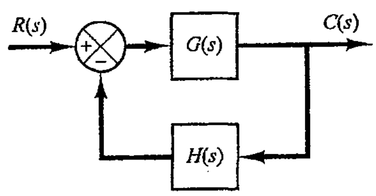

Figure 6–11
Control system.

1. *Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity).* From the factored form of the open-loop transfer function, locate the open-loop poles and zeros in the s plane. [Note that the open-loop zeros are the zeros of $G(s)H(s)$, while the closed-loop zeros consist of the zeros of $G(s)$ and the poles of $H(s)$.]

Note that the root loci are symmetrical about the real axis of the s plane, because the complex poles and complex zeros occur only in conjugate pairs.

A root-locus plot will have just as many branches as there are roots of the characteristic equation. Since the number of open-loop poles generally exceeds that of zeros, the number of branches equals that of poles. If the number of closed-loop poles is the same as the number of open-loop poles, then the number of individual root-locus branches terminating at finite open-loop zeros is equal to the number m of the open-loop zeros. The remaining $n - m$ branches terminate at infinity ($n - m$ implicit zeros at infinity) along asymptotes.

If we include poles and zeros at infinity, the number of open-loop poles is equal to that of open-loop zeros. Hence we can always state that the root loci start at the poles of $G(s)H(s)$ and end at the zeros of $G(s)H(s)$, as K increases from zero to infinity, where the poles and zeros include both those in the finite s plane and those at infinity.

2. *Determine the root loci on the real axis.* Root loci on the real axis are determined by open-loop poles and zeros lying on it. The complex-conjugate poles and zeros of the open-loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or zeros is 360° on the real axis. Each portion of the root locus on the real axis extends over a range from a pole or zero to another pole or zero. In constructing the root loci on the real axis, choose a test point on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.

3. *Determine the asymptotes of root loci.* If the test point s is located far from the origin, then the angle of each complex quantity may be considered the same. One open-loop zero and one open-loop pole then cancel the effects of the other. Therefore, the root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{n - m} \qquad (k = 0, 1, 2, \ldots)$$

where n = number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

Here, $k = 0$ corresponds to the asymptotes with the smallest angle with the real axis. Although k assumes an infinite number of values, as k is increased the angle repeats itself, and the number of distinct asymptotes is $n - m$.

All the asymptotes intersect on the real axis. The point at which they do so is obtained as follows: If both the numerator and denominator of the open-loop transfer function are expanded, the result is

$$G(s)H(s) = \frac{K\left[s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m\right]}{s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n}$$

If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)]s^{n-m-1} + \cdots}$$

or

$$G(s)H(s) = \frac{K}{\left[s + \frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m}\right]^{n-m}} \qquad (6\text{–}12)$$

The abscissa of the intersection of the asymptotes and the real axis is then obtained by setting the denominator of the right-hand side of Equation (6–12) equal to zero and solving for s, or

$$s = -\frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \qquad (6\text{–}13)$$

[Example 6–1 shows why Equation (6–13) gives the intersection.] Once this intersection is determined, the asymptotes can be readily drawn in the complex plane.

It is important to note that the asymptotes show the behavior of the root loci for $|s| \gg 1$. A root locus branch may lie on one side of the corresponding asymptote or may cross the corresponding asymptote from one side to the other side.

4. *Find the breakaway and break-in points.* Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in complex-conjugate pairs.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros. If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist no breakaway or break-in points or there may exist both breakaway and break-in points.

Suppose that the characteristic equation is given by

$$B(s) + KA(s) = 0$$

The breakaway points and break-in points correspond to multiple roots of the characteristic equation. Hence, as discussed in Example 6–1, the breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0 \qquad (6\text{–}14)$$

where the prime indicates differentiation with respect to s. It is important to note that the breakaway points and break-in points must be the roots of Equation (6–14), but not all roots of Equation (6–14) are breakaway or break-in points. If a real root of Equation (6–14) lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point. If a real root of Equation (6–14) is not on the root-locus portion of the real axis, then this root corresponds to neither a breakaway point nor a break-in point.

If two roots $s = s_1$ and $s = -s_1$ of Equation (6–14) are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value. If the value of K corresponding to a root $s = s_1$ of $dK/ds = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. (Since K is assumed to be nonnegative, if the value of K thus obtained is negative, or a complex quantity, then point $s = s_1$ is neither a breakaway nor break-in point.)

5. *Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero).* To sketch the root loci with reasonable accuracy, we must find the directions of the root loci near the complex poles and zeros. If a test point is chosen and moved in the very vicinity of a complex pole (or complex zero), the sum of the angular contributions from all other poles and zeros can be considered to remain the same. Therefore, the angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180° the sum of all the angles of vectors from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Angle of departure from a complex pole = 180°
− (sum of the angles of vectors to a complex pole in question from other poles)
+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
− (sum of the angles of vectors to a complex zero in question from other zeros)
+ (sum of the angles of vectors to a complex zero in question from poles)

The angle of departure is shown in Figure 6–12.

6. *Find the points where the root loci may cross the imaginary axis.* The points where the root loci intersect the $j\omega$ axis can be found easily by (a) use of Routh's stability criterion or (b) letting $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K. The values of ω thus found give the frequencies at which root loci cross the imaginary axis. The K value corresponding to each crossing frequency gives the gain at the crossing point.

7. *Taking a series of test points in the broad neighborhood of the origin of the s plane, sketch the root loci.* Determine the root loci in the broad neighborhood of the $j\omega$ axis and the origin. The most important part of the root loci is on neither the real axis nor the asymptotes, but the part in the broad neighborhood of the $j\omega$ axis and the origin. The

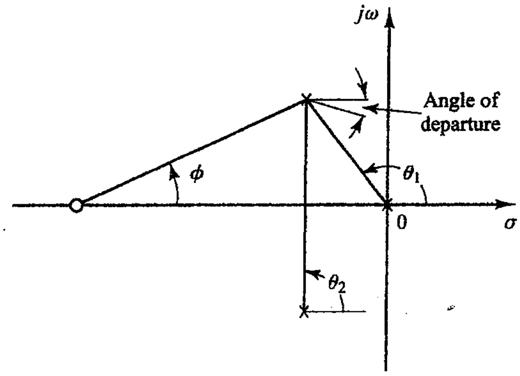

Figure 6–12 Construction of the root locus. [Angle of departure = 180° − $[\theta_1 + \theta_2] + \phi$.]

If two roots $s = s_1$ and $s = -s_1$ of Equation (6–14) are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value. If the value of K corresponding to a root $s = s_1$ of $dK/ds = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. (Since K is assumed to be nonnegative, if the value of K thus obtained is negative, or a complex quantity, then point $s = s_1$ is neither a breakaway nor break-in point.)

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Angle of departure from a complex pole = 180°
− (sum of the angles of vectors to a complex pole in question from other poles)
+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°
− (sum of the angles of vectors to a complex zero in question from other zeros)
+ (sum of the angles of vectors to a complex zero in question from poles)

The angle of departure is shown in Figure 6–12.

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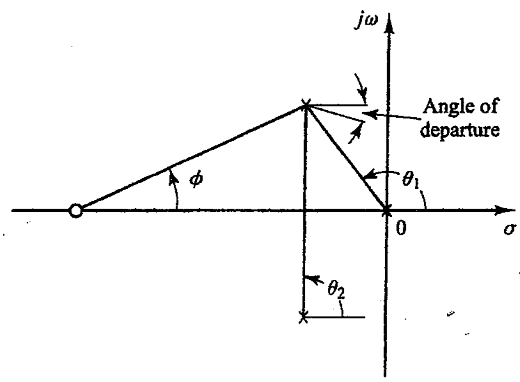

Figure 6–12 Construction of the root locus. [Angle of departure = 180° − $[\theta_1 + \theta_2] + \phi$.]

Cancellation of Poles of $G(s)$ with Zeros of $H(s)$. It is important to note that if the denominator of $G(s)$ and the numerator of $H(s)$ involve common factors then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. For example, consider the system shown in Figure 6–14(a). (This system has velocity feedback.) By modifying the block diagram of Figure 6–14(a) to that shown in Figure 6–14(b), it is clearly seen that $G(s)$ and $H(s)$ have a common factor $s + 1$. The closed-loop transfer function $C(s)/R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2) + K(s+1)}$$

The characteristic equation is

$$[s(s+2) + K](s+1) = 0$$

Because of the cancellation of the terms $(s + 1)$ appearing in $G(s)$ and $H(s)$, however, we have

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+1)(s+2)}$$
$$= \frac{s(s+2) + K}{s(s+2)}$$

The reduced characteristic equation is

$$s(s+2) + K = 0$$

The root-locus plot of $G(s)H(s)$ does not show all the roots of the characteristic equation, only the roots of the reduced equation.

To obtain the complete set of closed-loop poles, we must add the canceled pole of $G(s)H(s)$ to those closed-loop poles obtained from the root-locus plot of $G(s)H(s)$. The important thing to remember is that the canceled pole of $G(s)H(s)$ is a closed-loop pole of the system, as seen from Figure 6–14(c).

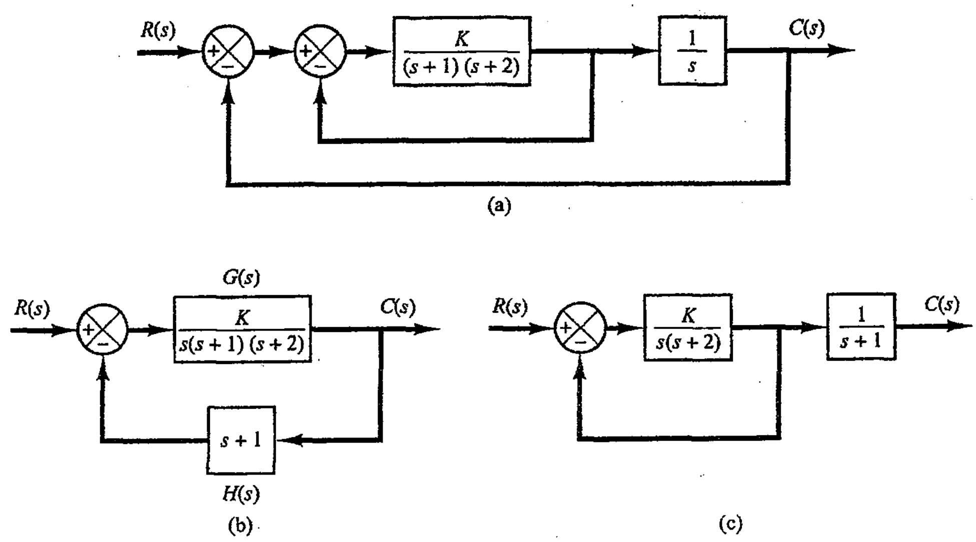

Figure 6–14 (a) Control system with velocity feedback; (b) and (c) modified block diagrams.

Cancellation of Poles of $G(s)$ with Zeros of $H(s)$. It is important to note that if the denominator of $G(s)$ and the numerator of $H(s)$ involve common factors then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. For example, consider the system shown in Figure 6–14(a). (This system has velocity feedback.) By modifying the block diagram of Figure 6–14(a) to that shown in Figure 6–14(b), it is clearly seen that $G(s)$ and $H(s)$ have a common factor $s + 1$. The closed-loop transfer function $C(s)/R(s)$ is

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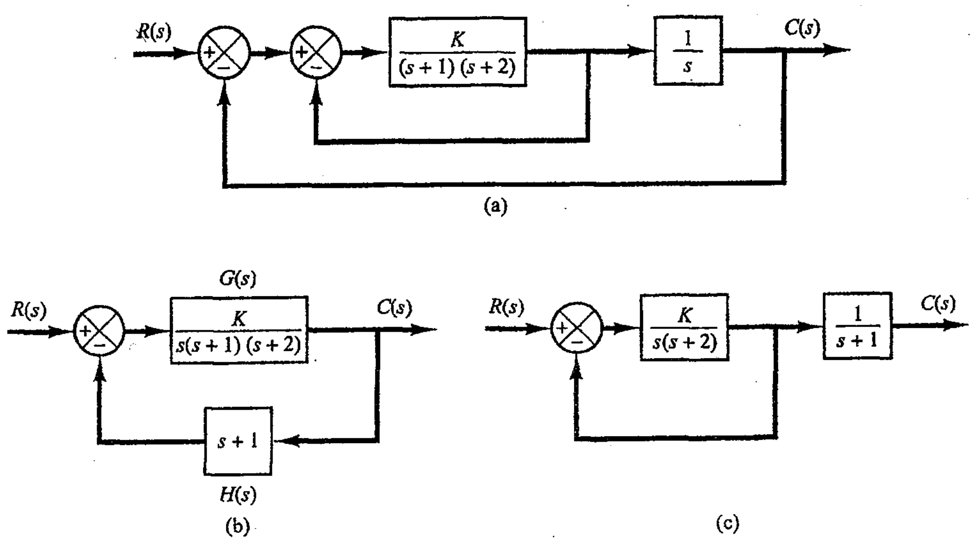


Figure 6–14
(a) Control system with velocity feedback; (b) and (c) modified block diagrams.

Note that once we have some experience with the method, we can easily evaluate the changes in the root loci due to the changes in the number and location of the open-loop poles and zeros by visualizing the root-locus plots resulting from various pole–zero configurations.

Summary. From the preceding discussions, it should be clear that it is possible to sketch a reasonably accurate root-locus diagram for a given system by following simple rules. (The reader should study the various root-locus diagrams shown in the solved problems at the end of the chapter.) At preliminary design stages, we may not need the precise locations of the closed-loop poles. Often their approximate locations are all that is needed to make an estimate of system performance. Thus, it is important that the designer have the capability of quickly sketching the root loci for a given system.

6–4 ROOT-LOCUS PLOTS WITH MATLAB

In this section we present the MATLAB approach to the generation of root-locus plots and finding relevant information from the root-locus plots.

Plotting Root Loci with MATLAB. In plotting root loci with MATLAB we deal with the system equation given in the form of Equation (6–11), which may be written as

$$1 + K\frac{\text{num}}{\text{den}} = 0$$

where num is the numerator polynomial and den is the denominator polynomial. That is,

$$\begin{aligned}\text{num} &= (s + z_1)(s + z_2)\cdots(s + z_m)\\ &= s^m + (z_1 + z_2 + \cdots + z_m)s^{m-1} + \cdots + z_1 z_2 \cdots z_m\\ \text{den} &= (s + p_1)(s + p_2)\cdots(s + p_n)\\ &= s^n + (p_1 + p_2 + \cdots + p_n)s^{n-1} + \cdots + p_1 p_2 \cdots p_n\end{aligned}$$

Note that both vectors num and den must be written in descending powers of s.

A MATLAB command commonly used for plotting root loci is

```
rlocus(num,den)
```

Using this command, the root-locus plot is drawn on the screen. The gain vector K is automatically determined. (The vector K contains all the gain values for which the closed-loop poles are to be computed.)

For the systems defined in state space, rlocus(A,B,C,D) plots the root locus of the system with the gain vector automatically determined.

Note that commands

```
rlocus(num,den,K)    and    rlocus(A,B,C,D,K)
```

use the user-supplied gain vector K.

If invoked with left-hand arguments

```
[r,K] = rlocus(num,den)
[r,K] = rlocus(num,den,K)
```

```
[r,K] = rlocus(A,B,C,D)
[r,K] = rlocus(A,B,C,D,K)
[r,K] = rlocus(sys)
```

the screen will show the matrix r and gain vector K. (r has length K rows and length den − 1 columns containing the complex root locations. Each row of the matrix corresponds to a gain from vector K.) The plot command

```
plot(r,'-')
```

plots the root loci.

If it is desired to plot the root loci with marks 'o' or 'x', it is necessary to use the following command:

```
r = rlocus(num,den)
plot(r,'o')    or    plot(r,'x')
```

Plotting root loci using marks o or x is instructive, since each calculated closed-loop pole is graphically shown; in some portion of the root loci those marks are densely placed and in another portion of the root loci they are sparsely placed. MATLAB supplies its own set of gain values used to calculate a root-locus plot. It does so by an internal adaptive step-size routine. Also, MATLAB uses the automatic axis-scaling feature of the plot command.

Finally, note that, since the gain vector is automatically determined, root-locus plots of

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+3)}$$

$$G(s)H(s) = \frac{10K(s+1)}{s(s+2)(s+3)}$$

$$G(s)H(s) = \frac{200K(s+1)}{s(s+2)(s+3)}$$

are all the same. The num and den set of the system is the same for all three systems. The num and den are

```
num = [0 0 1 1]
den = [1 5 6 0]
```

EXAMPLE 6–3 Consider the system shown in Figure 6–15. Plot root loci with a square aspect ratio so that a line with slope 1 is a true 45° line. Choose the region of root-locus plot to be

$$-6 \le x \le 6, \qquad -6 \le y \le 6$$

where x and y are the real-axis coordinate and imaginary-axis coordinate, respectively.

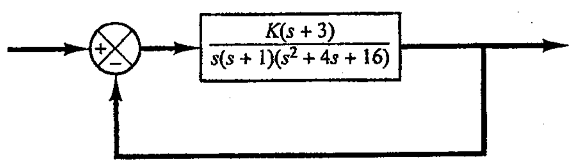


Figure 6–15
Control system.

To set the given plot region on the screen to be square, enter the command

```
v = [-6 6 -6 6]; axis (v); axis('square')
```

With this command, the region of the plot is as specified and a line with slope 1 is at a true 45°, not skewed by the irregular shape of the screen.

For this problem, the denominator is given as a product of first- and second-order terms. So we must multiply these terms to get a polynomial in s. The multiplication of these terms can be done easily by use of the convolution command, as shown next.

Define

$$a = s(s+1): \qquad a = [1 \quad 1 \quad 0]$$
$$b = s^2 + 4s + 16: \qquad b = [1 \quad 4 \quad 16]$$

Then we use the following command:

```
c = conv(a, b)
```

Note that conv(a, b) gives the product of two polynomials a and b. See the following computer output:

```
a = [1  1  0];
b = [1  4  16];
c = conv (a,b)
c =
     1  5  20  16  0
```

The denominator polynomial is thus found to be

```
den = [1  5  20  16  0]
```

To find the complex-conjugate open-loop poles (the roots of $s^2 + 4s + 16 = 0$), we may enter the roots command as follows:

```
r = roots(b)
r =
 -2.0000 + 3.4641i
 -2.0000 - 3.4641i
```

Thus, the system has the following open-loop zero and open-loop poles:

Open-loop zero: $s = -3$

Open-loop poles: $s = 0, \quad s = -1, \quad s = -2 \pm j3.4641$

MATLAB Program 6–1 will plot the root-locus diagram for this system. The plot is shown in Figure 6–16.

MATLAB Program 6–1

```
% --------- Root-locus plot ---------
num = [0  0  0  1  3];
den = [1  5  20  16  0];
rlocus(num,den)
v = [-6  6  -6  6];
axis(v); axis('square')
grid;
title ('Root-Locus Plot of G(s) = K(s + 3)/[s(s + 1)(s^2 + 4s + 16)]')
```

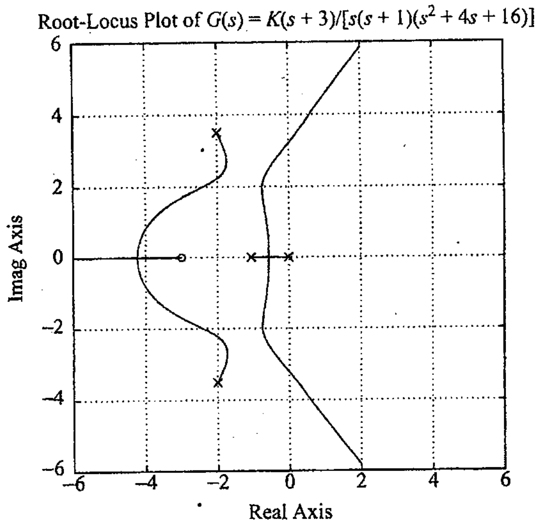


Figure 6–16
Root-locus plot.

Note that in MATLAB Program 6–1, instead of

```
den = [1  5  20  16  0]
```

we may enter

```
den = conv ([1  1  0], [1  4  16])
```

The results are the same.

EXAMPLE 6–4 Consider the system whose open-loop transfer function $G(s)H(s)$ is

$$G(s)H(s) = \frac{K}{s(s+0.5)(s^2+0.6s+10)} = \frac{K}{s^4+1.1s^3+10.3s^2+5s}$$

There are no open-loop zeros. Open-loop poles are located at $s = -0.3 + j3.1480$, $s = -0.3 - j3.1480$, $s = -0.5$, and $s = 0$.

Entering MATLAB Program 6–2 into the computer, we obtain the root-locus plot shown in Figure 6–17.

MATLAB Program 6–2

```
% --------- Root-locus plot ---------

num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
r = rlocus(num,den);
plot(r,'o')
v = [-6  6  -6  6]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s+10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```

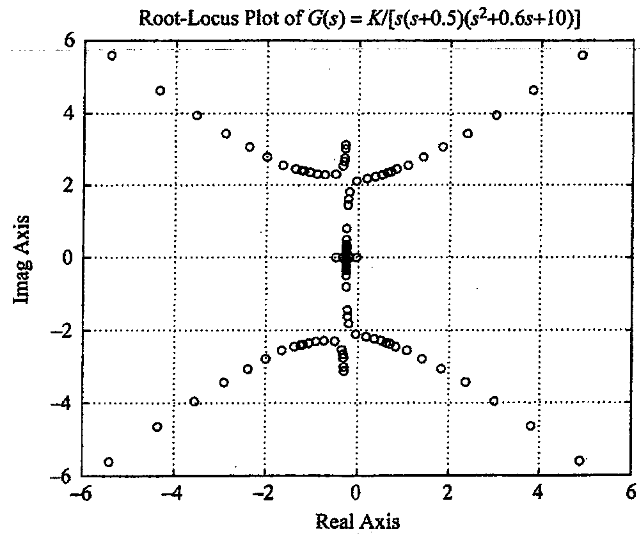


Figure 6–17
Root-locus plot.

Notice that in the regions near $x = -0.3$, $y = 2.3$ and $x = -0.3$, $y = -2.3$ two loci approach each other. We may wonder if these two branches should touch or not. To explore this situation, we may plot the root loci using smaller increments of K in the critical region.

By a conventional trial-and-error approach or using the command rlocfind to be presented later in this section, we find the particular region of interest to be $20 \leq K \leq 30$. By entering MATLAB Program 6–3, we obtain the root-locus plot shown in Figure 6–18. From this plot, it is clear that the two branches that approach in the upper half-plane (or in the lower half-plane) do not touch.

MATLAB Program 6–3

```
% --------- Root-locus plot ---------
num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
K1 = 0:0.2:20;
K2 = 20:0.1:30;
K3 = 30:5:1000;
K = [K1  K2  K3];
r = rlocus(num,den,K);
plot(r, 'o')
v = [-4  4  -4  4]; axis(v)
grid
title('Root-Locus Plot of G(s) = K/[s(s + 0.5)(s^2 + 0.6s + 10)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```

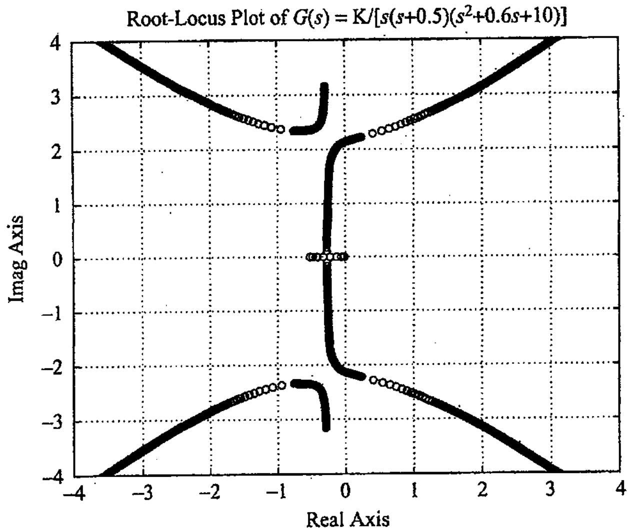


Figure 6–18
Root-locus plot.

EXAMPLE 6–5 Consider the system shown in Figure 6–19. The system equations are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$
$$u = r - y$$

In this example problem we shall obtain the root-locus diagram of the system defined in state space. As an example let us consider the case where matrices **A**, **B**, **C**, and D are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} \tag{6–15}$$

$$\mathbf{C} = [1 \quad 0 \quad 0], \qquad D = [0]$$

The root-locus plot for this system can be obtained with MATLAB by use of the following command:

```
rlocus(A,B,C,D)
```

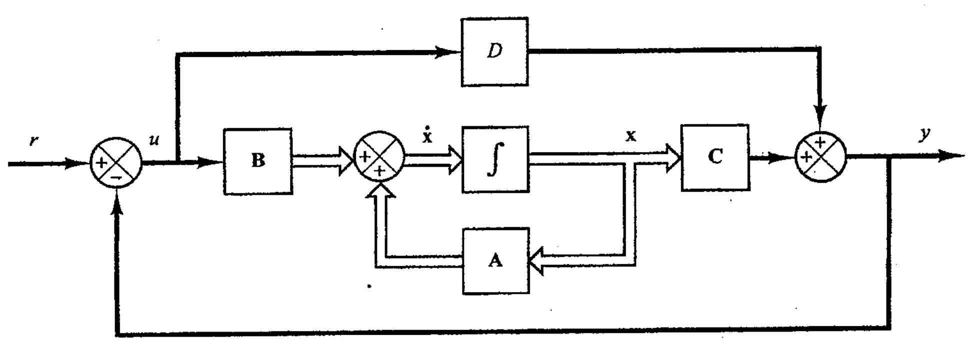


Figure 6–19
Closed-loop control system.

This command will produce the same root-locus plot as can be obtained by use of the rlocus (num,den) command, where num and den are obtained from

$$[\text{num,den}] = \text{ss2tf}(A,B,C,D)$$

as follows:

$$\text{num} = [0\ \ 0\ \ 1\ \ 0]$$
$$\text{den} = [1\ \ 14\ \ 56\ \ 160]$$

MATLAB Program 6–4 is a program that will generate the root-locus plot as shown in Figure 6–20.

MATLAB Program 6–4

```
% --------- Root-locus plot ---------
A = [0  1  0;0  0  1;-160  -56  -14];
B = [0;1;-14];
C = [1  0  0];
D = [0];
K = 0:0.1:400;
rlocus(A,B,C,D,K);
v = [-20  20  -20  20]; axis(v)
grid
title('Root-Locus Plot of System Defined in State Space')
```

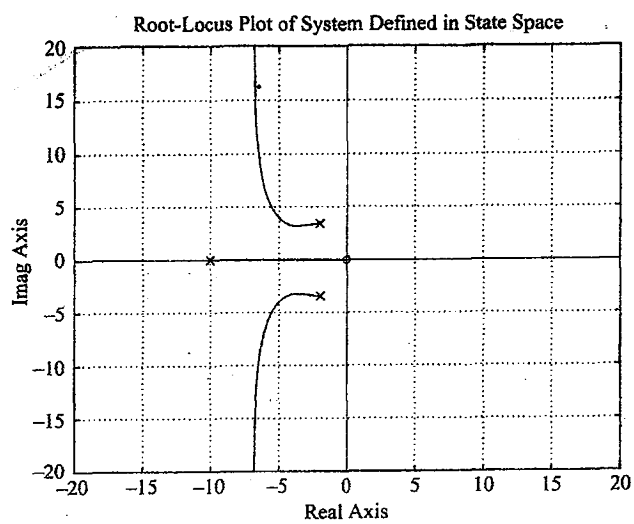


Figure 6–20 Root-locus plot of system defined in state space, where **A**, **B**, **C**, and *D* are as given by Equation (6–15).

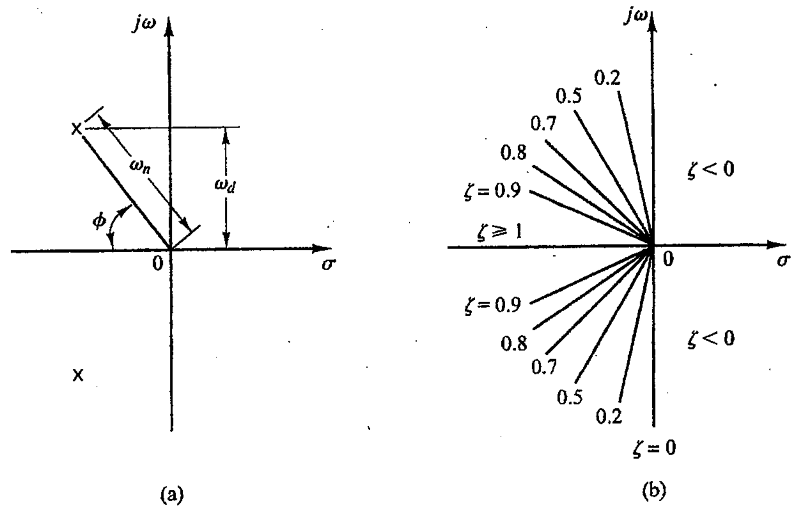


Figure 6–21
(a) Complex poles;
(b) lines of constant damping ratio ζ.

Constant ζ Loci and Constant ω_n Loci. Recall that in the complex plane the damping ratio ζ of a pair of complex-conjugate poles can be expressed in terms of the angle ϕ, which is measured from the negative real axis, as shown in Figure 6–21(a) with

$$\zeta = \cos\phi$$

In other words, lines of constant damping ratio ζ are radial lines passing through the origin as shown in Figure 6–21(b). For example, a damping ratio of 0.5 requires that the complex poles lie on the lines drawn through the origin making angles of $\pm 60°$ with the negative real axis. (If the real part of a pair of complex poles is positive, which means that the system is unstable, the corresponding ζ is negative.) the damping ratio determines the angular location of the poles, while the distance of the pole from the origin is determined by the undamped natural frequency ω_n. The constant ω_n loci are circles.

To draw constant ζ lines and constant ω_n circles on the root-locus diagram with MATLAB, use the command sgrid.

Plotting Polar Grids in the Root-Locus Diagam. The command

```
sgrid
```

overlays lines of constant damping ratio ($\zeta = 0 \sim 1$ with 0.1 increment) and circles of constant ω_n on the root-locus plot. See MATLAB Program 6–5 and the resulting diagram shown in Figure 6–22.

If only particular constant ζ lines (such as the $\zeta = 0.5$ line and $\zeta = 0.707$ line) and particular constant ω_n circles (such as the $\omega_n = 0.5$ circle, $\omega_n = 1$ circle, and $\omega_n = 2$ circle) are desired, use the following command:

```
sgrid([0.5, 0.707], [0.5, 1, 2])
```

MATLAB Program 6–5

```
sgrid
v = [-2 2 -2 2]; axis(v); axis('square')
title('Constant \zeta Lines and Constant \omega_n Circles')
xlabel(Real Axis')
ylabel('Imag Axis')
gtext('\zeta = 0.9')
gtext('0.8')
gtext('0.7')
gtext('0.6')
gtext('0.5')
gtext('0.4')
gtext('0.3')
gtext('0.2')
gtext('0.1')
gtext('\omega_n = 1')
gtext('\omega_n = 2')
```

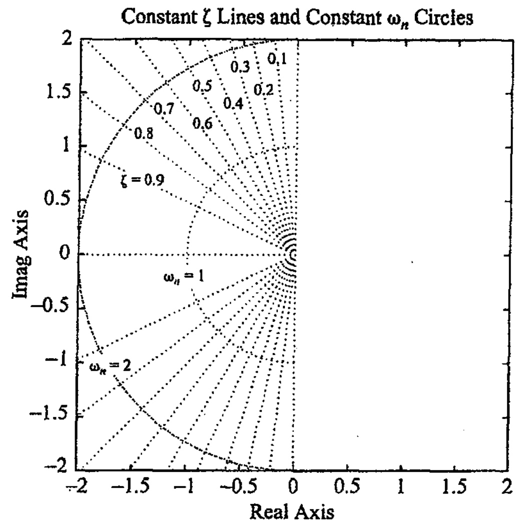


Figure 6–22
Constant ζ lines and constant ω_n circles.

If we wish to overlay lines of constant ζ and circles of constant ω_n as given above to a root-locus plot of a system with

```
num = [0 0 0 1]
den = [1 4 5 0]
```

then enter MATLAB Program 6–6 into the computer. The resulting root-locus plot is shown in Figure 6–23.

MATLAB Program 6–6

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den);
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid([0.5,0.707], [0.5,1,2])
title('Root-Locus Plot with \zeta = 0.5 and 0.707 Lines and \omega_n = 0.5, 1, and 2 Circles')
gtext('\zeta = 0.5')
gtext('\zeta = 0.707')
gtext('\omega_n = 2')
gtext('\omega_n = 1')
gtext('\omega_n = 0.5')
```

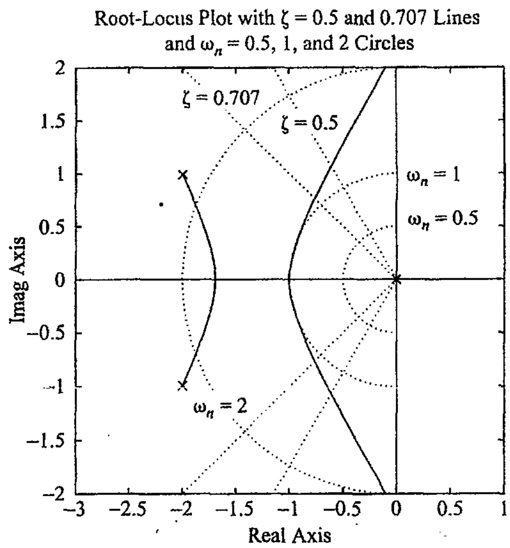


Figure 6–23
Constant ζ lines and constant ω_n circles superimposed on a root-locus plot.

If we want to omit either the entire constant ζ lines or entire constant ω_n circles, we may use empty brackets [ ] in the arguments of the sgrid command. For example, if we want to overlay only the constant damping ratio line corresponding to $\zeta = 0.5$ and no constant ω_n circles to the root-locus plot shown in Figure 6–23, then we may use the command

sgrid(0.5, [])

See MATLAB Program 6–7 and the resulting plot shown in Figure 6–24.

MATLAB Program 6–7

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den)
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid(0.5, [])
title('Root-Locus Plot and \zeta = 0.5 Line')
gtext('\zeta = 0.5')
```

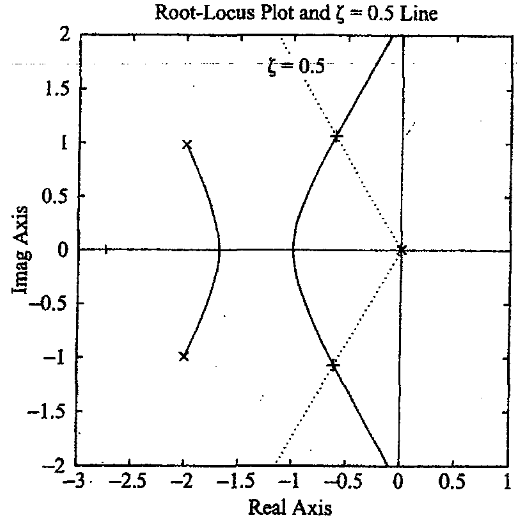


Figure 6–24 Root-locus plot with $\zeta = 0.5$ line.

Orthogonality of Root Loci and Constant-Gain Loci. Consider the system whose open-loop transfer function is $G(s)H(s)$. In the $G(s)H(s)$ plane, the loci of $|G(s)H(s)| =$ constant are circles centered at the origin, and the loci corresponding to $\angle G(s)H(s) = \pm 180°(2k + 1)$ $(k = 0, 1, 2, \ldots)$ lie on the negative real axis of the $G(s)H(s)$ plane, as shown in Figure 6–25. [Note that the complex plane employed here is not the s plane, but the $G(s)H(s)$ plane.]

The root loci and constant-gain loci in the s plane are conformal mappings of the loci of $\angle G(s)H(s) = \pm 180°(2k + 1)$ and of $|G(s)H(s)| =$ constant in the $G(s)H(s)$ plane.

Since the constant-phase and constant-gain loci in the $G(s)H(s)$ plane are orthogonal, the root loci and constant-gain loci in the s plane are orthogonal. Figure 6–26(a) shows the root loci and constant-gain loci for the following system:

$$G(s) = \frac{K(s + 2)}{s^2 + 2s + 3}, \qquad H(s) = 1$$

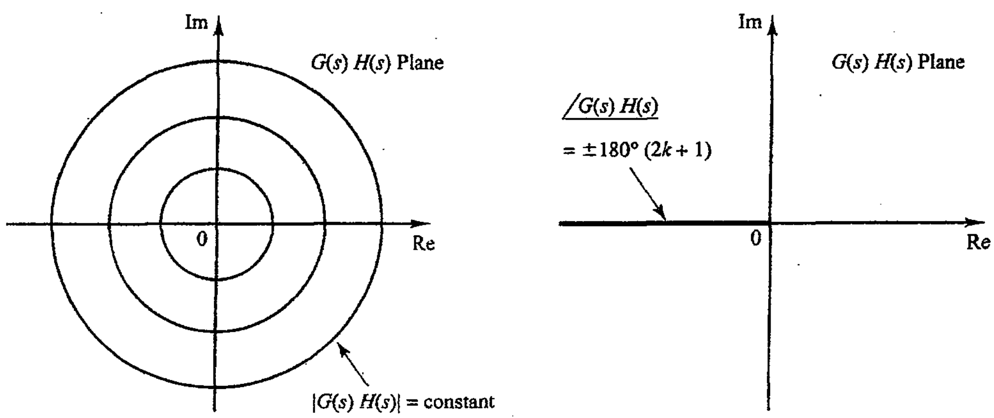


Figure 6–25 Plots of constant-gain and constant-phase loci in the $G(s)H(s)$ plane.

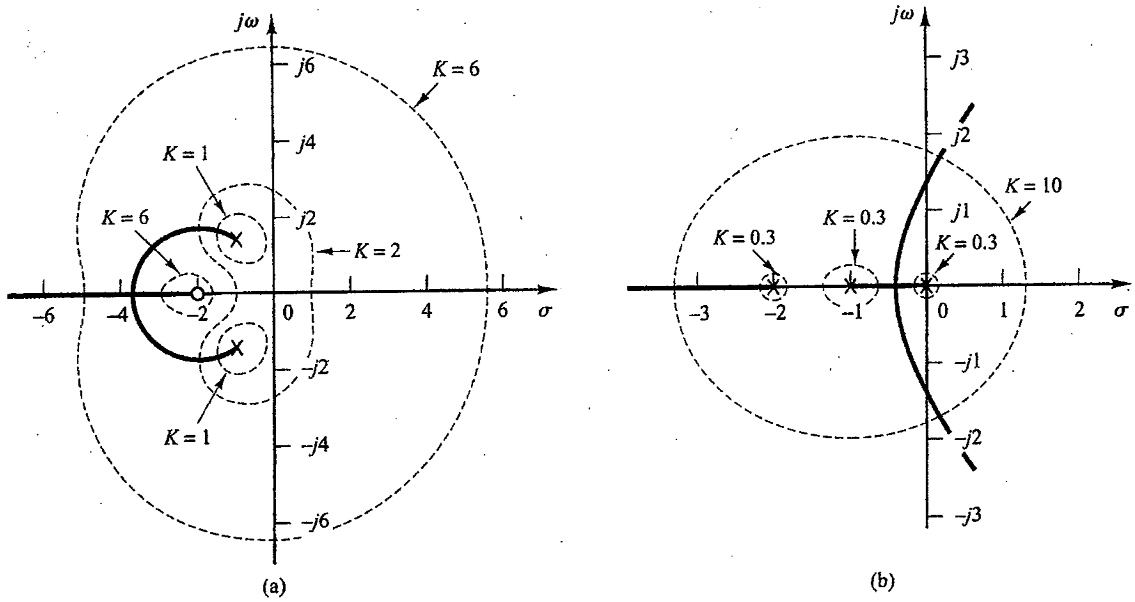


Figure 6–26
Plots of root loci and constant-gain loci. (a) System with $G(s) = K(s+2)/(s^2+2s+3)$, $H(s) = 1$; (b) system with $G(s) = K/[s(s+1)(s+2)]$, $H(s) = 1$.

Notice that since the pole–zero configuration is symmetrical about the real axis the constant-gain loci are also symmetrical about the real axis.

Figure 6–26(b) shows the root loci and constant-gain loci for the system:

$$G(s) = \frac{K}{s(s+1)(s+2)}, \qquad H(s) = 1$$

Notice that since the configuration of the poles in the s plane is symmetrical about the real axis and the line parallel to the imaginary axis passing through point ($\sigma = -1$, $\omega = 0$), the constant-gain loci are symmetrical about the $\omega = 0$ line (real axis) and the $\sigma = -1$ line.

From Figures 6–26(a) and (b), notice that every point in the s plane has the corresponding K value. If we use a command rlocfind (presented next), MATLAB will give the K value of the specified point as well as the nearest closed-loop poles corresponding to this K value.

Finding the Gain Value K at an Arbitrary Point on the Root Loci. In MATLAB analysis of closed-loop systems, it is frequently desired to find the gain value K at an arbitrary point on the root locus. This can be accomplished by using the following rlocfind command:

```
[K, r] = rlocfind(num, den)
```

The rlocfind command, which must follow an rlocus command, overlays movable x-y coordinates on the screen. Using the mouse, we position the origin of the x-y coordinates over the desired point on the root locus and press the mouse button. Then MATLAB displays on the screen the coordinates of that point, the gain value at that point, and the closed-loop poles corresponding to this gain value.

If the selected point is not on the root locus, the rlocfind command gives the coordinates of this selected point, the gain value of this point, and the locations of the closed-loop poles corresponding to this K value. [Note that every point on the s plane has a gain value. See, for example, Figures 6–26 (a) and (b).]

EXAMPLE 6–6 Consider the unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 4s + 5)}$$

Plot the root loci with MATLAB. Determine closed-loop poles that have the damping ratio of 0.5. Find the gain value K at this point.

We first plot a root-locus diagram as shown in Figure 6–27. Then enter the rlocfind command as shown in MATLAB Program 6–8. Position the origin of the x-y coordinates over the intersection of the upper root-locus branch and the $\zeta = 0.5$ line. Then press the button of the mouse. The screen shows the coordinates of this point, the gain value at this point, and the closed-loop poles corresponding to this gain value.

The plot shows the closed-loop poles by a plus sign (+). The three closed-loop poles obtained are

$$s = -2.7474, \qquad s = -0.6263 + j1.0800, \qquad s = -0.6263 - j1.0800$$

Note that the three closed-loop poles are slightly off the exact locations obtained by the analytic method. The reason is that we cannot position the origin of the movable x-y coordinates exactly at the intersection of the upper root-locus branch and the $\zeta = 0.5$ line.

MATLAB Program 6–8

```
num = [0 0 0 1];
den = [1 4 5 0];
rlocus(num, den);
v = [-3 1 -2 2]; axis(v); axis('square')
sgrid(0.5, [])
[K,r] = rlocfind(num, den)
Select a point in the graphics window

selected_point =

-0.6246 + 1.0792i

K =

  4.2823

r =

  -2.7474
  -0.6263 + 1.0800i
  -0.6263 - 1.0800i
```

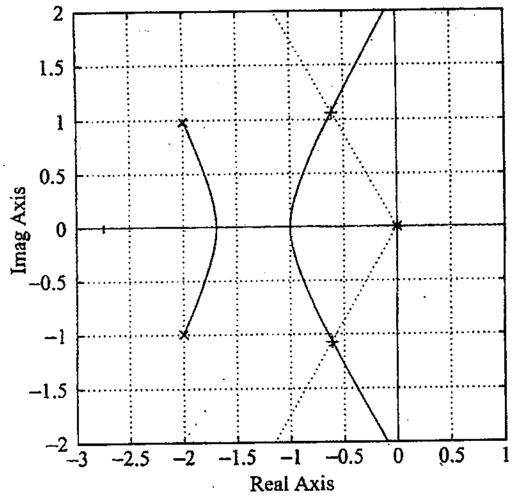


Figure 6–27
Root-locus plot with $\zeta = 0.5$ line.

Nonminimum-Phase Systems. If all the poles and zeros of a system lie in the left-half s plane, then the system is called *minimum phase.* If a system has at least one pole or zero in the right-half s plane, then the system is called *nonminimum phase.* The term nonminimum phase comes from the phase-shift characteristics of such a system when subjected to sinusoidal inputs.

Consider the system shown in Figure 6–28(a). For this system

$$G(s) = \frac{K(1 - T_a s)}{s(Ts + 1)} \qquad (T_a > 0), \qquad H(s) = 1$$

This is a nonminimum-phase system since there is one zero in the right-half s plane. For this system, the angle condition becomes

$$\begin{aligned}\underline{/G(s)} &= \underline{\Big/ -\frac{K(T_a s - 1)}{s(Ts + 1)}} \\ &= \underline{\Big/ \frac{K(T_a s - 1)}{s(Ts + 1)}} + 180^\circ \\ &= \pm 180^\circ(2k + 1) \qquad (k = 0, 1, 2, \dots)\end{aligned}$$

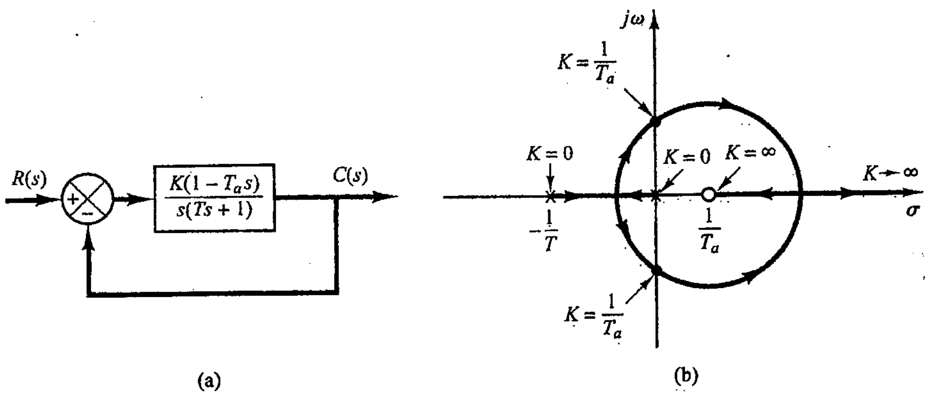


Figure 6–28
(a) Nonminimum-phase system;
(b) root-locus plot.

or

$$\angle \frac{K(T_a s - 1)}{s(Ts + 1)} = 0° \tag{6–16}$$

The root loci can be obtained from Equation (6–16). Figure 6–28(b) shows a root-locus plot for this system. From the diagram, we see that the system is stable if the gain K is less than $1/T_a$.

To obtain a root-locus plot with MATLAB, enter the numerator and denominator as usual. For example, if $T = 1$ sec and $T_a = 0.5$ sec, enter the following num and den in the program:

```
num = [0  -0.5  1]
den = [1  1  0]
```

MATLAB Program 6–9 gives the plot of the root loci shown in Figure 6–29.

MATLAB Program 6–9

```
num = [0  -0.5  1];
den = [1  1  0];
k1 = 0:0.01:30;
k2 = 30:1:100;
K3 = 100:5:500;
K = [k1  k2  k3];
rlocus(num,den,K)
v = [-2  6  -4  4]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(1 - 0.5s)/[s(s + 1)]')
```

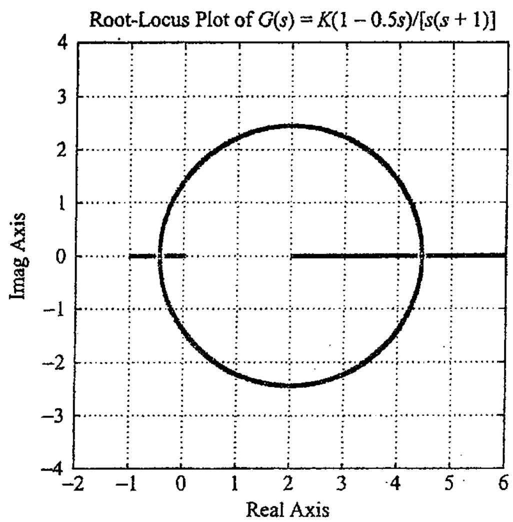


Figure 6–29 Root-locus plot of $G(s) = \frac{K(1 - 0.5s)}{s(s + 1)}$.

6–5 POSITIVE-FEEDBACK SYSTEMS

Root Loci for Positive-Feedback Systems.* In a complex control system, there may be a positive-feedback inner loop as shown in Figure 6–30. Such a loop is usually stabilized by the outer loop. In what follows, we shall be concerned only with the positive-feedback inner loop. The closed-loop transfer function of the inner loop is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

The characteristic equation is

$$1 - G(s)H(s) = 0 \qquad (6\text{–}17)$$

This equation can be solved in a manner similar to the development of the root-locus method in Section 6–2. The angle condition, however, must be altered.

Equation (6–17) can be rewritten as

$$G(s)H(s) = 1$$

which is equivalent to the following two equations:

$$\underline{/G(s)H(s)} = 0° \pm k360° \qquad (k = 0, 1, 2, \dots)$$

$$|G(s)H(s)| = 1$$

The total sum of all angles from the open-loop poles and zeros must be equal to $0° \pm k360°$. Thus the root locus follows a 0° locus in contrast to the 180° locus considered previously. The magnitude condition remains unaltered.

To illustrate the root-locus plot for the positive-feedback system, we shall use the following transfer functions $G(s)$ and $H(s)$ as an example.

$$G(s) = \frac{K(s+2)}{(s+3)(s^2+2s+2)}, \qquad H(s) = 1$$

The gain K is assumed to be positive.

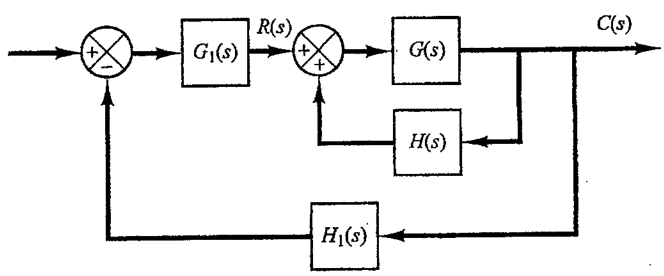


Figure 6–30
Control system.

* Reference W-4

The general rules for constructing root loci given in Section 6–3 must be modified in the following way:

Rule 2 is Modified as Follows: If the total number of real poles and real zeros to the right of a test point on the real axis is even, then this test point lies on the root locus.

Rule 3 is Modified as Follows:

$$\text{Angles of asymptotes} = \frac{\pm k360^\circ}{n - m} \qquad (k = 0, 1, 2, \dots)$$

where n = number of finite poles of $G(s)H(s)$
m = number of finite zeros of $G(s)H(s)$

Rule 5 is Modified as Follows: When calculating the angle of departure (or angle of arrival) from a complex open-loop pole (or at a complex zero), subtract from 0° the sum of all angles of the vectors from all the other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Other rules for constructing the root-locus plot remain the same. We shall now apply the modified rules to construct the root-locus plot.

1. Plot the open-loop poles ($s = -1 + j, s = -1 - j, s = -3$) and zero ($s = -2$) in the complex plane. As K is increased from 0 to ∞, the closed-loop poles start at the open-loop poles and terminate at the open-loop zeros (finite or infinite), just as in the case of negative-feedback systems.
2. Determine the root loci on the real axis. Root loci exist on the real axis between -2 and $+\infty$ and between -3 and $-\infty$.
3. Determine the asymptotes of the root loci. For the present system,

$$\text{Angles of asymptote} = \frac{\pm k360^\circ}{3 - 1} = \pm 180^\circ$$

   This simply means that asymptotes are on the real axis.
4. Determine the breakaway and break-in points. Since the characteristic equation is

$$(s + 3)(s^2 + 2s + 2) - K(s + 2) = 0$$

   we obtain

$$K = \frac{(s + 3)(s^2 + 2s + 2)}{s + 2}$$

   By differentiating K with respect to s, we obtain

$$\frac{dK}{ds} = \frac{2s^3 + 11s^2 + 20s + 10}{(s + 2)^2}$$

Note that

$$2s^3 + 11s^2 + 20s + 10 = 2(s + 0.8)(s^2 + 4.7s + 6.24)$$
$$= 2(s + 0.8)(s + 2.35 + j0.77)(s + 2.35 - j0.77)$$

Point $s = -0.8$ is on the root locus. Since this point lies between two zeros (a finite zero and an infinite zero), it is an actual break-in point. Points $s = -2.35 \pm j0.77$ do not satisfy the angle condition and, therefore, they are neither breakaway nor break-in points.

5. Find the angle of departure of the root locus from a complex pole. For the complex pole at $s = -1 + j$, the angle of departure θ is

$$\theta = 0° - 27° - 90° + 45°$$

or

$$\theta = -72°$$

(The angle of departure from the complex pole at $s = -1 - j$ is 72°.)

6. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin and apply the angle condition. Locate a sufficient number of points that satisfy the angle condition.

Figure 6–31 shows the root loci for the given positive-feedback system. The root loci are shown with dashed lines and a curve.

Note that if

$$K > \frac{(s + 3)(s^2 + 2s + 2)}{s + 2}\bigg|_{s=0} = 3$$

one real root enters the right-half s plane. Hence, for values of K greater than 3, the system becomes unstable. (For $K > 3$, the system must be stabilized with an outer loop.)

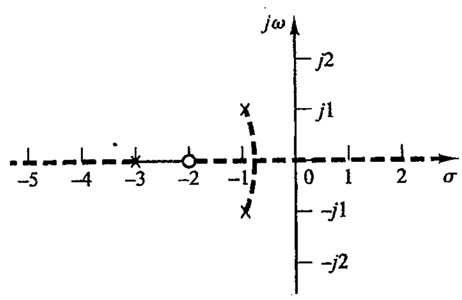


Figure 6–31 Root-locus plot for the positive-feedback system with $G(s) = K(s + 2)/[(s + 3)(s^2 + 2s + 2)]$, $H(s) = 1$.

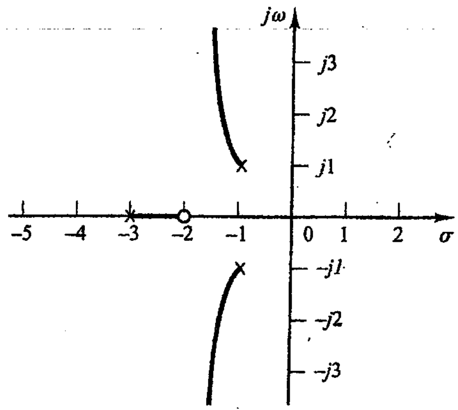


Figure 6–32
Root-locus plot for the negative-feedback system with $G(s) = K(s+2)/[(s+3)(s^2+2s+2)]$, $H(s) = 1$.

Note that the closed-loop transfer function for the positive-feedback system is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$

$$= \frac{K(s+2)}{(s+3)(s^2+2s+2) - K(s+2)}$$

To compare this root-locus plot with that of the corresponding negative-feedback system, we show in Figure 6–32 the root loci for the negative-feedback system whose closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K(s+2)}{(s+3)(s^2+2s+2) + K(s+2)}$$

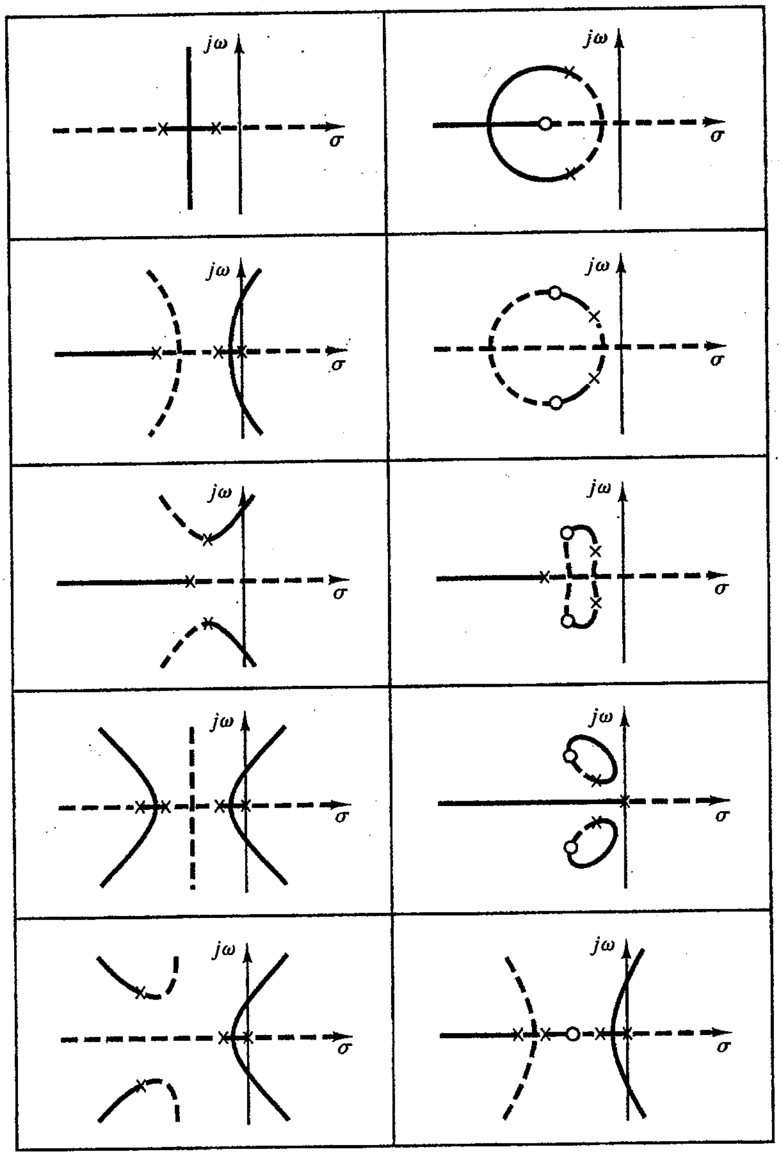
Table 6–2 shows various root-locus plots of negative-feedback and positive-feedback systems. The closed-loop transfer functions are given by

$$\frac{C}{R} = \frac{G}{1+GH}, \qquad \text{for negative-feedback systems}$$

$$\frac{C}{R} = \frac{G}{1-GH}, \qquad \text{for positive-feedback systems}$$

where GH is the open-loop transfer function. In Table 6–2, the root loci for negative-feedback systems are drawn with heavy lines and curves, and those for positive-feedback systems are drawn with dashed lines and curves.

Table 6–2 Root-Locus Plots of Negative-Feedback and Positive-Feedback Systems



Heavy lines and curves correspond to negative-feedback systems; dashed lines and curves correspond to positive-feedback systems.

6–6 CONDITIONALLY STABLE SYSTEMS

Consider the system shown in Figure 6–33. We can plot the root loci for this system by applying the general rules and procedure for constructing root loci, or use MATLAB to get root-locus plots. MATLAB Program 6–10 will plot the root-locus diagram for the system. The plot is shown in Figure 6–34.

It can be seen from the root-locus plot of Figure 6–34 that this system is stable only for limited ranges of the value of K—that is, $0 < K < 12$ and $73 < K < 154$. The system becomes unstable for $12 < K < 73$ and $154 < K$. (If K assumes a value corre-

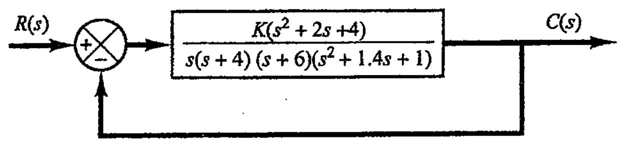


Figure 6–33
Control system.

MATLAB Program 6–10

```
num = [0 0 0 1 2 4];
den = conv(conv([1 4 0],[1 6]), [1 1.4 1]);
rlocus(num, den)
v = [-7 3 -5 5]; axis(v); axis('square')
grid
title('Root-Locus Plot of G(s) = K(s^2 + 2s + 4)/[s(s + 4)(s + 6)(s^2 + 1.4s + 1)]')
text(1.0, 0.55,'K = 12')
text(1.0,3.0,'K = 73')
text(1.0,4.15,'K = 154')
```

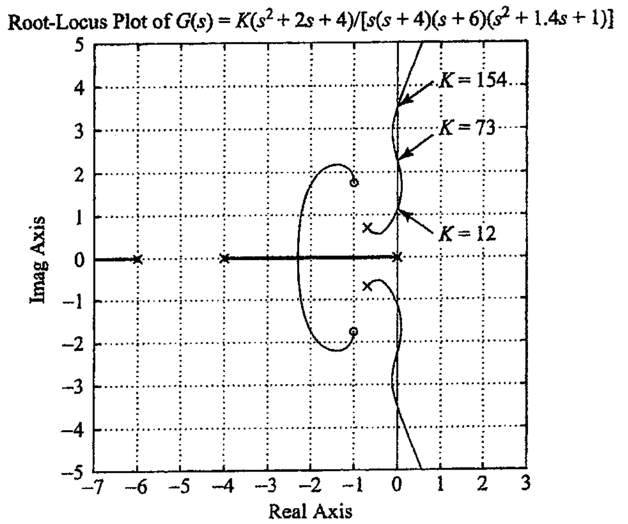


Figure 6–34
Root-locus plot of conditionally stable system.

sponding to unstable operation, the system may break down or may become nonlinear due to a saturation nonlinearity that may exist.) Such a system is called conditionally stable.

In practice, conditionally stable systems are not desirable. Conditional stability is dangerous but does occur in certain systems, in particular, a system that has an unstable feedforward path. Such an unstable feedforward path may occur if the system has a minor loop. It is advisable to avoid such conditional stability since, if the gain drops beyond the critical value for any reason, the system becomes unstable. Note that the addition of a proper compensating network will eliminate conditional stability. [An addition of a zero will cause the root loci to bend to the left.(See Section 7–2.) Hence conditional stability may be eliminated by adding proper compensation.]

6–7 ROOT LOCI FOR SYSTEMS WITH TRANSPORT LAG

Figure 6–35 shows a thermal system in which hot air is circulated to keep the temperature of a chamber constant. In this system, the measuring element is placed downstream a distance L ft from the furnace, the air velocity is v ft/sec, and $T = L/v$ sec would elapse before any change in the furnace temperature is sensed by the thermometer. Such a delay in measuring, delay in controller action, or delay in actuator operation, and the like, is called *transport lag* or *dead time*. Dead time is present in most process control systems.

The input $x(t)$ and the output $y(t)$ of a transport-lag or dead-time element are related by

$$y(t) = x(t - T)$$

where T is dead time. The transfer function of transport lag or dead time is given by

$$\text{Transfer function of transport lag or dead time} = \frac{\mathcal{L}[x(t-T)1(t-T)]}{\mathcal{L}[x(t)1(t)]} = \frac{X(s)e^{-Ts}}{X(s)} = e^{-Ts}$$

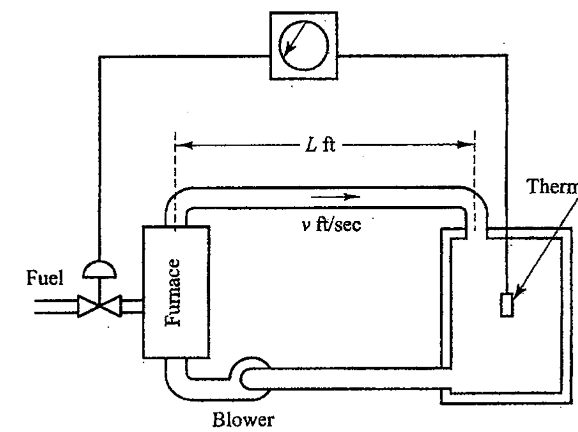


Figure 6–35
Thermal system.

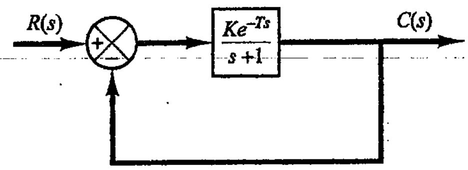


Figure 6–36
Block diagram of the system shown in Figure 6–35.

Suppose that the feedforward transfer function of this thermal system can be approximated by

$$G(s) = \frac{Ke^{-Ts}}{s+1}$$

as shown in Figure 6–36. Let us construct a root-locus plot for this system. The characteristic equation for this closed-loop system is

$$1 + \frac{Ke^{-Ts}}{s+1} = 0 \tag{6–18}$$

It is noted that for systems with transport lag the rules of construction presented earlier need to be modified. For example, the number of the root-locus branches is infinite, since the characteristic equation has an infinite number of roots. The number of asymptotes is infinite. They are all parallel to the real axis of the s plane, as will be seen later.

From Equation (6–18), we obtain

$$\frac{Ke^{-Ts}}{s+1} = -1$$

Thus, the angle condition becomes

$$\underline{/\frac{Ke^{-Ts}}{s+1}} = \underline{/e^{-Ts}} - \underline{/s+1} = \pm 180°(2k+1) \qquad (k = 0, 1, 2, \dots) \tag{6–19}$$

To find the angle of e^{-Ts}, substitute $s = \sigma + j\omega$. Then we obtain

$$e^{-Ts} = e^{-T\sigma - j\omega T}$$

Since $e^{-T\sigma}$ is a real quantity, the angle of $e^{-T\sigma}$ is zero. Hence

$$\begin{aligned} \underline{/e^{-Ts}} &= \underline{/e^{-j\omega T}} = \underline{/\cos\omega T - j\sin\omega T} \\ &= -\omega T \qquad \text{(radians)} \\ &= -57.3\omega T \qquad \text{(degrees)} \end{aligned}$$

Since T is a given constant, the angle of e^{-Ts} is a function of ω only. The angle condition, Equation (6–19), then becomes

$$-57.3\omega T - \underline{/s+1} = \pm 180°(2k+1)$$

We shall next determine the angle contribution due to e^{-Ts} as given by Equation (6–19). For $k = 0$, the angle condition may be written

$$\underline{/s+1} = \pm 180° - 57.3°\omega T \tag{6–20}$$

Since the angle contribution of e^{-Ts} is zero for $\omega = 0$, the real axis from -1 to $-\infty$ forms a part of the root loci. Now assume a value ω_1 for ω and compute $57.3°\omega_1 T$. At point -1 on the negative real axis, draw a line that makes an angle of $180° - 57.3°\omega_1 T$ with the real axis. Find the intersection of this line and the horizontal line $\omega = \omega_1$. This intersection, point P in Figure 6–37(a), is a point satisfying Equation (6–20) and hence is on a root locus. Continuing the same process, we obtain the root-locus plot as shown in Figure 6–37(b).

Note that as s approaches minus infinity, the open-loop transfer function

$$\frac{Ke^{-Ts}}{s+1}$$

approaches minus infinity since

$$\lim_{s=-\infty} \frac{Ke^{-Ts}}{s+1} = \left. \frac{\dfrac{d}{ds}\left(Ke^{-Ts}\right)}{\dfrac{d}{ds}(s+1)} \right|_{s=-\infty}$$
$$= -KTe^{-Ts}\big|_{s=-\infty}$$
$$= -\infty$$

Therefore, $s = -\infty$ is a pole of the open-loop transfer function. Thus, root loci start from $s = -1$ or $s = -\infty$ and terminate at $s = \infty$, as K increases from zero to infinity. Since the right-hand side of the angle condition given by Equation (6–19) has an infinite number of values, there are an infinite number of root loci, as the value of k $(k = 0, 1, 2, \ldots)$ goes from zero to infinity. For example, if $k = 1$, the angle condition becomes

$$\underline{/s+1} = \pm 540° - 57.3°\omega T \qquad \text{(degrees)}$$
$$= \pm 3\pi - \omega T \qquad \text{(radians)}$$

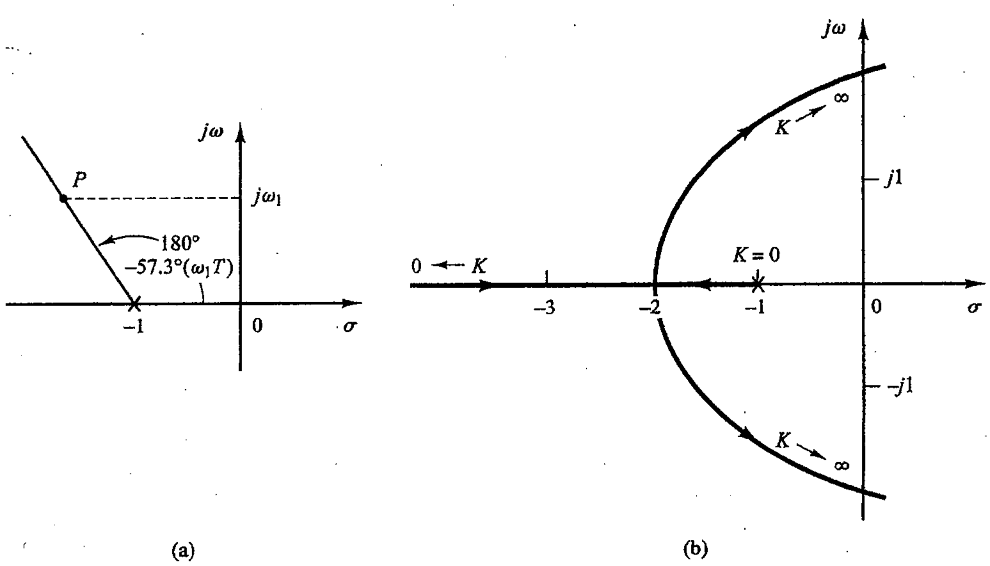


Figure 6–37
(a) Construction of the root locus;
(b) root-locus plot.

The construction of the root loci for $k = 1$ is the same as that for $k = 0$. A plot of root loci for $k = 0, 1,$ and 2 when $T = 1$ sec is shown in Figure 6–38.

The magnitude condition states that

$$\left| \frac{Ke^{-Ts}}{s+1} \right| = 1$$

Since the magnitude of e^{-Ts} is equal to that of $e^{-T\sigma}$ or

$$|e^{-Ts}| = |e^{-T\sigma}| \cdot |e^{-j\omega T}| = e^{-T\sigma}$$

the magnitude condition becomes

$$|s+1| = Ke^{-T\sigma}$$

The root loci shown in Figure 6–38 are graduated in terms of K when $T = 1$ sec.

Although there are an infinite number of root-locus branches, the primary branch that lies between $-j\pi$ and $j\pi$ is most important. Referring to Figure 6–38, the critical value of K at the primary branch is equal to 2, while the critical values of K at other branches are much higher (8, 14, . . .). Therefore, the critical value $K = 2$ on the primary branch is most significant from the stability viewpoint. The transient response of the system is determined by the roots located closest to the $j\omega$ axis and lie on the primary branch. In summary, the root-locus branch corresponding to $k = 0$ is the dominant one; other branches corresponding to $k = 1, 2, 3, \ldots$ are not so important and may be neglected.

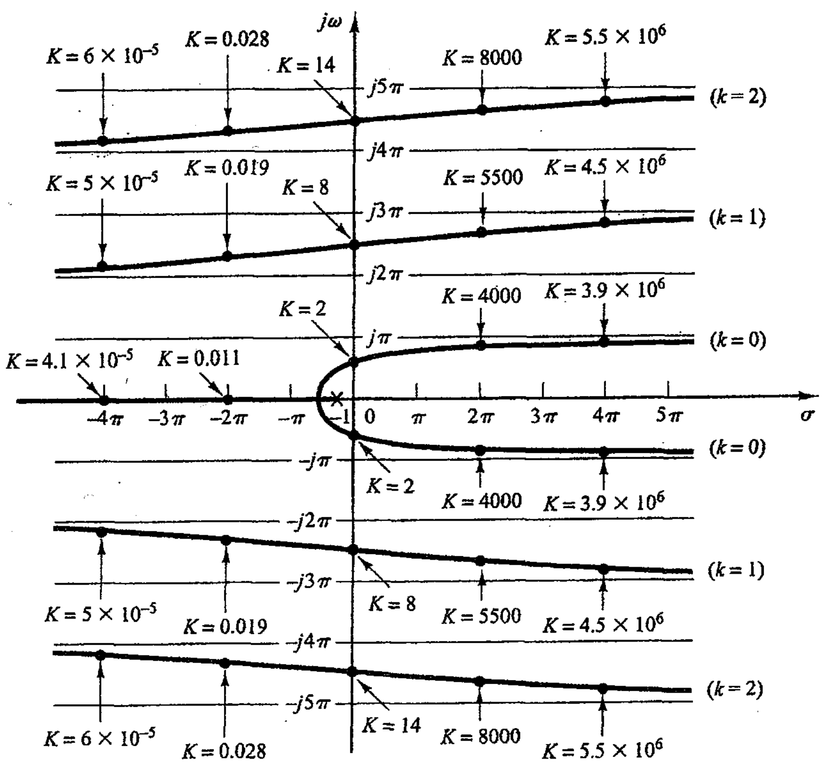


Figure 6–38 Root-locus plot for the system shown in Figure 6–36 ($T = 1$ sec).

This example illustrates the fact that dead time can cause instability even in the first-order system because the root loci enter the right-half s plane for large values of K. Therefore, although the gain K of the first-order system can be set at a high value in the absence of dead time, it cannot be set too high if dead time is present. (For the system considered here, the value of gain K must be considerably less than 2 for a satisfactory operation.)

Approximation of Transport Lag or Dead Time. If the dead time T is very small, then e^{-Ts} is frequently approximated by

$$e^{-Ts} \doteqdot 1 - Ts$$

or

$$e^{-Ts} \doteqdot \frac{1}{Ts + 1}$$

Such approximations are good if the dead time is very small and, in addition, the input time function $f(t)$ to the dead-time element is smooth and continuous. [This means that the second- and higher-order derivatives of $f(t)$ are small.]

A more elaborate expression to approximate e^{-Ts} is available and is

$$e^{-Ts} = \frac{1 - \frac{Ts}{2} + \frac{(Ts)^2}{8} - \frac{(Ts)^3}{48} + \cdots}{1 + \frac{Ts}{2} + \frac{(Ts)^2}{8} + \frac{(Ts)^3}{48} + \cdots}$$

If only the first two terms in the numerator and denominator are taken, then

$$e^{-Ts} \doteqdot \frac{1 - \frac{Ts}{2}}{1 + \frac{Ts}{2}} = \frac{2 - Ts}{2 + Ts}$$

This approximation is also used frequently.

MATLAB Approximation of Dead Time. To handle dead time e^{-sT}, MATLAB uses the pade approximation. For example, if $T = 0.1$ sec, then using the third-order transfer function as an approximation to e^{-sT}, enter the following MATLAB program into the computer.

```
[num,den] = pade(0.1, 3);
printsys(num, den, 's')
  num/den =
  -1s^3 + 120s^2 - 6000s + 120000
  -------------------------------
   s^3 + 120s^2 + 6000s + 120000
```

Similarly, the program for the fourth-order transfer function approximation with $T = 0.1$ sec is

```
[num,den] = pade(0.1, 4);
printsys(num, den, 's')

 num/den =

 s^4 - 200s^3 + 18000s^2 - 840000s + 16800000
 ---------------------------------------------
 s^4 + 200s^3 + 18000s^2 + 840000s + 16800000
```

Notice that the pade approximation depends on the dead time T and the desired order for the approximating transfer function.

EXAMPLE PROBLEMS AND SOLUTIONS

A–6–1. Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and −1 and between −2 and −3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

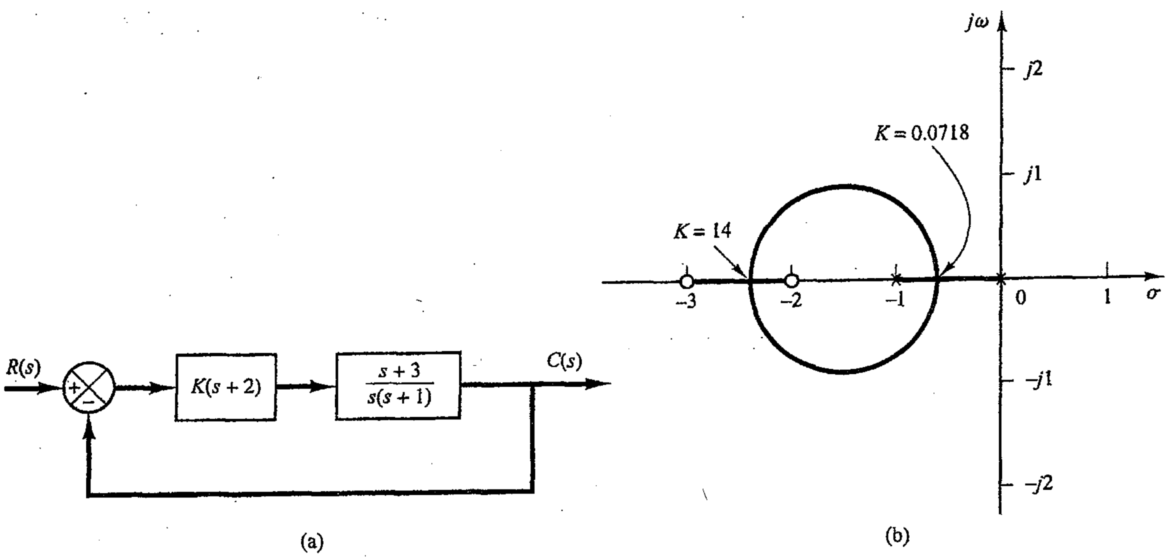


Figure 6–39
(a) Control system; (b) root-locus plot.

3. Determine the breakaway and break-in points. The characteristic equation for the system is

$$1 + \frac{K(s + 2)(s + 3)}{s(s + 1)} = 0$$

or

$$K = -\frac{s(s + 1)}{(s + 2)(s + 3)}$$

The breakaway and break-in points are determined from

$$\begin{aligned} \frac{dK}{ds} &= -\frac{(2s + 1)(s + 2)(s + 3) - s(s + 1)(2s + 5)}{[(s + 2)(s + 3)]^2} \\ &= -\frac{4(s + 0.634)(s + 2.366)}{[(s + 2)(s + 3)]^2} \\ &= 0 \end{aligned}$$

as follows:

$$s = -0.634, \qquad s = -2.366$$

Notice that both points are on root loci. Therefore, they are actual breakaway or break-in points. At point $s = -0.634$, the value of K is

$$K = -\frac{(-0.634)(0.366)}{(1.366)(2.366)} = 0.0718$$

Similarly, at $s = -2.366$,

$$K = -\frac{(-2.366)(-1.366)}{(-0.366)(0.634)} = 14$$

(Because point $s = -0.634$ lies between two poles, it is a breakaway point, and because point $s = -2.366$ lies between two zeros, it is a break-in point.)

4. Determine a sufficient number of points that satisfy the angle condition. (It can be found that the root loci involve a circle with center at −1.5 that passes through the breakaway and break-in points.) The root-locus plot for this system is shown in Figure 6–39(b).

Note that this system is stable for any positive value of K since all the root loci lie in the left-half s plane.

Small values of K ($0 < K < 0.0718$) correspond to an overdamped system. Medium values of K ($0.0718 < K < 14$) correspond to an underdamped system. Finally, large values of K ($14 < K$) correspond to an overdamped system. With a large value of K, the steady state can be reached in much shorter time than with a small value of K.

The value of K should be adjusted so that system performance is optimum according to a given performance index.

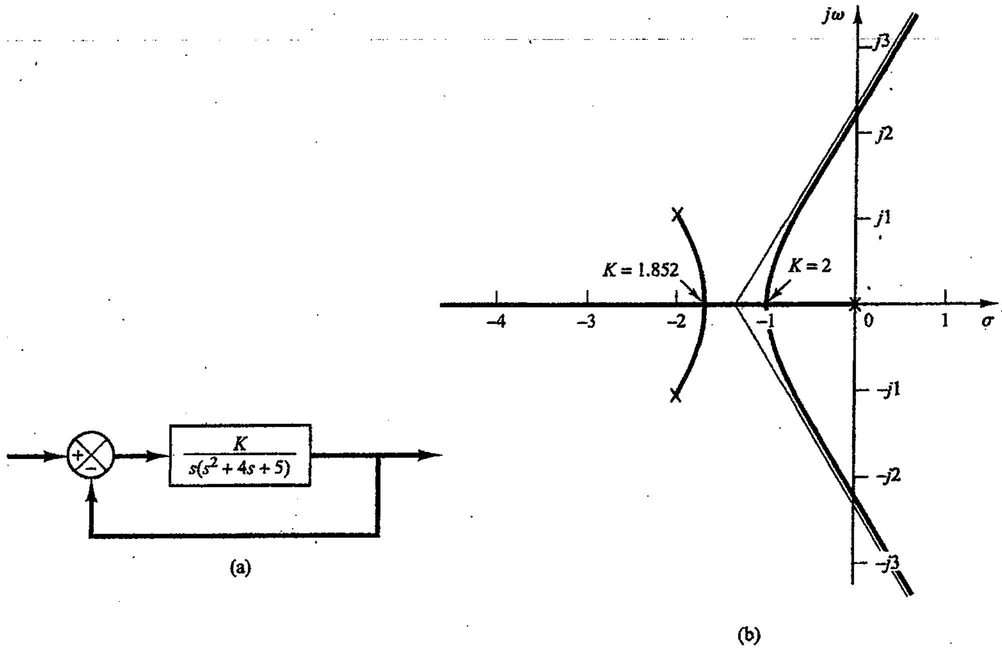


Figure 6–41
(a) Control system;
(b) root-locus plot.

The breakaway and break-in points are found from $dK/ds = 0$. Since the characteristic equation is

$$s^3 + 4s^2 + 5s + K = 0$$

we have

$$K = -(s^3 + 4s^2 + 5s)$$

Now we set

$$\frac{dK}{ds} = -(3s^2 + 8s + 5) = 0$$

which yields

$$s = -1, \qquad s = -1.6667$$

Since these points are on root loci, they are actual breakaway or break-in points. (At point $s = -1$, the value of K is 2, and at point $s = -1.6667$, the value of K is 1.852.)

The angle of departure from a complex pole in the upper half s plane is obtained from

$$\theta = 180° - 153.43° - 90°$$

or

$$\theta = -63.43°$$

The root-locus branch from the complex pole in the upper half s plane breaks into the real axis at $s = -1.6667$.

Next we determine the points where root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 4(j\omega)^2 + 5(j\omega) + K = 0$$

or

$$(K - 4\omega^2) + j\omega(5 - \omega^2) = 0$$

from which we obtain

$$\omega = \pm\sqrt{5}, \quad K = 20 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = \sqrt{5}$ and $\omega = -\sqrt{5}$. The root-locus branch on the real axis touches the $j\omega$ axis at $\omega = 0$. A sketch of the root loci for the system is shown in Figure 6–41(b).

Note that since this system is of third order, there are three closed-loop poles. The nature of the system response to a given input depends on the locations of the closed-loop poles.

For $0 < K < 1.852$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. For $1.852 \le K \le 2$, there are three real closed-loop poles. For example, the closed-loop poles are located at

$$s = -1.667, \quad s = -1.667, \quad s = -0.667, \quad \text{for } K = 1.852$$

$$s = -1, \quad s = -1, \quad s = -2, \quad \text{for } K = 2$$

For $2 < K$, there are a set of complex-conjugate closed-loop poles and a real closed-loop pole. Thus, small values of K $(0 < K < 1.852)$ correspond to an underdamped system. (Since the real closed-loop pole dominates, only a small ripple may show up in the transient response.) Medium values of K $(1.852 \le K \le 2)$ correspond to an overdamped system. Large values of K $(2 < K)$ correspond to an underdamped system. With a large value of K, the system responds much faster than with a smaller value of K.

A–6–4. Sketch the root loci for the system shown in Figure 6–42(a).

Solution. The open-loop poles are located at $s = 0$, $s = -1$, $s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{4} = 45°, -45°, 135°, -135°$$

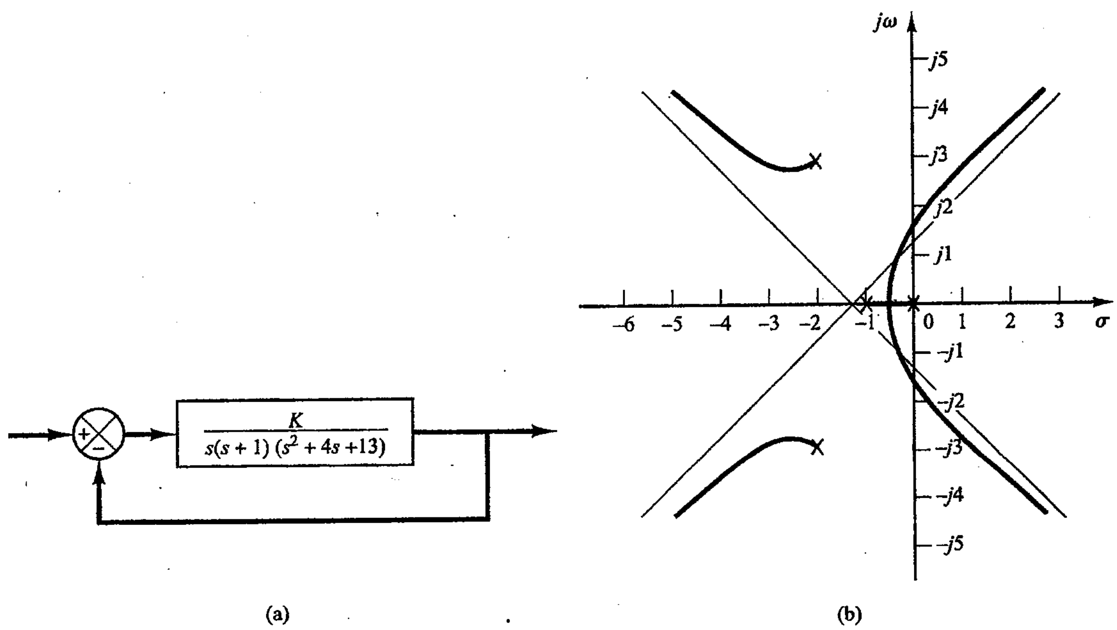


Figure 6–42
(a) Control system; (b) root-locus plot.

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0 + 1 + 2 + 2}{4} = -1.25$$

The breakaway and break-in points are found from $dK/ds = 0$. Noting that

$$K = -s(s + 1)(s^2 + 4s + 13) = -(s^4 + 5s^3 + 17s^2 + 13s)$$

we have

$$\frac{dK}{ds} = -(4s^3 + 15s^2 + 34s + 13) = 0$$

from which we get

$$s = -0.467, \qquad s = -1.642 + j2.067, \qquad s = -1.642 - j2.067$$

Point $s = -0.467$ is on a root locus. Therefore, it is an actual breakaway point. The gain values K corresponding to points $s = -1.642 \pm j2.067$ are complex quantities. Since the gain values are not real positive, these points are neither breakaway nor break-in points.

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 123.69° - 108.44° - 90°$$

or

$$\theta = -142.13°$$

Next we shall find the points where root loci may cross the $j\omega$ axis. Since the characteristic equation is

$$s^4 + 5s^3 + 17s^2 + 13s + K = 0$$

by substituting $s = j\omega$ into it we obtain

$$(j\omega)^4 + 5(j\omega)^3 + 17(j\omega)^2 + 13(j\omega) + K = 0$$

or

$$(K + \omega^4 - 17\omega^2) + j\omega(13 - 5\omega^2) = 0$$

from which we obtain

$$\omega = \pm 1.6125, \qquad K = 37.44 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$

The root-locus branches that extend to the right-half s plane cross the imaginary axis at $\omega = \pm 1.6125$. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–42(b) shows a sketch of the root loci for the system. Notice that each root-locus branch that extends to the right half s plane crosses its own asymptote.

A–6–5. Sketch the root loci for the system shown in Figure 6–43(a).

Solution. A root locus exists on the real axis between points $s = -1$ and $s = -3.6$. The asymptotes can be determined as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3-1} = 90°, -90°$$

The intersection of the asymptotes and the real axis is found from

$$s = -\frac{0+0+3.6-1}{3-1} = -1.3$$

Since the characteristic equation is

$$s^3 + 3.6s^2 + K(s+1) = 0$$

we have

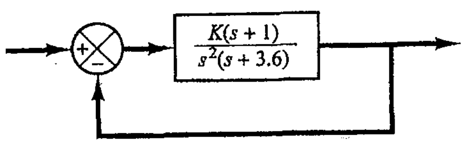
$$K = -\frac{s^3 + 3.6s^2}{s+1}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s+1) - (s^3 + 3.6s^2)}{(s+1)^2} = 0$$

or

$$s^3 + 3.3s^2 + 3.6s = 0$$



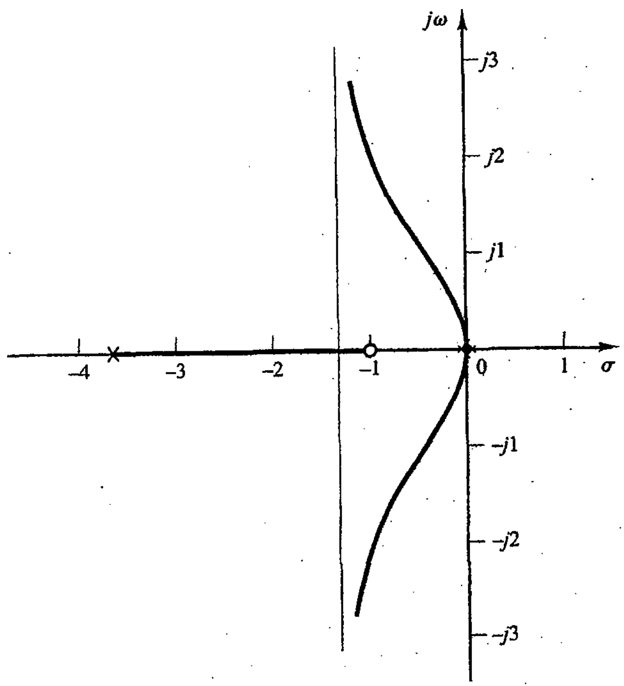


Figure 6–43
(a) Control system; (b) root-locus plot.

from which we get

$$s = 0, \quad s = -1.65 + j0.9367, \quad s = -1.65 - j0.9367$$

Point $s = 0$ corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ are neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.

To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

$$(K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

Notice that this equation can be satisfied only if $\omega = 0, K = 0$. Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$ axis at $\omega = 0$. The root-locus branches do not cross the $j\omega$ axis. Figure 6–43(b) is a sketch of the root loci for this system.

A–6–6. Sketch the root loci for the system shown in Figure 6–44(a).

Solution. A root locus exists on the real axis between point $s = -0.4$ and $s = -3.6$. The angles of asymptotes can be found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3 - 1} = 90°, -90°$$

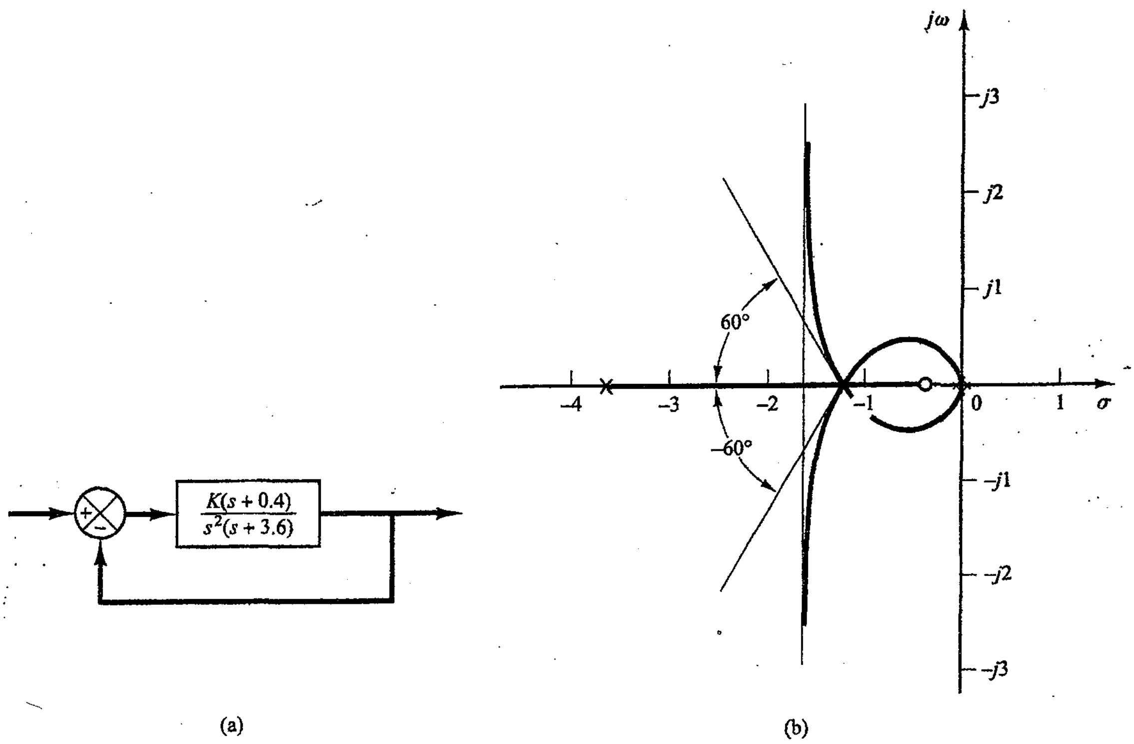


Figure 6–44
(a) Control system; (b) root-locus plot.

The intersection of the asymptotes and the real axis is obtained from

$$s = -\frac{0 + 0 + 3.6 - 0.4}{3 - 1} = -1.6$$

Next we shall find the breakaway points. Since the characteristic equation is

$$s^3 + 3.6s^2 + Ks + 0.4K = 0$$

we have

$$K = -\frac{s^3 + 3.6s^2}{s + 0.4}$$

The breakaway and break-in points are found from

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 0.4) - (s^3 + 3.6s^2)}{(s + 0.4)^2} = 0$$

from which we get

$$s^3 + 2.4s^2 + 1.44s = 0$$

or

$$s(s + 1.2)^2 = 0$$

Thus, the breakaway or break-in points are at $s = 0$ and $s = -1.2$. Note that $s = -1.2$ is a double root. When a double root occurs in $dK/ds = 0$ at point $s = -1.2$, $d^2K/(ds^2) = 0$ at this point. The value of gain K at point $s = -1.2$ is

$$K = -\frac{s^3 + 3.6s^2}{s + 4}\bigg|_{s=-1.2} = 4.32$$

This means that with $K = 4.32$ the characteristic equation has a triple root at point $s = -1.2$. This can be easily verified as follows:

$$s^3 + 3.6s^2 + 4.32s + 1.728 = (s + 1.2)^3 = 0$$

Hence, three root-locus branches meet at point $s = -1.2$. The angles of departures at point $s = -1.2$ of the root locus branches that approach the asymptotes are $\pm 180°/3$, that is, 60° and −60°. (See Problem **A–6–7**.)

Finally, we shall examine if root-locus branches cross the imaginary axis. By substituting $s = j\omega$ into the characteristic equation, we have

$$(j\omega)^3 + 3.6(j\omega)^2 + K(j\omega) + 0.4K = 0$$

or

$$(0.4K - 3.6\omega^2) + j\omega(K - \omega^2) = 0$$

This equation can be satisfied only if $\omega = 0$, $K = 0$. At point $\omega = 0$, the root locus is tangent to the $j\omega$ axis because of the presence of a double pole at the origin. There are no points that root-locus branches cross the imaginary axis.

A sketch of the root loci for this system is shown in Figure 6–44(b).

8

Frequency-Response Analysis

8–1 INTRODUCTION

By the term *frequency response,* we mean the steady-state response of a system to a sinusoidal input. In frequency-response methods, we vary the frequency of the input signal over a certain range and study the resulting response.

In this and the next chapter we present frequency-response approaches to the analysis and design of control systems. The information we get from such analysis is different from what we get from root-locus analysis. In fact, the frequency response and root-locus approaches complement each other. One advantage of the frequency-response approach is that we can use the data obtained from measurements on the physical system without deriving its mathematical model. In many practical designs of control systems both approaches are employed. Control engineers must be familiar with both.

Frequency-response methods were developed in 1930s and 1940s by Nyquist, Bode, Nichols, and many others. The frequency-response methods are most powerful in conventional control theory. They are also indispensable to robust control theory.

The Nyquist stability criterion enables us to investigate both the absolute and relative stabilities of linear closed-loop systems from a knowledge of their open-loop frequency-response characteristics. An advantage of the frequency-response approach is that frequency-response tests are, in general, simple and can be made accurately by use of readily available sinusoidal signal generators and precise measurement equipment. Often the transfer functions of complicated components can be determined experimentally by frequency-response tests. In addition, the frequency-response approach has the advantages that a system may be designed so that the effects of undesirable noise are negligible and that such analysis and design can be extended to certain nonlinear control systems.

Although the frequency response of a control system presents a qualitative picture of the transient response, the correlation between frequency and transient responses is indirect, except for the case of second-order systems. In designing a closed-loop system, we adjust the frequency-response characteristic of the open-loop transfer function by using several design criteria in order to obtain acceptable transient-response characteristics for the system.

Obtaining Steady-State Outputs to Sinusoidal Inputs. We shall show that the steady-state output of a transfer function system can be obtained directly from the sinusoidal transfer function, that is, the transfer function in which s is replaced by $j\omega$, where ω is frequency.

Consider the stable, linear, time-invariant system shown in Figure 8–1. The input and output of the system, whose transfer function is $G(s)$, are denoted by $x(t)$ and $y(t)$, respectively. If the input $x(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency, but with possibly different magnitude and phase angle.

Let us assume that the input signal is given by

$$x(t) = X \sin \omega t$$

Suppose that the transfer function $G(s)$ can be written as a ratio of two polynomials in s; that is,

$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s + s_1)(s + s_2)\cdots(s + s_n)}$$

The Laplace-transformed output $Y(s)$ is then

$$Y(s) = G(s)X(s) = \frac{p(s)}{q(s)} X(s) \tag{8–1}$$

where $X(s)$ is the Laplace transform of the input $x(t)$.

It will be shown that, after waiting until steady-state conditions are reached, the frequency response can be calculated by replacing s in the transfer function by $j\omega$. It will also be shown that the steady-state response can be given by

$$G(j\omega) = Me^{j\phi} = M\underline{/\phi}$$

where M is the amplitude ratio of the output and input sinusoids and ϕ is the phase shift between the input sinusoid and the output sinusoid. In the frequency-response test, the input frequency ω is varied until the entire frequency range of interest is covered.

The steady-state response of a stable, linear, time-invariant system to a sinusoidal input does not depend on the initial conditions. (Thus, we can assume the zero initial condition.) If $Y(s)$ has only distinct poles, then the partial fraction expansion of Equation (8–1) yields

$$\begin{aligned} Y(s) &= G(s)X(s) = G(s)\frac{\omega X}{s^2 + \omega^2} \\ &= \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \frac{b_1}{s + s_1} + \frac{b_2}{s + s_2} + \cdots + \frac{b_n}{s + s_n} \end{aligned} \tag{8–2}$$

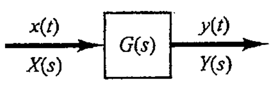


Figure 8–1
Stable, linear, time-invariant system.

where a and the b_i (where $i = 1, 2, \ldots, n$) are constants and $\bar{a}$ is the complex conjugate of a. The inverse Laplace transform of Equation (8–2) gives

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + b_1e^{-s_1t} + b_2e^{-s_2t} + \cdots + b_ne^{-s_nt} \qquad (t \geq 0) \qquad (8\text{–}3)$$

For a stable system, $-s_1, -s_2, \ldots, -s_n$ have negative real parts. Therefore, as t approaches infinity, the terms $e^{-s_1t}, e^{-s_2t}, \ldots,$ and e^{-s_nt} approach zero. Thus, all the terms on the right-hand side of Equation (8–3), except the first two, drop out at steady state.

If $Y(s)$ involves multiple poles s_j of multiplicity m_j, then $y(t)$ will involve terms such as $t^{h_j}e^{-s_jt}(h_j = 0, 1, 2, \ldots, m_j - 1)$. For a stable system, the terms $t^{h_j}e^{-s_jt}$ approach zero as t approaches infinity.

Thus, regardless of whether the system is of the distinct-pole type, the steady-state response becomes

$$y_{ss}(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \qquad (8\text{–}4)$$

where the constant a can be evaluated from Equation (8–2) as follows:

$$a = G(s)\frac{\omega X}{s^2 + \omega^2}(s + j\omega)\bigg|_{s=-j\omega} = -\frac{XG(-j\omega)}{2j}$$

Note that

$$\bar{a} = G(s)\frac{\omega X}{s^2 + \omega^2}(s - j\omega)\bigg|_{s=j\omega} = \frac{XG(j\omega)}{2j}$$

Since $G(j\omega)$ is a complex quantity, it can be written in the following form:

$$G(j\omega) = |G(j\omega)|e^{j\phi}$$

where $|G(j\omega)|$ represents the magnitude and ϕ represents the angle of $G(j\omega)$; that is,

$$\phi = \underline{/G(j\omega)} = \tan^{-1}\left[\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right]$$

The angle ϕ may be negative, positive, or zero. Similarly, we obtain the following expression for $G(-j\omega)$:

$$G(-j\omega) = |G(-j\omega)|e^{-j\phi} = |G(j\omega)|e^{-j\phi}$$

Then, noting that

$$a = -\frac{X|G(j\omega)|e^{-j\phi}}{2j}, \qquad \bar{a} = \frac{X|G(j\omega)|e^{j\phi}}{2j}$$

Equation (8–4) can be written

$$\begin{aligned} y_{ss}(t) &= X|G(j\omega)|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\ &= X|G(j\omega)|\sin(\omega t + \phi) \\ &= Y\sin(\omega t + \phi) \qquad (8\text{–}5) \end{aligned}$$

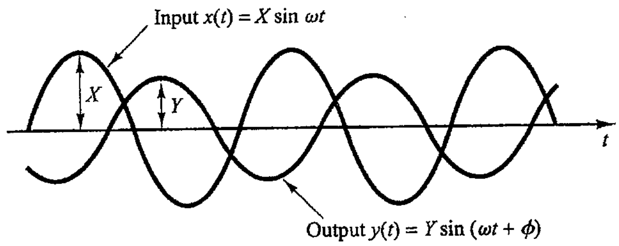


Figure 8–2
Input and output sinusoidal signals.

where $Y = X|G(j\omega)|$. We see that a stable, linear, time-invariant system subjected to a sinusoidal input will, at steady state, have a sinusoidal output of the same frequency as the input. But the amplitude and phase of the output will, in general, be different from those of the input. In fact, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$, while the phase angle differs from that of the input by the amount $\phi = \angle G(j\omega)$. An example of input and output sinusoidal signals is shown in Figure 8–2.

On the basis of this, we obtain this important result: For sinusoidal inputs,

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right| = \text{amplitude ratio of the output sinuisoid to the input sinusoid}$$

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{X(j\omega)} = \text{phase shift of the output sinusoid with respect to the input sinusoid}$$

Hence, the steady-state response characteristics of a system to a sinusoidal input can be obtained directly from

$$\frac{Y(j\omega)}{X(j\omega)} = G(j\omega)$$

The function $G(j\omega)$ is called the sinusoidal transfer function. It is the ratio of $Y(j\omega)$ to $X(j\omega)$, is a complex quantity, and can be represented by the magnitude and phase angle with frequency as a parameter. The sinusoidal transfer function of any linear system is obtained by substituting $j\omega$ for s in the transfer function of the system.

A positive phase angle is called phase lead, and a negative phase angle is called phase lag. A network that has phase-lead characteristics is called a lead network, while a network that has phase-lag characteristics is called a lag network.

EXAMPLE 8–1 Consider the system shown in Figure 8–3. The transfer function $G(s)$ is

$$G(s) = \frac{K}{Ts + 1}$$

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be found as follows: Substituting $j\omega$ for s in $G(s)$ yields

$$G(j\omega) = \frac{K}{jT\omega + 1}$$

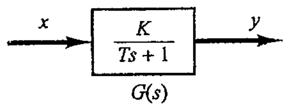


Figure 8–3
First-order system.

The amplitude ratio of the output to the input is

$$|G(j\omega)| = \frac{K}{\sqrt{1 + T^2\omega^2}}$$

while the phase angle ϕ is

$$\phi = \underline{/G(j\omega)} = -\tan^{-1} T\omega$$

Thus, for the input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be obtained from Equation (8–5) as follows:

$$y_{ss}(t) = \frac{XK}{\sqrt{1 + T^2\omega^2}} \sin(\omega t - \tan^{-1} T\omega) \tag{8–6}$$

From Equation (8–6), it can be seen that for small ω, the amplitude of the steady-state output $y_{ss}(t)$ is almost equal to K times the amplitude of the input. The phase shift of the output is small for small ω. For large ω, the amplitude of the output is small and almost inversely proportional to ω. The phase shift approaches $-90°$ as ω approaches infinity. This is a phase-lag network.

EXAMPLE 8–2 Consider the network given by

$$G(s) = \frac{s + \frac{1}{T_1}}{s + \frac{1}{T_2}}$$

Determine whether this network is a lead network or lag network.

For the sinusoidal input $x(t) = X \sin \omega t$, the steady-state output $y_{ss}(t)$ can be found as follows: Since

$$G(j\omega) = \frac{j\omega + \frac{1}{T_1}}{j\omega + \frac{1}{T_2}} = \frac{T_2(1 + T_1 j\omega)}{T_1(1 + T_2 j\omega)}$$

we have

$$|G(j\omega)| = \frac{T_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}}$$

and

$$\phi = \underline{/G(j\omega)} = \tan^{-1} T_1\omega - \tan^{-1} T_2\omega$$

Thus the steady-state output is

$$y_{ss}(t) = \frac{XT_2\sqrt{1 + T_1^2\omega^2}}{T_1\sqrt{1 + T_2^2\omega^2}} \sin(\omega t + \tan^{-1} T_1\omega - \tan^{-1} T_2\omega)$$

From this expression, we find that if $T_1 > T_2$, then $\tan^{-1} T_1\omega - \tan^{-1} T_2\omega > 0$. Thus, if $T_1 > T_2$, then the network is a lead network. If $T_1 < T_2$, then the network is a lag network.

Presenting Frequency-Response Characteristics in Graphical Forms. The sinusoidal transfer function, a complex function of the frequency ω, is characterized by its magnitude and phase angle, with frequency as the parameter. There are three commonly used representations of sinusoidal transfer functions:

1. Bode diagram or logarithmic plot
2. Nyquist plot or polar plot
3. Log-magnitude-versus-phase plot (Nichols plots)

We shall discuss these representations in detail in this chapter. We shall also discuss the MATLAB approach to obtain Bode diagrams, Nyquist plots, and Nichols plots.

Outline of the Chapter. Section 8–1 has presented introductory material on the frequency response. Section 8–2 presents Bode diagrams of various transfer-function systems. Section 8–3 discusses a computational approach to obtain Bode diagrams with MATLAB, Section 8–4 treats polar plots of sinusoidal transfer functions, and Section 8–5 discusses drawing Nyquist plots with MATLAB. Section 8–6 briefly presents log-magnitude-versus-phase plots. Section 8–7 gives a detailed account of the Nyquist stability criterion, Section 8–8 discusses the stability analysis of closed-loop systems using the Nyquist stability criterion, and Section 8–9 treats the relative stability analysis of closed-loop systems. Measures of relative stability such as phase margin and gain margin are introduced here. The correlation between the transient response and frequency response is also discussed. Section 8–10 presents a method for obtaining the closed-loop frequency response from the open-loop frequency response by use of the M and N circles. Use of the Nichols chart is also discussed for obtaining the closed-loop frequency response. Finally, Section 8–11 deals with the determination of the transfer function based on an experimental Bode diagram.

8–2 BODE DIAGRAMS

Bode Diagrams or Logarithmic Plots. A Bode diagram consists of two graphs: One is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of the phase angle; both are plotted against the frequency on a logarithmic scale.

The standard representation of the logarithmic magnitude of $G(j\omega)$ is $20 \log|G(j\omega)|$, where the base of the logarithm is 10. The unit used in this representation of the magnitude is the decibel, usually abbreviated dB. In the logarithmic representation, the curves are drawn on semilog paper, using the log scale for frequency and the linear scale for either magnitude (but in decibels) or phase angle (in degrees). (The frequency range of interest determines the number of logarithmic cycles required on the abscissa.)

The main advantage of using the Bode diagram is that multiplication of magnitudes can be converted into addition. Furthermore, a simple method for sketching an approximate log-magnitude curve is available. It is based on asymptotic approximations. Such approximation by straight-line asymptotes is sufficient if only rough information on the frequency-response characteristics is needed. Should the exact curve be desired, corrections can be made easily to these basic asymptotic plots. Expanding the low-frequency range by use of a logarithmic scale for the frequency is highly advantageous since characteristics at low frequencies are most important in practical systems. Although it is not possible to plot the curves right down to zero frequency because of the logarithmic frequency ($\log 0 = -\infty$), this does not create a serious problem.

Note that the experimental determination of a transfer function can be made simple if frequency-response data are presented in the form of a Bode diagram.

Basic Factors of $G(j\omega)H(j\omega)$. As stated earlier, the main advantage in using the logarithmic plot is the relative ease of plotting frequency-response curves. The basic factors that very frequently occur in an arbitrary transfer function $G(j\omega)H(j\omega)$ are

1. Gain K
2. Integral and derivative factors $(j\omega)^{\mp 1}$
3. First-order factors $(1 + j\omega T)^{\mp 1}$
4. Quadratic factors $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{\mp 1}$

Once we become familiar with the logarithmic plots of these basic factors, it is possible to utilize them in constructing a composite logarithmic plot for any general form of $G(j\omega)H(j\omega)$ by sketching the curves for each factor and adding individual curves graphically, because adding the logarithms of the gains corresponds to multiplying them together.

The Gain K. A number greater than unity has a positive value in decibels, while a number smaller than unity has a negative value. The log-magnitude curve for a constant gain K is a horizontal straight line at the magnitude of $20 \log K$ decibels. The phase angle of the gain K is zero. The effect of varying the gain K in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

A number–decibel conversion line is given in Figure 8–4. The decibel value of any number can be obtained from this line. As a number increases by a factor of 10, the corresponding decibel value increases by a factor of 20. This may be seen from the following:

$$20 \log(K \times 10) = 20 \log K + 20$$

Similarly,

$$20 \log(K \times 10^n) = 20 \log K + 20n$$

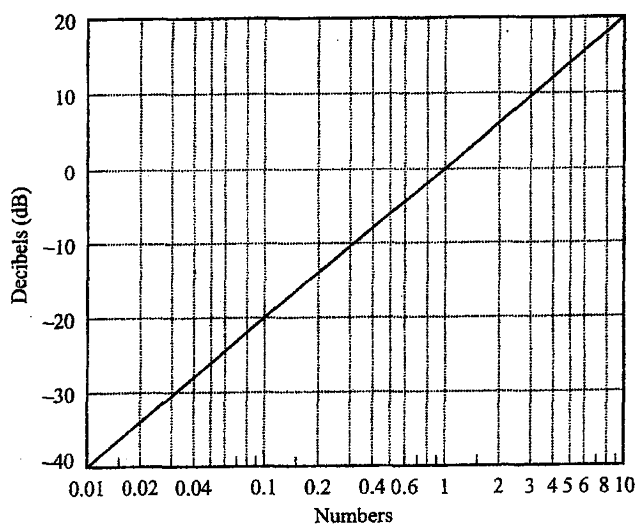


Figure 8–4 Number–decibel conversion line.

Note that, when expressed in decibels, the reciprocal of a number differs from its value only in sign; that is, for the number K,

$$20 \log K = -20 \log \frac{1}{K}$$

Integral and Derivative Factors $(j\omega)^{\mp 1}$. The logarithmic magnitude of $1/j\omega$ in decibels is

$$20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega \text{ dB}$$

The phase angle of $1/j\omega$ is constant and equal to $-90°$.

In Bode diagrams, frequency ratios are expressed in terms of octaves or decades. An octave is a frequency band from ω_1 to $2\omega_1$, where ω_1 is any frequency value. A decade is a frequency band from ω_1 to $10\omega_1$, where again ω_1 is any frequency. (On the logarithmic scale of semilog paper, any given frequency ratio can be represented by the same horizontal distance. For example, the horizontal distance from $\omega = 1$ to $\omega = 10$ is equal to that from $\omega = 3$ to $\omega = 30$.)

If the log magnitude $-20 \log \omega$ dB is plotted against ω on a logarithmic scale, it is a straight line. To draw this straight line, we need to locate one point (0 dB, $\omega = 1$) on it. Since

$$(-20 \log 10\omega) \text{ dB} = (-20 \log \omega - 20) \text{ dB}$$

the slope of the line is -20 dB/decade (or -6 dB/octave).

Similarly, the log magnitude of $j\omega$ in decibels is

$$20 \log |j\omega| = 20 \log \omega \text{ dB}$$

The phase angle of $j\omega$ is constant and equal to 90°. The log-magnitude curve is a straight line with a slope of 20 dB/decade. Figures 8–5(a) and (b) show frequency-response curves for $1/j\omega$ and $j\omega$, respectively. We can clearly see that the differences in the frequency responses of the factors $1/j\omega$ and $j\omega$ lie in the signs of the slopes of the log-magnitude curves and in the signs of the phase angles. Both log magnitudes become equal to 0 dB at $\omega = 1$.

If the transfer function contains the factor $(1/j\omega)^n$ or $(j\omega)^n$, the log magnitude becomes, respectively,

$$20 \log \left| \frac{1}{(j\omega)^n} \right| = -n \times 20 \log |j\omega| = -20n \log \omega \text{ dB}$$

or

$$20 \log |(j\omega)^n| = n \times 20 \log |j\omega| = 20n \log \omega \text{ dB}$$

The slopes of the log-magnitude curves for the factors $(1/j\omega)^n$ and $(j\omega)^n$ are thus $-20n$ dB/decade and $20n$ dB/decade, respectively. The phase angle of $(1/j\omega)^n$ is equal to $-90° \times n$ over the entire frequency range, while that of $(j\omega)^n$ is equal to $90° \times n$ over the entire frequency range. The magnitude curves will pass through the point (0 dB, $\omega = 1$).

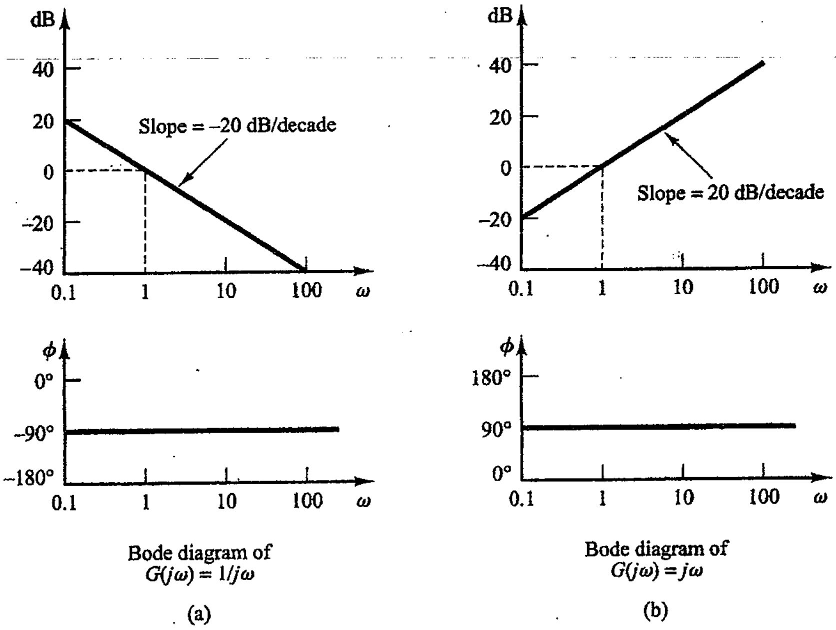


Figure 8–5
(a) Bode diagram of $G(j\omega) = 1/j\omega$;
(b) Bode diagram of $G(j\omega) = j\omega$.

First-Order Factors $(1 + j\omega T)^{\mp 1}$. The log magnitude of the first-order factor $1/(1 + j\omega T)$ is

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}$$

For low frequencies, such that $\omega \ll 1/T$, the log magnitude may be approximated by

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log 1 = 0 \text{ dB}$$

Thus, the log-magnitude curve at low frequencies is the constant 0-dB line. For high frequencies, such that $\omega \gg 1/T$,

$$-20 \log \sqrt{1 + \omega^2 T^2} \doteq -20 \log \omega T \text{ dB}$$

This is an approximate expression for the high-frequency range. At $\omega = 1/T$, the log magnitude equals 0 dB; at $\omega = 10/T$, the log magnitude is -20 dB. Thus, the value of $-20 \log \omega T$ dB decreases by 20 dB for every decade of ω. For $\omega \gg 1/T$, the log-magnitude curve is thus a straight line with a slope of -20 dB/decade (or -6 dB/octave).

Our analysis shows that the logarithmic representation of the frequency-response curve of the factor $1/(1 + j\omega T)$ can be approximated by two straight-line asymptotes, one a straight line at 0 dB for the frequency range $0 < \omega < 1/T$ and the other a straight line with slope -20 dB/decade (or -6 dB/octave) for the frequency range $1/T < \omega < \infty$. The exact log-magnitude curve, the asymptotes, and the exact phase-angle curve are shown in Figure 8–6.

The frequency at which the two asymptotes meet is called the *corner* frequency or *break* frequency. For the factor $1/(1 + j\omega T)$, the frequency $\omega = 1/T$ is the corner frequency since at $\omega = 1/T$ the two asymptotes have the same value. (The low-frequency asymptotic expression at $\omega = 1/T$ is 20 log 1 dB = 0 dB, and the high-frequency

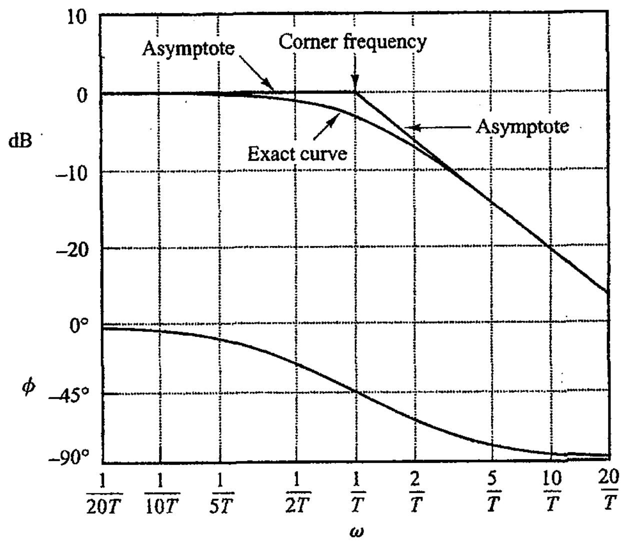


Figure 8–6 Log-magnitude curve, together with the asymptotes, and phase-angle curve of $1/(1 + j\omega T)$.

asymptotic expression at $\omega = 1/T$ is also 20 log 1 dB = 0 dB.) The corner frequency divides the frequency-response curve into two regions: a curve for the low-frequency region and a curve for the high-frequency region. The corner frequency is very important in sketching logarithmic frequency-response curves.

The exact phase angle ϕ of the factor $1/(1 + j\omega T)$ is

$$\phi = -\tan^{-1}\omega T$$

At zero frequency, the phase angle is 0°. At the corner frequency, the phase angle is

$$\phi = -\tan^{-1}\frac{T}{T} = -\tan^{-1}1 = -45°$$

At infinity, the phase angle becomes −90°. Since the phase angle is given by an inverse-tangent function, the phase angle is skew symmetric about the inflection point at $\phi = -45°$.

The error in the magnitude curve caused by the use of asymptotes can be calculated. The maximum error occurs at the corner frequency and is approximately equal to −3 dB since

$$-20\log\sqrt{1+1} + 20\log 1 = -10\log 2 = -3.03 \text{ dB}$$

The error at the frequency one octave below the corner frequency, that is, at $\omega = 1/(2T)$, is

$$-20\log\sqrt{\frac{1}{4}+1} + 20\log 1 = -20\log\frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$

The error at the frequency one octave above the corner frequency, that is, at $\omega = 2/T$, is

$$-20\log\sqrt{2^2+1} + 20\log 2 = -20\log\frac{\sqrt{5}}{2} = -0.97 \text{ dB}$$

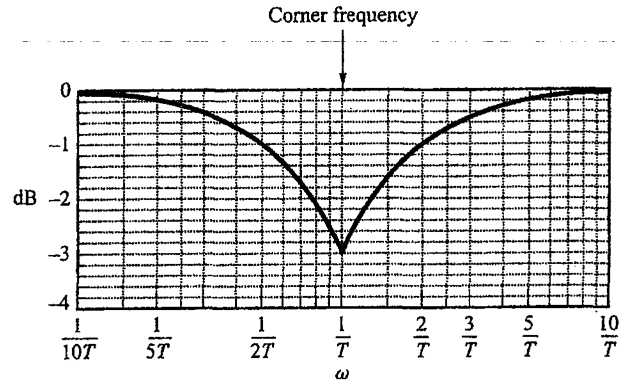


Figure 8–7 Log-magnitude error in the asymptotic expression of the frequency-response curve of $1/(1 + j\omega T)$.

Thus, the error at one octave below or above the corner frequency is approximately equal to −1 dB. Similarly, the error at one decade below or above the corner frequency is approximately −0.04 dB. The error in decibels involved in using the asymptotic expression for the frequency-response curve of $1/(1 + j\omega T)$ is shown in Figure 8–7. The error is symmetric with respect to the corner frequency.

Since the asymptotes are quite easy to draw and are sufficiently close to the exact curve, the use of such approximations in drawing Bode diagrams is convenient in establishing the general nature of the frequency-response characteristics quickly with a minimum amount of calculation and may be used for most preliminary design work. If accurate frequency-response curves are desired, corrections may easily be made by referring to the curve given in Figure 8–7. In practice, an accurate frequency-response curve can be drawn by introducing a correction of 3 dB at the corner frequency and a correction of 1 dB at points one octave below and above the corner frequency and then connecting these points by a smooth curve.

Note that varying the time constant T shifts the corner frequency to the left or to the right, but the shapes of the log-magnitude and the phase-angle curves remain the same.

The transfer function $1/(1 + j\omega T)$ has the characteristics of a low-pass filter. For frequencies above $\omega = 1/T$, the log magnitude falls off rapidly toward $-\infty$. This is essentially due to the presence of the time constant. In the low-pass filter, the output can follow a sinusoidal input faithfully at low frequencies. But as the input frequency is increased, the output cannot follow the input because a certain amount of time is required for the system to build up in magnitude. Thus, at high frequencies, the amplitude of the output approaches zero and the phase angle of the output approaches −90°. Therefore, if the input function contains many harmonics, then the low-frequency components are reproduced faithfully at the output, while the high-frequency components are attenuated in amplitude and shifted in phase. Thus, a first-order element yields exact, or almost exact, duplication only for constant or slowly varying phenomena.

An advantage of the Bode diagram is that for reciprocal factors—for example, the factor $1 + j\omega T$—the log-magnitude and the phase-angle curves need only be changed in sign, since

$$20 \log|1 + j\omega T| = -20 \log\left|\frac{1}{1 + j\omega T}\right|$$

and

$$\underline{/1 + j\omega T} = \tan^{-1}\omega T = -\Big/\frac{1}{1 + j\omega T}$$

The corner frequency is the same for both cases. The slope of the high-frequency asymptote of $1 + j\omega T$ is 20 dB/decade, and the phase angle varies from 0° to 90° as the frequency ω is increased from zero to infinity. The log-magnitude curve, together with the asymptotes, and the phase-angle curve for the factor $1 + j\omega T$ are shown in Figure 8–8.

To draw a phase curve accurately, we have to locate several points on the curve. The phase angles of $(1 + j\omega T)^{\mp 1}$ are

$$\mp 45^\circ \quad \text{at} \quad \omega = \frac{1}{T}$$

$$\mp 26.6^\circ \quad \text{at} \quad \omega = \frac{1}{2T}$$

$$\mp 5.7^\circ \quad \text{at} \quad \omega = \frac{1}{10T}$$

$$\mp 63.4^\circ \quad \text{at} \quad \omega = \frac{2}{T}$$

$$\mp 84.3^\circ \quad \text{at} \quad \omega = \frac{10}{T}$$

For the case where a given transfer function involves terms like $(1 + j\omega T)^{\mp n}$, a similar asymptotic construction may be made. The corner frequency is still at $\omega = 1/T$, and the asymptotes are straight lines. The low-frequency asymptote is a horizontal straight line

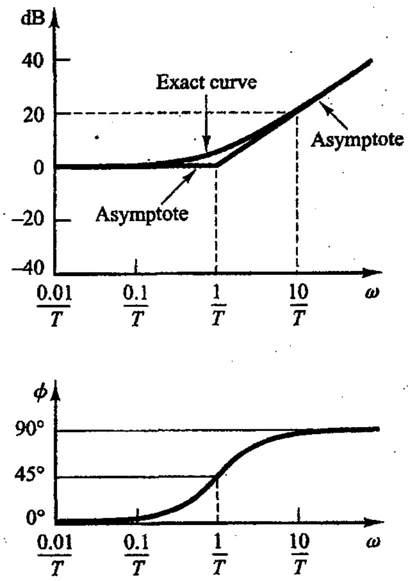


Figure 8–8 Log-magnitude curve, together with the asymptotes, and phase-angle curve for $1 + j\omega T$.

at 0 dB, while the high-frequency asymptote has the slope of $-20n$ dB/decade or $20n$ dB/decade. The error involved in the asymptotic expressions is n times that for $(1 + j\omega T)^{\mp 1}$. The phase angle is n times that of $(1 + j\omega T)^{\mp 1}$ at each frequency point.

Quadratic Factors $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{\mp 1}$. Control systems often possess quadratic factors of the form

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \tag{8-7}$$

If $\zeta > 1$, this quadratic factor can be expressed as a product of two first-order factors with real poles. If $0 < \zeta < 1$, this quadratic factor is the product of two complex-conjugate factors. Asymptotic approximations to the frequency-response curves are not accurate for a factor with low values of ζ. This is because the magnitude and phase of the quadratic factor depend on both the corner frequency and the damping ratio ζ.

The asymptotic frequency-response curve may be obtained as follows: Since

$$20\log\left|\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

for low frequencies such that $\omega \ll \omega_n$, the log magnitude becomes

$$-20\log 1 = 0 \text{ dB}$$

The low-frequency asymptote is thus a horizontal line at 0 dB. For high frequencies such that $\omega \gg \omega_n$, the log magnitude becomes

$$-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n} \text{ dB}$$

The equation for the high-frequency asymptote is a straight line having the slop -40 dB/decade since

$$-40\log\frac{10\omega}{\omega_n} = -40 - 40\log\frac{\omega}{\omega_n}$$

The high-frequency asymptote intersects the low-frequency one at $\omega = \omega_n$ since at th frequency

$$-40\log\frac{\omega_n}{\omega_n} = -40\log 1 = 0 \text{ dB}$$

This frequency, ω_n, is the corner frequency for the quadratic factor considered.

The two asymptotes just derived are independent of the value of ζ. Near th frequency $\omega = \omega_n$, a resonant peak occurs, as may be expected from Equation (8–7 The damping ratio ζ determines the magnitude of this resonant peak. Errors obv ously exist in the approximation by straight-line asymptotes. The magnitude of th error depends on the value of ζ. It is large for small values of ζ. Figure 8–9 shows th exact log-magnitude curves, together with the straight-line asymptotes and the exa

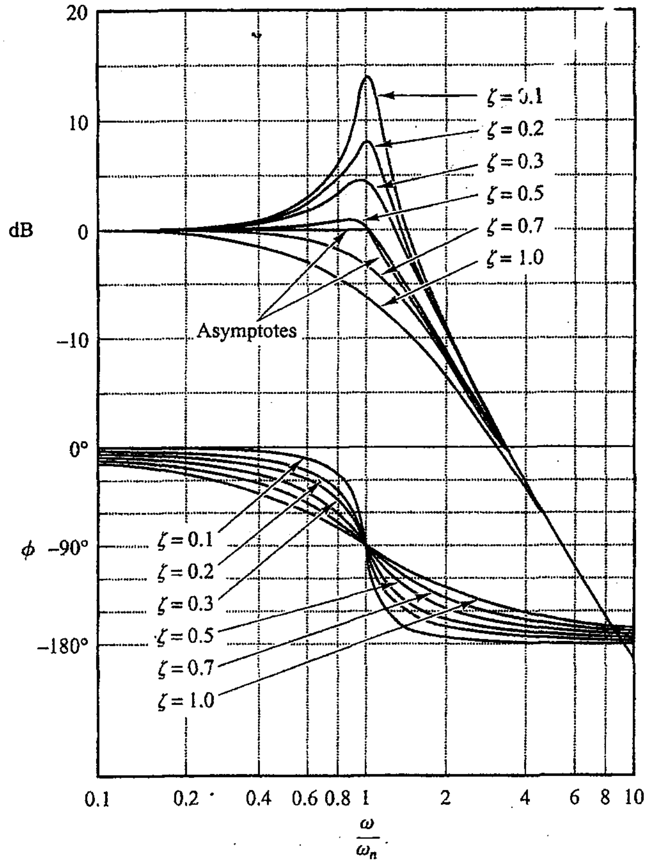


Figure 8–9 Log-magnitude curves, together with the asymptotes, and phase-angle curves of the quadratic transfer function given by Equation (8–7).

phase-angle curves for the quadratic factor given by Equation (8–7) with several values of ζ. If corrections are desired in the asymptotic curves, the necessary amounts of correction at a sufficient number of frequency points may be obtained from Figure 8–9.

The phase angle of the quadratic factor $\left[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2\right]^{-1}$ is

$$\phi = \bigg/ \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \tag{8–8}$$

The phase angle is a function of both ω and ζ. At $\omega = 0$, the phase angle equals 0°. At the corner frequency $\omega = \omega_n$, the phase angle is −90° regardless of ζ, since

$$\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}\infty = -90^\circ$$

At $\omega = \infty$, the phase angle becomes −180°. The phase-angle curve is skew symmetric about the inflection point—the point where $\phi = -90°$. There are no simple ways to sketch such phase curves. We need to refer to the phase-angle curves shown in Figure 8–9.

The frequency-response curves for the factor

$$1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2$$

can be obtained by merely reversing the sign of the log magnitude and that of the phase angle of the factor

$$\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

To obtain the frequency-response curves of a given quadratic transfer function, we must first determine the value of the corner frequency ω_n and that of the damping ratio ζ. Then, by using the family of curves given in Figure 8–9, the frequency-response curves can be plotted.

The Resonant Frequency ω_r and the Resonant Peak Value M_r. The magnitude of

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

is

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \tag{8–9}$$

If $|G(j\omega)|$ has a peak value at some frequency, this frequency is called the *resonant* frequency. Since the numerator of $|G(j\omega)|$ is constant, a peak value of $|G(j\omega)|$ will occur when

$$g(\omega) = \left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2 \tag{8–10}$$

is a minimum. Since Equation (8–10) can be written

$$g(\omega) = \left[\frac{\omega^2 - \omega_n^2(1 - 2\zeta^2)}{\omega_n^2}\right]^2 + 4\zeta^2(1 - \zeta^2) \tag{8–11}$$

the minimum value of $g(\omega)$ occurs at $\omega = \omega_n\sqrt{1 - 2\zeta^2}$. Thus the resonant frequency ω_r is

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}, \qquad \text{for } 0 \le \zeta \le 0.707 \tag{8–12}$$

As the damping ratio ζ approaches zero, the resonant frequency approaches ω_n. For $0 < \zeta \le 0.707$, the resonant frequency ω_r is less than the damped natural frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, which is exhibited in the transient response. From Equation (8–12), it can be seen that for $\zeta > 0.707$, there is no resonant peak. The magnitude $|G(j\omega)|$ decreases monotonically with increasing frequency ω. (The magnitude is less than 0 dB for all values of $\omega > 0$. Recall that, for $0.7 < \zeta < 1$, the step response is oscillatory, but the oscillations are well damped and are hardly perceptible.)

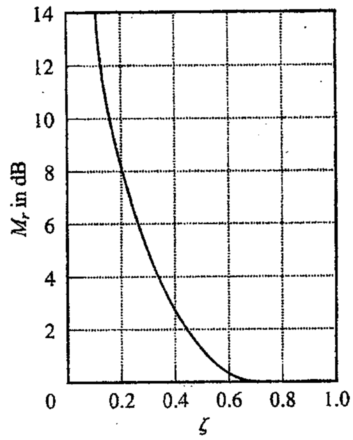


Figure 8–10
M_r-versus-ζ curve for the second-order system $1/[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]$.

The magnitude of the resonant peak, M_r, can be found by substituting Equation (8–12) into Equation (8–9). For $0 \leq \zeta \leq 0.707$,

$$M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{8–13}$$

For $\zeta > 0.707$,

$$M_r = 1 \tag{8–14}$$

As ζ approaches zero, M_r approaches infinity. This means that if the undamped system is excited at its natural frequency, the magnitude of $G(j\omega)$ becomes infinity. The relationship between M_r and ζ is shown in Figure 8–10.

The phase angle of $G(j\omega)$ at the frequency where the resonant peak occurs can be obtained by substituting Equation (8–12) into Equation (8–8). Thus, at the resonant frequency ω_r,

$$\underline{/G(j\omega_r)} = -\tan^{-1}\frac{\sqrt{1 - 2\zeta^2}}{\zeta} = -90° + \sin^{-1}\frac{\zeta}{\sqrt{1 - \zeta^2}}$$

General Procedure for Plotting Bode Diagrams. MATLAB provides an easy way to plot Bode diagrams. (See Section 8–3.) Here, however, we consider the case where we want to draw Bode diagrams manually without using MATLAB.

First rewrite the sinusoidal transfer function $G(j\omega)H(j\omega)$ as a product of basic factors discussed above. Then identify the corner frequencies associated with these basic factors. Finally, draw the asymptotic log-magnitude curves with proper slopes between the corner frequencies. The exact curve, which lies close to the asymptotic curve, can be obtained by adding proper corrections.

The phase-angle curve of $G(j\omega)H(j\omega)$ can be drawn by adding the phase-angle curves of individual factors.

The use of Bode diagrams employing asymptotic approximations requires much less time than other methods that may be used for computing the frequency response of a transfer function. The ease of plotting the frequency-response curves for a given transfer function and the ease of modification of the frequency-response curve as compensation is added are the main reasons why Bode diagrams are very frequently used in practice.

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$$1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2$$

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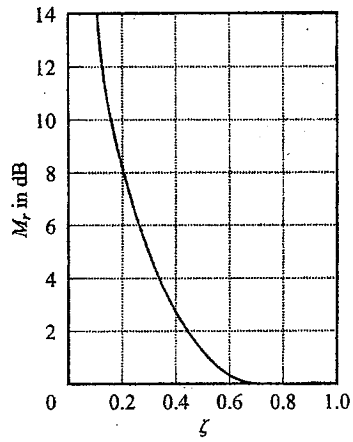

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The phase-angle curve of $G(j\omega)H(j\omega)$ can be drawn by adding the phase-angle curves of individual factors.

The use of Bode diagrams employing asymptotic approximations requires much less time than other methods that may be used for computing the frequency response of a transfer function. The ease of plotting the frequency-response curves for a given transfer function and the ease of modification of the frequency-response curve as compensation is added are the main reasons why Bode diagrams are very frequently used in practice.

EXAMPLE 8–5 Draw the Bode diagram for the following transfer function:

$$G(j\omega) = \frac{10(j\omega + 3)}{(j\omega)(j\omega + 2)[(j\omega)^2 + j\omega + 2]}$$

Make corrections so that the log-magnitude curve is accurate.

To avoid any possible mistakes in drawing the log-magnitude curve, it is desirable to put $G(j\omega)$ in the following normalized form, where the low-frequency asymptotes for the first-order factors and the second-order factor are the 0-dB line:

$$G(j\omega) = \frac{7.5\left(\frac{j\omega}{3} + 1\right)}{(j\omega)\left(\frac{j\omega}{2} + 1\right)\left[\frac{(j\omega)^2}{2} + \frac{j\omega}{2} + 1\right]}$$

This function is composed of the following factors:

$$7.5, \quad (j\omega)^{-1}, \quad 1 + j\frac{\omega}{3}, \quad \left(1 + j\frac{\omega}{2}\right)^{-1}, \quad \left[1 + j\frac{\omega}{2} + \frac{(j\omega)^2}{2}\right]^{-1}$$

The corner frequencies of the third, fourth, and fifth terms are $\omega = 3$, $\omega = 2$, and $\omega = \sqrt{2}$, respectively. Note that the last term has the damping ratio of 0.3536.

To plot the Bode diagram, the separate asymptotic curves for each of the factors are shown in Figure 8–11. The composite curve is then obtained by algebraically adding the individual curves, also shown in Figure 8–11. Note that when the individual asymptotic curves are added at each frequency, the slope of the composite curve is cumulative. Below $\omega = \sqrt{2}$, the plot has the slope of -20 dB/decade. At the first corner frequency $\omega = \sqrt{2}$, the slope changes to -60 dB/decade and continues to the next corner frequency $\omega = 2$, where the slope becomes -80 dB/decade. At the last corner frequency $\omega = 3$, the slope changes to -60 dB/decade.

Once such an approximate log-magnitude curve has been drawn, the actual curve can be obtained by adding corrections at each corner frequency and at frequencies one octave below and above the corner frequencies. For first-order factors $(1 + j\omega T)^{\mp 1}$, the corrections are ± 3 dB at the corner frequency and ± 1 dB at the frequencies one octave below and above the corner frequency. Corrections necessary for the quadratic factor are obtained from Figure 8–9. The exact log-magnitude curve for $G(j\omega)$ is shown by a dashed curve in Figure 8–11.

Note that any change in the slope of the magnitude curve is made only at the corner frequencies of the transfer function $G(j\omega)$. Therefore, instead of drawing individual magnitude curves and adding them up, as shown, we may sketch the magnitude curve without sketching individual curves. We may start drawing the lowest-frequency portion of the straight line (that is, the straight line with the slope -20 dB/decade for $\omega < \sqrt{2}$). As the frequency is increased, we get the effect of the complex-conjugate poles (quadratic term) at the corner frequency $\omega = \sqrt{2}$. The complex-conjugate poles cause the slopes of the magnitude curve to change from -20 to -60 dB/decade. At the next corner frequency, $\omega = 2$, the effect of the pole is to change the slope to -80 dB/decade. Finally, at the corner frequency $\omega = 3$, the effect of the zero is to change the slope from -80 to -60 dB/decade.

For plotting the complete phase-angle curve, the phase-angle curves for all factors have to be sketched. The algebraic sum of all phase-angle curves provides the complete phase-angle curve, as shown in Figure 8–11.

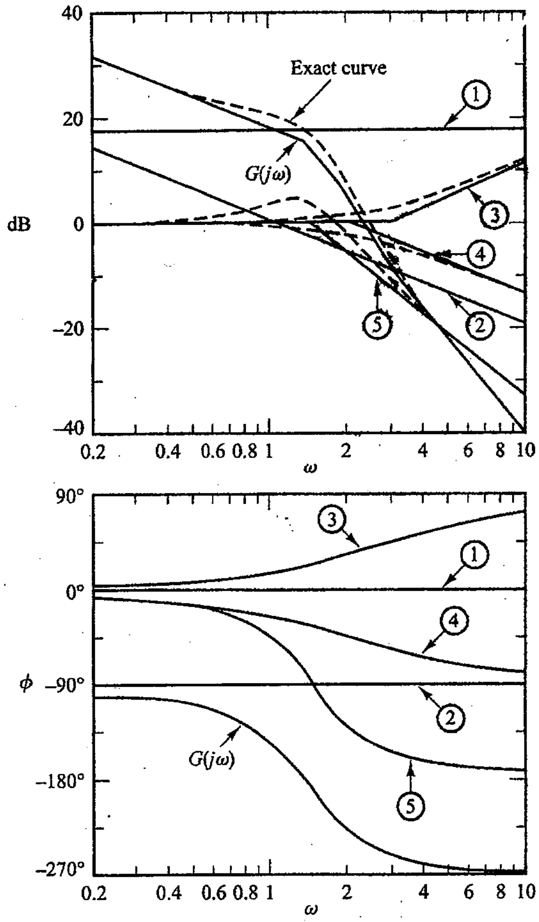


Figure 8–11
Bode diagram of the system considered in Example 8–3.

Minimum-Phase Systems and Nonminimum-Phase Systems. Transfer functions having neither poles nor zeros in the right-half s plane are minimum-phase transfer functions, whereas those having poles and/or zeros in the right-half s plane are nonminimum-phase transfer functions. Systems with minimum-phase transfer functions are called *minimum-phase* systems, whereas those with nonminimum-phase transfer functions are called *nonminimum-phase* systems.

For systems with the same magnitude characteristic, the range in phase angle of the minimum-phase transfer function is minimum among all such systems, while the range in phase angle of any nonminimum-phase transfer function is greater than this minimum.

It is noted that for a minimum-phase system, the transfer function can be uniquely determined from the magnitude curve alone. For a nonminimum-phase system, this is not the case. Multiplying any transfer function by all-pass filters does not alter the magnitude curve, but the phase curve is changed.

Consider as an example the two systems whose sinusoidal transfer functions are, respectively,

$$G_1(j\omega) = \frac{1 + j\omega T}{1 + j\omega T_1}, \qquad G_2(j\omega) = \frac{1 - j\omega T}{1 + j\omega T_1}, \qquad 0 < T < T_1$$

A–6–2 Sketch the root loci of the control system shown in Figure 6–40(a).

Solution. The open-loop poles are located at $s = 0$, $s = -3 + j4$, and $s = -3 - j4$. A root locus branch exists on the real axis between the origin and $-\infty$. There are three asymptotes for the root loci. The angles of asymptotes are

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{3} = 60°, -60°, 180°$$

Referring to Equation (6–13), the intersection of the asymptotes and the real axis is obtained as

$$s = -\frac{0+3+3}{3} = -2$$

Next we check the breakaway and break-in points. For this system we have

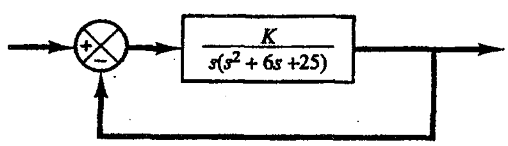
$$K = -s(s^2 + 6s + 25)$$

Now we set

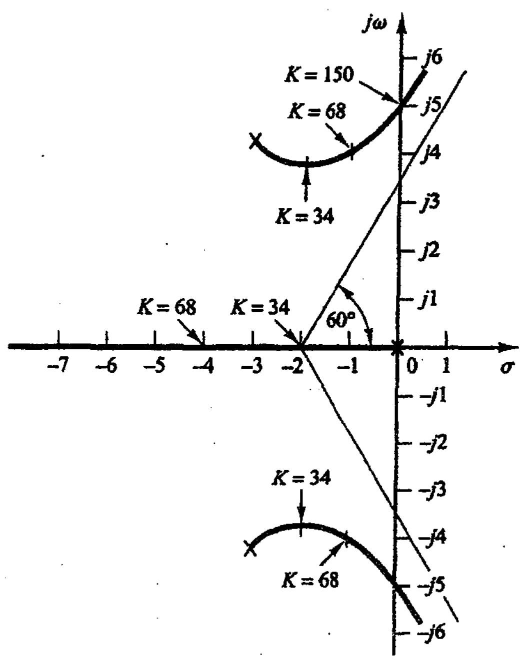
$$\frac{dK}{ds} = -(3s^2 + 12s + 25) = 0$$

which yields

$$s = -2 + j2.0817, \qquad s = -2 - j2.0817$$



(a)



(b)

Figure 6–40
(a) Control system; (b) root-locus plot.

Notice that at points $s = -2 \pm j2.0817$ the angle condition is not satisfied. Hence, they are neither breakaway nor break-in points. In fact, if we calculate the value of K, we obtain

$$K = -s(s^2 + 6s + 25)\Big|_{s=-2\pm j2.0817} = 34 \pm j18.04$$

(To be an actual breakaway or break-in point, the corresponding value of K must be real and positive.)

The angle of departure from the complex pole in the upper half s plane is

$$\theta = 180° - 126.87° - 90°$$

or

$$\theta = -36.87°$$

The points where root-locus branches cross the imaginary axis may be found by substituting $s = j\omega$ into the characteristic equation and solving the equation for ω and K as follows: Noting that the characteristic equation is

$$s^3 + 6s^2 + 25s + K = 0$$

we have

$$(j\omega)^3 + 6(j\omega)^2 + 25(j\omega) + K = (-6\omega^2 + K) + j\omega(25 - \omega^2) = 0$$

which yields

$$\omega = \pm 5, \quad K = 150 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Root-locus branches cross the imaginary axis at $\omega = 5$ and $\omega = -5$. The value of gain K at the crossing points is 150. Also, the root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. Figure 6–40(b) shows a root-locus plot for the system.

It is noted that if the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more, and if some of the closed-loop poles move on the root locus toward the right as gain K is increased, then other closed-loop poles must move toward the left as gain K is increased. This fact can be seen clearly in this problem. If the gain K is increased from $K = 34$ to $K = 68$, the complex-conjugate closed-loop poles are moved from $s = -2 + j3.65$ to $s = -1 + j4$; the third pole is moved from $s = -2$ (which corresponds to $K = 34$) to $s = -4$ (which corresponds to $K = 68$). Thus, the movements of two complex-conjugate closed-loop poles to the right by one unit cause the remaining closed-loop pole (real pole in this case) to move to the left by two units.

A–6–3. Consider the system shown in Figure 6–41(a). Sketch the root loci for the system. Observe that for small or large values of K the system is underdamped and for medium values of K it is overdamped.

Solution. A root locus exists on the real axis between the origin and $-\infty$. The angles of asymptotes of the root-locus branches are obtained as

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k + 1)}{3} = 60°, -60°, -180°$$

The intersection of the asymptotes and the real axis is located on the real axis at

$$s = -\frac{0 + 2 + 2}{3} = -1.3333$$