

Chapter Eleven

Strain Energy

11.1 Introduction

Strain energy is defined as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

$$\text{Strain energy } U = \text{work done}$$

Thus for a gradually applied load, the work done in straining the material will be given by the shaded area under the load-extension graph of Fig. 11.1.

$$U = \frac{1}{2} P \delta$$

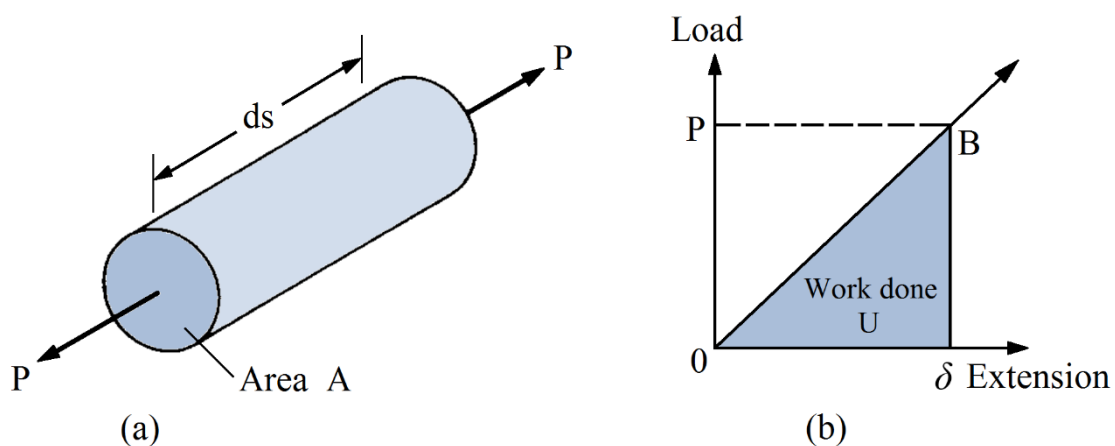


Fig. 11.1 Work done by a gradually applied load.

The strain energy per unit volume is often referred to as the **resilience**. The unshaded area above the line OB of Fig. 11.1 is called the **complementary energy**, a quantity that is utilized in some advanced energy methods of solution and is not considered within the terms of reference for our study.

11.2 Strain energy - tension or compression

(a) Neglecting the weight of the bar

Consider a small element of a bar, length ds , shown in Fig. 11.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

$$\text{strain energy } U = \frac{1}{2} P \delta \quad (\text{E.1})$$

$$\text{Now} \quad \text{Young's modulus } E = \frac{\text{stress}}{\text{strain}} = \frac{P}{A} \times \frac{ds}{\delta}$$

$$\therefore \delta = \frac{Pds}{AE} \quad (\text{E.2})$$

Now, substituting eqn. (E.2) in (E.1)

$$\therefore \text{for the bar element } U = \frac{P^2 ds}{2 AE}$$

$$\therefore \text{total strain energy for a bar of length } L, \quad U = \int_0^L \frac{P^2 ds}{2 AE}$$

Thus, assuming that the area of the bar remains constant along the length,

$$U = \frac{P^2 L}{2 AE} \quad (11.1)$$

Multiplying by A/A yields

$$U = \frac{P^2 A L}{2 A^2 E} = \frac{\sigma^2 A L}{2 E}$$

$$U = \frac{\sigma^2}{2 E} \times \text{volume of bar} \quad (11.2)$$

Where the stress $\sigma = P/A$,

The **strain energy**, or **resilience**, **per unit volume** of a bar subjected to direct load, tensile or compressive

$$U = \frac{\sigma^2}{2 E} \quad (11.3)$$

or, alternatively,

$$= \frac{1}{2} \sigma \times \frac{\sigma}{E} = \frac{1}{2} \sigma \epsilon$$

$$\text{resilience} = \frac{1}{2} \text{stress} \times \text{strain}$$

(b) Including the weight of the bar

Consider a bar of length L mounted vertically, as shown in Fig. 11.2. At any section AB the total load on the section will be the external load P together with the weight of the bar material below AB .

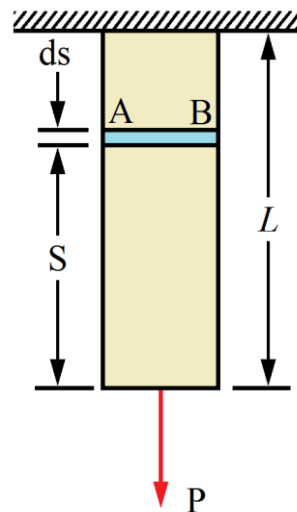


Fig. 11.2 Direct load - tension or compression.

Assuming a uniform cross-section of area A with density ρ ,

$$\text{load on section } AB = P \pm \rho g As$$

The positive sign being used when P is tensile, and the negative sign when P is compressive. Thus, for a tensile force P the extension of the element ds is given by the definition of Young's modulus E to be

$$\delta = \frac{\sigma ds}{E}$$

$$\delta = \frac{(P \pm \rho g As)}{AE} ds$$

But

$$\begin{aligned}
 \text{work done} &= \frac{1}{2} \times \text{load} \times \text{extension} \\
 &= \frac{1}{2} (P \pm \rho g A s) \frac{(P \pm \rho g A s)}{AE} ds \\
 &= \frac{P^2}{2AE} ds + \frac{P \rho g}{E} s ds + \frac{(\rho g)^2 A}{2 E} s^2 ds
 \end{aligned}$$

∴ total strain energy or work done

$$\begin{aligned}
 &= \int_0^L \frac{P^2}{2AE} ds + \int_0^L \frac{P \rho g}{E} s ds + \int_0^L \frac{(\rho g)^2 A}{2 E} s^2 ds \\
 U &= \frac{P^2 L}{2AE} + \frac{P \rho g L^2}{2 E} + \frac{(\rho g)^2 A L^3}{6 E} \quad (11.4)
 \end{aligned}$$

11.3 Strain energy – shear

Consider the elemental bar subjected to a shear load Q at one end causing deformation through the angle γ (the shear strain) and a shear deflection δ , as shown in Fig. 11.3.

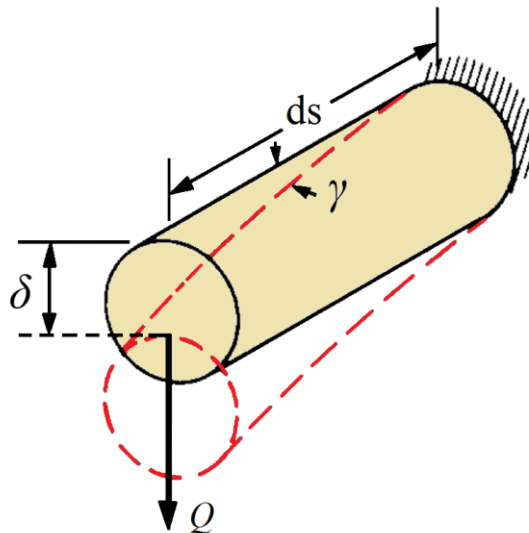


Fig. 11.3 Shear Strain Energy.

$$\text{Strain energy } U = \text{work done} = \frac{1}{2} Q \delta = \frac{1}{2} Q \gamma ds \quad (\text{E.1})$$

But modulus of rigidity $G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$

$$\therefore \gamma = \frac{Q}{AG} \quad (\text{E.2})$$

Substitute eqn. (E.2) in (E.1), yields **shear strain energy**

$$\therefore = \frac{1}{2} Q \times \frac{Q}{AG} \times ds$$

$$\therefore \text{shear strain energy} = \frac{Q^2}{2AG} ds$$

\therefore total strain energy resulting from shear

$$= \int_0^L \frac{Q^2 ds}{2AG}$$

$$\text{total strain energy } U = \frac{Q^2 L}{2AG} \quad (11.5)$$

Multiplying by A/A yields

$$U = \frac{Q^2 A L}{2A^2 G}$$

Where the shear stress $\tau = (Q/A)$,

$$= \frac{\tau^2 A L}{2G}$$

$$U = \frac{\tau^2}{2G} \times \text{volume of bar} \quad (11.6)$$

11.4 Strain energy – bending

Let the element is subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle $d\theta$ at the center (Fig. 11.4). The beam will also have moved through an angle $d\theta$.

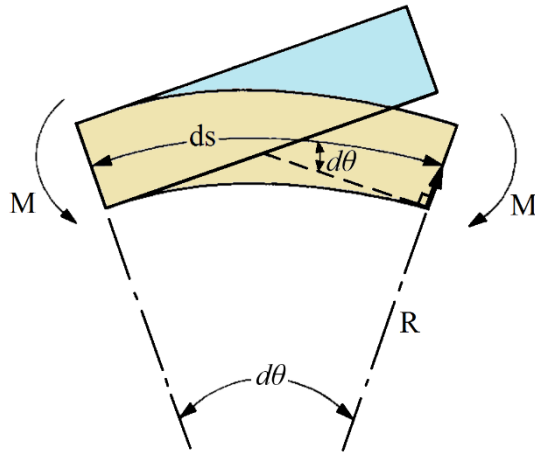


Fig. 11.4 Strain Energy in Bending.

$$\begin{aligned}\text{Strain energy } U &= \text{work done} = \frac{1}{2} \times \text{moment} \times \text{angle turned through (in radians)} \\ &= \frac{1}{2} M d\theta\end{aligned}\quad (\text{E.1})$$

$$\text{But} \quad ds = R d\theta \quad \text{and} \quad \frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad d\theta = \frac{ds}{R} = \frac{M}{EI} ds \quad (\text{E.2})$$

Substitute eqn. (E.2) in (E.1),

$$\text{Strain energy} = \frac{1}{2} M \times \frac{M}{EI} ds = \frac{M^2 ds}{2EI}$$

Total strain energy resulting from bending,

$$U = \int_0^L \frac{M^2 ds}{2EI} \quad (11.7)$$

If the bending moment is constant, this reduces to

$$U = \frac{M^2 L}{2EI}$$

11.5 Strain energy - torsion

Considered the element is subjected to a torque T as shown in Fig. 11.5, producing an angle of twist $d\theta$ radians.

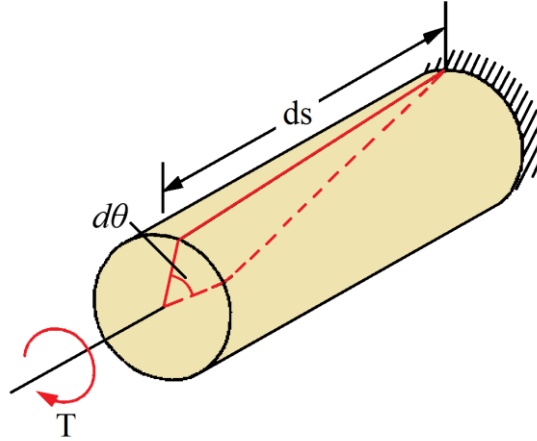


Fig. 11.5 Strain Energy in Torsion.

$$\text{Strain energy } U = \text{work done} = \frac{1}{2} T d\theta \quad (\text{E.1})$$

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{G d\theta}{ds} \quad \text{and} \quad d\theta = \frac{T ds}{G J} \quad (\text{E.2})$$

Substitute eqn. (E.2) in (E.1),

$$U = \int_0^L \frac{T^2 ds}{2G J}$$

\therefore total strain energy resulting from torsion,

$$U = \frac{T^2 L}{2G J} \quad (11.8)$$

It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

$$\text{Strain energy } U = \frac{(\text{applied load})^2 \times L}{2 \times \text{product of two related constants}}$$

11.6 Suddenly applied loads

If a load P is applied gradually to a bar to produce an extension δ the load-extension graph will be as shown in Fig. 11.1 and repeated in Fig. 11.6, the work done being given by,

$$U = \frac{1}{2} P \delta$$

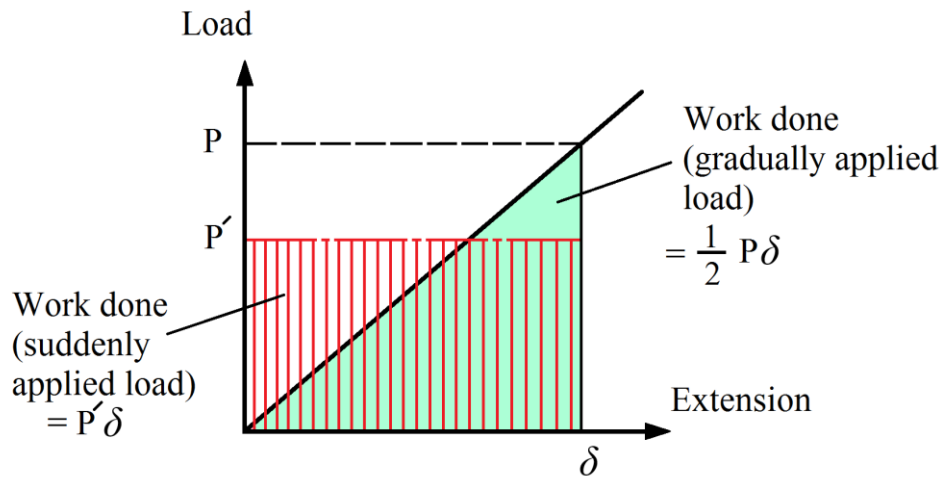


Fig. 11.6 Work done by a suddenly applied load.

If a load P' is suddenly applied to produce the same extension δ , the graph will appear as a horizontal straight line with a work done or strain energy $= P' \delta$.

The bar will be strained by an equal amount δ in both cases and the energy stored must be equal, and equate the area of the rectangle with that of triangle yields;

$$P' \delta = \frac{1}{2} P \delta$$

$$P' = \frac{1}{2} P$$

That means the suddenly applied load is half that of statically applied load to produce the same amount of energy.

The rule states, ***“A load P which is suddenly applied will produce twice the effect of the same load statically applied”.***

11.7 Impact loads - axial load application

Consider the vertically bar shown in Fig. 11.7 with a rigid collar is attached at the end. The load W is free to slide vertically and is suspended by some means at a distance h above the collar. When the load is dropped, it will produce a maximum instantaneous extension δ of the bar, and will have done work (neglecting the mass of the bar and collar).

$$\text{Work done} = \text{force} \times \text{distance} = W(h + \delta) \quad (\text{E.1})$$

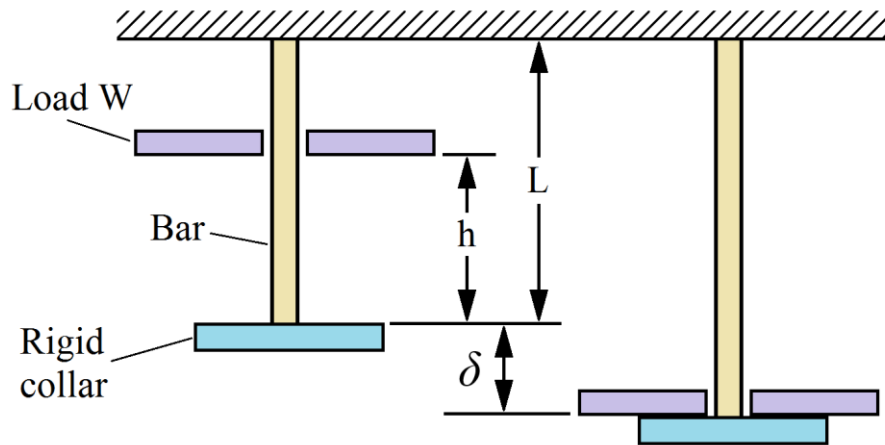


Fig. 11.7 Impact load - axial application.

This work will be stored as strain energy and is given by eqn. (11.2):

$$U = \frac{\sigma^2 A L}{2E} \quad (\text{E.2})$$

where σ is the instantaneous stress set up.

Equating E.1 and E.2 yields,

$$\frac{\sigma^2 A L}{2E} = W(h + \delta) \quad (11.9)$$

If the extension δ is small compared with h it may be ignored and then, approximately,

$$\sigma^2 = \frac{2W E h}{A L}$$

$$\sigma = \sqrt{\frac{2W E h}{A L}} \quad (11.10)$$

If δ is not small compared with h it must be expressed in terms of σ , thus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma L}{\delta} \quad \text{and} \quad \delta = \frac{\sigma L}{E}$$

Substituting in eqn. (11.9)

$$\begin{aligned} \frac{\sigma^2 A L}{2E} &= W h + \frac{W \sigma L}{E} \\ \frac{\sigma^2 A L}{2E} - \sigma \frac{W L}{E} - W h &= 0 \\ \sigma^2 - \frac{2W}{A} \sigma - \frac{2W E h}{A L} &= 0 \end{aligned}$$

Solving by “*the quadratic formula*” and ignoring the negative sign,

$$\begin{aligned} \sigma &= \frac{1}{2} \left\{ \frac{2W}{A} + \sqrt{\left(\frac{2W}{A}\right)^2 + 4 \left(\frac{2W E h}{A L}\right)} \right\} \\ \sigma &= \frac{W}{A} + \sqrt{\left(\frac{W}{A}\right)^2 + \frac{2W E h}{A L}} \end{aligned} \quad (11.11)$$

11.8 Castigliano’s first theorem assumption for deflection

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

$$\delta = \frac{\partial U}{\partial W}$$

Consider the beam or structure shown in Fig. 11.8 with forces P_A, P_B, P_C , etc., acting at points A, B, C , etc.

If a, b, c , etc., are the deflections in the direction of the loads then the total strain energy of the system is equal to the work done.

$$U = \frac{1}{2} P_A a + \frac{1}{2} P_B b + \frac{1}{2} P_C c + \dots \quad (11.12)$$

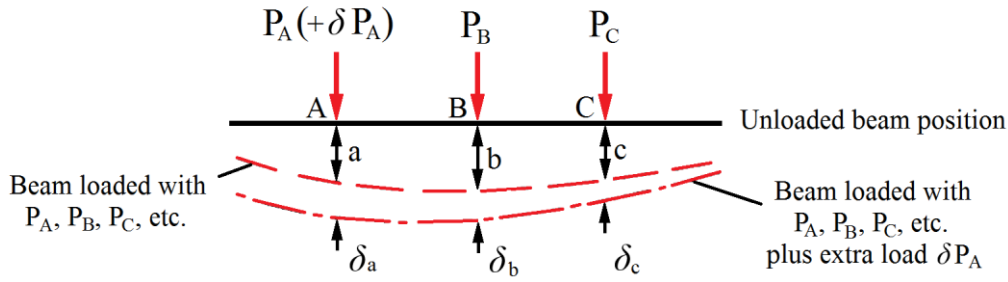


Fig. 11.8 any beam or structure subjected to a system of applied concentrated loads $P_A, P_B, P_C \dots P_N$, etc.

The partial differential of the strain energy U with respect to P_A gives the deflection under and in the direction of P_A .

$$\frac{\partial U}{\partial P_A} = a \quad \text{Similarly} \quad \frac{\partial U}{\partial P_B} = b \quad \text{and} \quad \frac{\partial U}{\partial P_C} = c, \quad \text{etc.}$$

In most beam applications, the strain energy and hence the deflection resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$\therefore \quad U = \int \frac{M^2}{2EI} ds \quad (11.13)$$

$$\frac{\partial U}{\partial P} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P} = \int \frac{2M}{2EI} ds \times \frac{\partial M}{\partial P}$$

$$\delta = \frac{\partial U}{\partial P} = \int \frac{M}{EI} \frac{\partial M}{\partial P} ds \quad (11.14)$$

11.13 Application of Castigliano's theorem to angular movements

If the total strain energy expressed in terms of the external moments, were partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,

$$\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M_i} ds \quad (11.15)$$

where M_i is the imaginary or applied moment at the point where θ is required.

Example 11.1

Determine the diameter of an aluminum shaft, which is designed to store the same amount of strain energy per unit volume as a 50 mm diameter steel shaft of the same length. Both shafts are subjected to equal compressive axial loads.

What will be the ratio of the stresses set up in the two shafts?

$$E_{\text{steel}} = 200 \text{ GN/m}^2; E_{\text{aluminum}} = 67 \text{ GN/m}^2.$$

Solution:

The strain energy per unit volume from eqn. (11.3)

$$\text{Strain energy per unit volume} = \frac{\sigma^2}{2E}$$

Since the strain energy/unit volume in the two shafts is equal,

$$\frac{\sigma_A^2}{2E_A} = \frac{\sigma_S^2}{2E_S}$$

$$\therefore \frac{\sigma_A^2}{\sigma_S^2} = \frac{E_A}{E_S} = \frac{67}{200} = \frac{1}{3} \quad (\text{approximately}) \quad (\text{E.1})$$

$$\therefore 3\sigma_A^2 = \sigma_S^2 \quad (\text{E.2})$$

Now $\sigma = \frac{P}{\text{area}}$ where P is the applied load

Therefore from (E.2) $3\left(\frac{P_A}{\frac{\pi}{4}D_A^2}\right)^2 = \left(\frac{P_S}{\frac{\pi}{4}D_S^2}\right)^2$ But $P_A = P_S = P$

$$\therefore 3D_S^4 = D_A^4$$

$$\therefore D_A^4 = 3 \times (50)^4$$

$$\therefore D_A = \sqrt[4]{1875 \times 10^4} \quad , \quad D_A = 65.8 \text{ mm}$$

The stresses ratio set up in the two shafts from (E.2) is

$$3\sigma_A^2 = \sigma_S^2$$

$$\frac{\sigma_S}{\sigma_A} = \sqrt{3}$$

Example 11.2

Two shafts are of the same material, length and weight. One is solid and 100 mm diameter, the other is hollow. If the hollow shaft is to store 25 % more energy than the solid shaft when transmitting torque, what must be its internal and external diameters?

Assume the same maximum shear stress applies to both shafts.

Solution:

Let A be the solid shaft and B the hollow shaft. If they are the same weight and the same material, their volume must be equal.

$$\begin{aligned} \therefore \quad \frac{\pi}{4} D_A^2 \times L &= \frac{\pi}{4} (D_B^2 - d_B^2) L \\ \therefore \quad D_A^2 &= D_B^2 - d_B^2 = (100 \times 10^{-3})^2 \text{ m}^2 = 0.01 \text{ m}^2 \\ \therefore \quad D_B^2 - d_B^2 &= 0.01 \text{ m}^2 \end{aligned} \quad (\text{E.1})$$

Now for the same maximum shear stress

$$\begin{aligned} \tau &= \frac{T r}{J} = \frac{T D}{2 J} \\ \frac{T_A D_A}{J_A} &= \frac{T_B D_B}{J_B} \\ \therefore \quad \frac{T_A}{T_B} &= \frac{D_B J_A}{D_A J_B} \end{aligned} \quad (\text{E.2})$$

But the strain energy of $B = 1.25 \times$ strain energy of A .

$$\begin{aligned} \therefore \quad \text{since } U &= \frac{T^2 L}{2 G J} \\ \frac{T_B^2 L}{2 G J_B} &= 1.25 \frac{T_A^2 L}{2 G J_A} \quad \text{or} \quad \frac{T_A^2}{T_B^2} = \frac{J_A}{1.25 J_B} \end{aligned} \quad (\text{E.3})$$

Now substitute eqn. (E.2) in (E.3),

$$\begin{aligned} \frac{D_B^2}{D_A^2} &= \frac{J_B}{1.25 J_A} \\ \frac{D_B^2}{D_A^2} &= \frac{\frac{\pi}{32} (D_B^4 - d_B^4)}{1.25 \frac{\pi}{32} D_A^4} = \frac{D_B^4 - d_B^4}{1.25 D_A^4} \end{aligned}$$

$$D_B^2 - d_B^2 = 0.01 \quad , \quad d_B^2 = D_B^2 - 0.01 \quad , \quad d_B^2 = D_B^2 - 10 \times 10^{-3}$$

$$\begin{aligned} D_B^2 &= \frac{D_B^4 - d_B^4}{1.25 D_A^2} \\ &= \frac{D_B^4 - (D_B^2 - 10 \times 10^{-3})^2}{1.25 \times (100 \times 10^{-3})^2} \end{aligned}$$

$$1.25 \times 10^{-2} D_B^2 = D_B^4 - D_B^4 + 20 \times 10^{-3} D_B^2 - 100 \times 10^{-6}$$

$$\therefore 7.5 \times 10^{-3} D_B^2 = 100 \times 10^{-6}$$

$$\therefore D_B^2 = \frac{100 \times 10^{-6}}{7.5 \times 10^{-3}} = 13.3 \times 10^{-3}$$

$$D_B = 115.47 \text{ mm}$$

$$d_B^2 = D_B^2 - D_A^2 = (13.3 - 10) \times 10^{-3}$$

$$\therefore d_B = 57.74 \text{ mm}$$

The internal and external diameters of the hollow tube are 57.7 mm and 115.5 mm respectively.

Example 11.3

Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Fig. 11.9, (b) the deflection at the center of a simply supported beam carrying a uniformly distributed load.

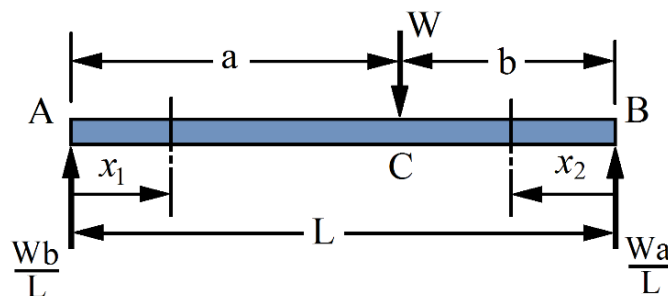


Fig. 11.9

Solution:

(a) For the beam shown in Fig. 11.9,

$$\begin{aligned}
 \delta &= \int_B^A \frac{M}{EI} \frac{\partial M}{\partial W} ds \\
 &= \int_A^C \frac{M}{EI} \frac{\partial M}{\partial W} ds + \int_C^B \frac{M}{EI} \frac{\partial M}{\partial W} ds \\
 &= \frac{1}{EI} \int_0^a \frac{Wbx_1}{L} \times \frac{bx_1}{L} \times dx_1 + \frac{1}{EI} \int_0^b \frac{Wax_2}{L} \times \frac{ax_2}{L} \times dx_2 \\
 &= \frac{Wb^2}{L^2 EI} \int_0^a x_1^2 dx_1 + \frac{Wa^2}{L^2 EI} \int_0^b x_2^2 dx_2 \\
 &= \frac{Wb^2 a^3}{3 L^2 EI} + \frac{Wa^2 b^3}{3 L^2 EI} = \frac{Wa^2 b^2}{3 L^2 EI} (a + b) = \frac{Wa^2 b^2}{3 LEI}
 \end{aligned}$$

(b) For the u.d.l. beam shown in Fig. 11.10 an imaginary load P must be introduced at mid-span; then the mid-span deflection will be

$$\begin{aligned}
 \delta &= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial W} ds = 2 \int_0^{L/2} \frac{M}{EI} \frac{\partial M}{\partial W} ds \\
 M_{xx} &= \frac{(wL + P)}{2} x - \frac{wx^2}{2} \quad \text{and} \quad \frac{\partial M}{\partial P} = \frac{x}{2} \\
 \delta &= \frac{2}{EI} \int_0^{L/2} \left[\frac{(wL + P)}{2} x - \frac{wx^2}{2} \right] \frac{x}{2} dx \\
 \delta &= \frac{1}{2EI} \int_0^{L/2} (wLx^2 - wx^3) dx \quad \text{since } P = 0 \\
 \delta &= \frac{w}{2EI} \left[\frac{Lx^3}{3} - \frac{x^4}{4} \right]_0^{L/2} \\
 \delta &= \frac{wL^4}{2EI} \left[\frac{1}{24} - \frac{1}{64} \right] = \frac{wL^4}{2EI} \left[\frac{8-3}{192} \right] = \frac{5WL^4}{384EI}
 \end{aligned}$$

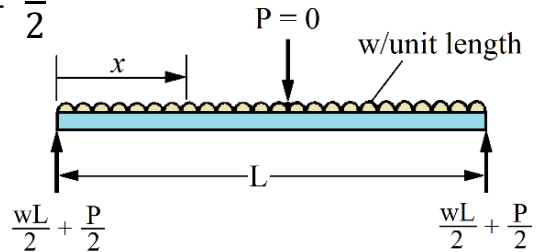
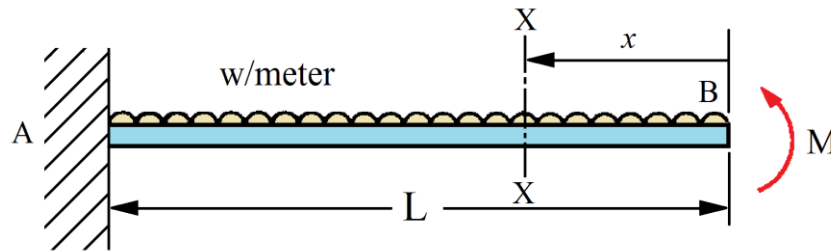


Fig. 11.10

Example 11.4

Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.

**Fig. 11.11****Solution:**

Using Castigliano's procedure, apply an imaginary moment M_i in a positive direction at point B where the slope rotation is required.

BM at XX due to applied loading and imaginary couple

$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

from Castigliano's theorem

$$\begin{aligned}\theta &= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_i} dx \\ &= \frac{1}{EI} \int_0^L \left(M_i - \frac{wx^2}{2} \right) (1) dx\end{aligned}$$

Which, with $M_i = 0$ in the absence of any applied moment at B , becomes

$$\theta = \frac{-w}{2EI} \int_0^L x^2 dx = \frac{wL^3}{6EI} \text{ radian}$$

The negative sign indicates that rotation of the free end is in the opposite direction to that taken for the imaginary moment, i.e. the beam will slope downwards at B as should have been expected.

Example 11.5

Determine, for the cranked member shown in Fig. 11.12:

- the magnitude of the force P necessary to produce a vertical movement of P of 25 mm.
- the angle, in degrees, by which the tip of the member diverges when the force P is applied.

The member has a uniform width of 50 mm throughout. $E = 200 \text{ GN/m}^2$.

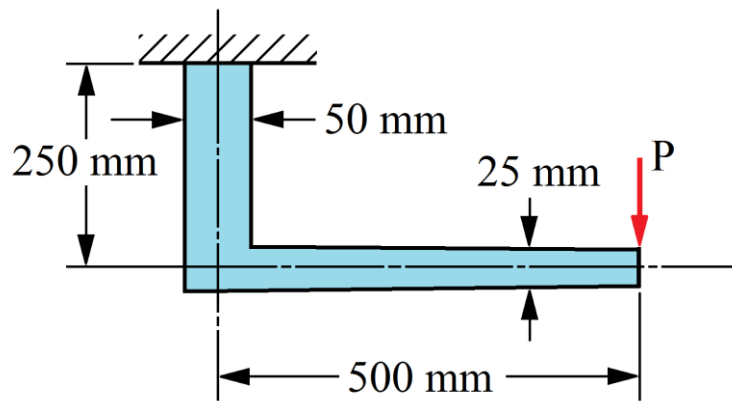


Fig. 11.12

Solution:

(a)

Horizontal beam:

$$M = Px, \quad \frac{\partial M}{\partial P} = x$$

$$\delta = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx$$

$$I_h = \frac{bh^3}{12} = \frac{(0.05)(0.025)^3}{12} = 6.51 \times 10^{-8} \text{ m}^4$$

$$\frac{1}{(EI)_h} = \frac{1}{6.51 \times 10^{-8} \times 200 \times 10^9} = 7.68 \times 10^{-5}$$

$$\delta_{\text{horizontal}} = 7.68 \times 10^{-5} \int_0^{0.5} Px^2 dx$$

$$\delta_{\text{horizontal}} = 7.68 \times 10^{-5} P \left[\frac{x^3}{3} \right]_0^{0.5} = 3.2 \times 10^{-6} P$$

Vertical beam:

$$M = 0.5 P , \quad \frac{\partial M}{\partial P} = 0.5$$

$$I_v = \frac{bh^3}{12} = \frac{(0.05)(0.05)^3}{12} = 5.208 \times 10^{-7} \text{ m}^4$$

$$\frac{1}{(EI)_v} = \frac{1}{5.208 \times 10^{-7} \times 200 \times 10^9} = 9.6 \times 10^{-6}$$

$$\delta_{vertical} = 9.6 \times 10^{-6} \int_0^{0.25} 0.5 P (0.5) dx$$

$$\delta_{vertical} = 9.6 \times 10^{-6} [0.25 Px]_0^{0.25} = 0.6 \times 10^{-6} P$$

$$\delta_{total} = \delta_{horizontal} + \delta_{vertical}$$

$$0.025 = 3.2 \times 10^{-6} P + 0.6 \times 10^{-6} P$$

$$P = 6.579 \text{ kN}$$

(b)**Horizontal beam:**

$$\theta = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_i} dx$$

$$M = Px + M_i , \quad \frac{\partial M}{\partial M_i} = 1$$

$$\theta_{horizontal} = 7.68 \times 10^{-5} \int_0^{0.5} (Px + M_i)(1) dx$$

$$\theta_{horizontal} = 7.68 \times 10^{-5} \left[\frac{Px^2}{2} + M_i x \right]_0^{0.5} = 7.68 \times 10^{-5} \frac{0.5^2 P}{2}$$

Where $M_i = 0$ and $P = 6.579 \text{ kN}$

$$\theta_{horizontal} = 0.0632 \text{ rad}$$

Vertical beam:

$$M = 0.5P + M_i, \quad \frac{\partial M}{\partial M_i} = 1$$

$$\theta_{vertical} = 9.6 \times 10^{-6} \int_0^{0.25} (0.5P + M_i)(1) dx$$

$$\theta_{vertical} = 9.6 \times 10^{-6} [0.5Px + M_i x]_0^{0.25} = 9.6 \times 10^{-6} \times (0.5)(0.25)P$$

Where $M_i = 0$ and $P = 6.579 \text{ kN}$

$$\theta_{vertical} = 0.007895 \text{ rad}$$

$$\theta_{total} = \theta_{horizontal} + \theta_{vertical}$$

$$\theta_{total} = 0.0632 + 0.007895 = 0.0711 \text{ rad} = 4.1^\circ$$