## Chapter Ten

## Thick Cylinders

### 10.1 Difference between thin and thick cylinders

The main differences between thin and thick cylinders are:

1. The radial stress is not neglected.
2. The hoop stress is not constant like thin cylinders.
3. The longitudinal is calculated more accurate.
4. The hoop and longitudinal stress are not constant.

The thick cylinders for which the variation of hoop and radial stresses is shown in Fig. 10.1, their values being given by the Lamé equations:

$$
\sigma_{H}=A+\frac{B}{r^{2}} \quad \text { and } \quad \sigma_{r}=A-\frac{B}{r^{2}}
$$

The three stresses being termed radial, hoop (tangential or circumferential) and longitudinal (axial) stresses.


Fig. 10.1 Thick cylinder subjected to internal pressure.

### 10.2 Development of the Lamé theory

Consider the thick cylinder shown in Fig. 10.2. The stresses acting on an element of unit length at radius $r$, as shown in Fig. 10.3, the radial stress increasing from $\sigma_{r}$ to $\sigma_{r}+d \sigma_{r}$ over the element thickness $d r$ (all stresses are assumed tensile).

For radial equilibrium of the element:

$$
\left(\sigma_{r}+d \sigma_{r}\right)(r+d r) d \theta \times 1-\sigma_{r} \times r d \theta \times 1=2 \sigma_{H} \times d r \times 1 \times \sin \frac{d \theta}{2}
$$



Fig. 10.2


Fig. 10.3
For small angles:

$$
\sin \frac{d \theta}{2} \approx \frac{d \theta}{2} \text { radian }
$$

Therefore, neglecting second-order small quantities,

$$
r d \sigma_{r}+\sigma_{r} d r=\sigma_{H} d r
$$

$$
\begin{array}{ll}
\therefore & \sigma_{r}+r \frac{d \sigma_{r}}{d r}=\sigma_{H} \\
\text { or } & \sigma_{\boldsymbol{H}}-\sigma_{r}=\boldsymbol{r} \frac{\boldsymbol{d} \sigma_{r}}{\boldsymbol{d r}}
\end{array}
$$

The longitudinal strain $\epsilon_{L}$ is constant across the wall of the cylinder,

$$
\begin{aligned}
\epsilon_{L} & =\frac{1}{E}\left[\sigma_{L}-v \sigma_{r}-v \sigma_{H}\right] \\
& =\frac{1}{E}\left[\sigma_{L}-v\left(\sigma_{r}+\sigma_{H}\right)\right]=\mathrm{constant}
\end{aligned}
$$

It is assumed that the longitudinal stress $\sigma_{L}$ is constant across the cylinder walls at points remote from the ends.

$$
\begin{equation*}
\therefore \quad \sigma_{r}+\sigma_{H}=\mathrm{constant}=2 A(\text { say }) \tag{10.2}
\end{equation*}
$$

Substituting in (10.1) for $\sigma_{H}$

$$
2 A-\sigma_{r}-\sigma_{r}=r \frac{d \sigma_{r}}{d r}
$$

Multiplying through by $r$ and rearranging,

$$
\begin{gathered}
2 \sigma_{r} r+r^{2} \frac{d \sigma_{r}}{d r}-2 A r=0 \\
\frac{d}{d r}\left(\sigma_{r} r^{2}-A r^{2}\right)=0
\end{gathered}
$$

Therefore, integrating,

$$
\begin{align*}
\sigma_{r} r^{2}-A r^{2} & =\text { constant }=-B(\text { say }) \\
\therefore \quad \boldsymbol{\sigma}_{r} & =\boldsymbol{A}-\frac{\boldsymbol{B}}{\boldsymbol{r}^{2}} \tag{10.3}
\end{align*}
$$

and from eqn. (10.2)

$$
\begin{equation*}
\sigma_{H}=A+\frac{B}{r^{2}} \tag{10.4}
\end{equation*}
$$

The above equations yield the radial and hoop stresses at any radius $\boldsymbol{r}$ in terms of constants $A$ and $B$.

### 10.3 Thick cylinder - internal pressure only

Consider the thick cylinder shown in Fig. 10.4 subjected to an internal pressure $P$, the external pressure being zero.


Fig. 10.4 Cylinder cross-section.

The two known conditions of stress, which enable the Lamé constants $A$ and $B$ to be determined are:

$$
\text { At } r=R_{1} \quad \sigma_{r}=-\mathrm{P} \quad \text { and } \quad \text { at } \quad r=R_{2} \quad \sigma_{r}=0
$$

The internal pressure is considered as a negative radial stress since it will produce a radial compression of the cylinder walls and the normal stress convention takes compression as negative.

Substituting the above conditions in eqn. (10.3),

$$
\begin{gathered}
-P=A-\frac{B}{R_{1}^{2}} \\
0=A-\frac{B}{R_{2}^{2}} \\
A=\frac{P R_{1}^{2}}{\left(R_{2}^{2}-R_{1}^{2}\right)} \quad \text { and } \quad B=\frac{P R_{1}^{2} R_{2}^{2}}{\left(R_{2}^{2}-R_{1}^{2}\right)}
\end{gathered}
$$

### 10.4 Longitudinal stress

Consider the cross-section of a thick cylinder with closed ends subjected to an internal pressure $P_{1}$ and an external pressure $P_{2}$ (Fig. 10.5).


Fig. 10.5 Cylinder longitudinal section.
For horizontal equilibrium:

$$
P_{1} \times \pi R_{1}^{2}-P_{2} \times \pi R_{2}^{2}=\sigma_{L} \times \pi\left(R_{2}^{2}-R_{1}^{2}\right)
$$

where $\sigma_{L}$ is the longitudinal stress set up in the cylinder walls,
$\therefore \quad$ longitudinal stress $\sigma_{L}=\frac{P_{1} R_{1}^{2}-P_{2} R_{2}^{2}}{R_{2}^{2}-R_{1}^{2}}$
This can be verified for the "internal pressure only" by substituting $P_{2}=0$ in eqn. (10.5) above.

For combined internal and external pressures, the relationship $\sigma_{L}=A$ also applies.

### 10.5 Maximum shear stress

The stresses on an element at any point in the cylinder wall are principal stresses. The maximum shear stress at any point will be given by eq. as

$$
\tau_{\max }=\frac{\sigma_{1}-\sigma_{2}}{2}
$$

Therefore, in the case of the thick cylinder, normally,

$$
\tau_{\max }=\frac{\sigma_{H}-\sigma_{r}}{2}
$$

Since $\sigma_{H}$ is normally tensile, whilst $\sigma_{r}$ is compressive and both exceed $\sigma_{L}$ in magnitude.

$$
\begin{align*}
\tau_{\max } & =\frac{1}{2}\left[\left(A+\frac{B}{r^{2}}\right)-\left(A-\frac{B}{r^{2}}\right)\right] \\
\boldsymbol{\tau}_{\max } & =\frac{\boldsymbol{B}}{\boldsymbol{r}^{2}} \tag{10.6}
\end{align*}
$$

The greatest value of $\tau_{\max }$ thus normally occurs at the inside radius where $r=R_{1}$.

### 10.6 Change of cylinder dimensions

## (a) Change of diameter

The diametral strain on a cylinder equals the hoop or circumferential strain.
Therefore, $\quad$ change of diameter $=$ diametral strain $\times$ original diameter

$$
=\text { circumferential strain } \times \text { original diameter }
$$

The change of diameter at any radius $r$ of the cylinder is given by

$$
\begin{equation*}
\Delta D=\frac{2 r}{E}\left[\sigma_{H}-v \sigma_{r}-v \sigma_{L}\right] \tag{10.7}
\end{equation*}
$$

## (b) Change of length

Similarly, the change of length of the cylinder is given by

$$
\begin{equation*}
\Delta L=\frac{L}{E}\left[\sigma_{L}-v \sigma_{r}-v \sigma_{H}\right] \tag{10.8}
\end{equation*}
$$

### 10.7 Compound thick cylinders

If this compound cylinder is subjected to internal pressure, the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage. Therefore, the cylinders are often built up by shrinking one tube on to the outside of another as drawn in Fig. 10.6.


Fig. 10.6 Compound cylinders-combined internal pressure and shrinkage effects.

## (a) Same materials

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:
(a) Shrinkage pressure only on the inside cylinder.
(b) Shrinkage pressure only on the outside cylinder.
(c) Internal pressure only on the complete cylinder, as shown in Fig. 10.7.

(a) Shrinkage - internal cylinder

(b) Shrinkage - external cylinder

Fig. 10.7 Method of solution for compound cylinders.
For each of the resulting load conditions there are two known values of radial stress which enable the Lamé constants to be determined in each case.
condition (a) shrinkage - internal cylinder:
At $r=R_{1}, \quad \sigma_{r}=0$
At $r=R_{\mathrm{c}}, \quad \sigma_{r}=-P \quad$ (compressive since it tends to reduce the wall thickness)
condition (b) shrinkage - external cylinder:
At $r=R_{2}, \quad \sigma_{r}=0$
At $r=R_{\mathrm{c}}, \quad \sigma_{r}=-P$
condition (c) internal pressure - compound cylinder:
At $r=R_{2}, \quad \sigma_{r}=0$
At $r=R_{1}, \quad \sigma_{r}=-P_{1}$
Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied.

For force and shrink fits of cylinders made of different materials, the total interference or shrinkage allowance (on radius) is

$$
\left[\epsilon_{H_{o}}-\epsilon_{H_{i}}\right] r
$$

Where, $\epsilon_{H_{o}}$ and $\epsilon_{H_{i}}$ are the hoop strains existing in the outer and inner cylinders respectively at the common radius $r$. For cylinders of the same material this equation reduces to

$$
\begin{equation*}
\text { shrinkage allowance }=\frac{r}{E}\left[\sigma_{H_{o}}-\sigma_{H_{i}}\right] \tag{10.9}
\end{equation*}
$$

## Example 10.1

A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of $60 \mathrm{MN} / \mathrm{m}^{2}$ and an external pressure of $30 \mathrm{MN} / \mathrm{m}^{2}$. Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

## Solution:



Fig. 10.8
The internal and external pressures both have the effect of decreasing the thickness of the cylinder; the radial stresses at both the inside and outside radii are thus compressive, i.e. negative (Fig. 10.8).

| $\therefore$ | at $r=0.1 \mathrm{~m}$, | $\sigma_{r}=-60 \mathrm{MN} / \mathrm{m}^{2}$ |
| :--- | :--- | :--- |
| and | at $r=0.15 \mathrm{~m}$, | $\sigma_{r}=-30 \mathrm{MN} / \mathrm{m}^{2}$ |

Therefore, from eqn. (10.3), with stress units of $\mathrm{MN} / \mathrm{m}^{2}$,
$\therefore \quad-60=A-100 B$
and
$-30=A-44.5 B$

Subtracting (2) from (1),

$$
\begin{aligned}
-30 & =-55.5 B \\
B & =0.54
\end{aligned}
$$

Therefore, from (1)

$$
\begin{aligned}
& A=-60+(100 \times 0.54) \\
& A=-6
\end{aligned}
$$

Therefore, at $r=0.1 \mathrm{~m}$, from eqn. (10.4)

$$
\begin{aligned}
\sigma_{H} & =A+\frac{B}{r^{2}}=-6+0.54 \times 100 \\
& =\mathbf{4 8} \mathbf{M N} / \mathbf{m}^{2}
\end{aligned}
$$

and at $r=0.15 \mathrm{~m}$,

$$
\begin{aligned}
\sigma_{H} & =-6+0.54 \times 44.5=-6+24 \\
& =\mathbf{1 8} \mathbf{M N} / \mathbf{m}^{2}
\end{aligned}
$$

From eqn. (10.5) the longitudinal stress is given by

$$
\begin{aligned}
\sigma_{L}=\frac{P_{1} R_{1}^{2}-P_{2} R_{2}^{2}}{R_{2}^{2}-R_{1}^{2}} & =\frac{\left(60 \times 0.1^{2}-30 \times 0.15^{2}\right)}{\left(0.15^{2}-0.1^{2}\right)} \\
& =-6 \mathbf{M N} / \mathbf{m}^{2} \quad(\text { compressive })
\end{aligned}
$$

## Example 10.2

An external pressure of $10 \mathrm{MN} / \mathrm{m}^{2}$ is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm . If the maximum hoop stress permitted on the inside wall of the cylinder is limited to $30 \mathrm{MN} / \mathrm{m}^{2}$, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied $? E=207 \mathrm{GN} / \mathrm{m}^{2}, v=0.29$.

## Solution:

The conditions for the cylinder are:
When

$$
r=0.08 \mathrm{~m}, \quad \sigma_{r}=-P \quad \text { and } \quad \frac{1}{r^{2}}=156
$$

When

$$
\begin{array}{lll}
\text { When } & r=0.16 \mathrm{~m}, & \sigma_{r}=-10 \mathrm{MN} / \mathrm{m} \\
\text { and when } & r=0.08 \mathrm{~m}, & \sigma_{H}=30 \mathrm{MN} / \mathrm{m}^{2}
\end{array}
$$

since the maximum hoop stress occurs at the inside surface of the cylinder.
Using the latter two conditions in eqns. (10.3) and (10.4) with units of $\mathrm{MN} / \mathrm{m}^{2}$,

$$
\begin{align*}
-10 & =A-39 B  \tag{1}\\
30 & =A+156 B \tag{2}
\end{align*}
$$

Subtracting (1) from (2),

$$
40=195 B \quad \therefore B=0.205
$$

Substituting in (1),

$$
\begin{aligned}
& A
\end{aligned}=-10+(39 \times 0.205)
$$

Therefore, at $r=0.08$, from eqn. (10.3),

$$
\begin{aligned}
\sigma_{r} & =-P=A-156 B \\
& =-2-(156 \times 0.205) \\
\sigma_{r} & =-\mathbf{3 4} \mathbf{M N} / \mathbf{m}^{2}
\end{aligned}
$$

The allowable internal pressure is $34 \mathrm{MN} / \mathrm{m}^{2}$.
From eqn. (10.7) the change in diameter is given by

$$
\Delta D=\frac{2 r_{o}}{E}\left(\sigma_{H}-v \sigma_{r}-v \sigma_{L}\right)
$$

Now at the outside surface

$$
\begin{aligned}
\sigma_{r}=-10 \mathrm{MN} / \mathrm{m}^{2} \text { and } \sigma_{H} & =A+\frac{B}{r^{2}} \\
& =-2+(39 \times 0.205) \\
\boldsymbol{\sigma}_{\boldsymbol{H}} & =\mathbf{6} \mathbf{M N} / \mathbf{m}^{2}
\end{aligned}
$$

$$
\begin{gathered}
\sigma_{L}=\frac{P_{1} R_{1}^{2}-P_{2} R_{2}^{2}}{R_{2}^{2}-R_{1}^{2}}=\frac{\left(34 \times 0.08^{2}-10 \times 0.16^{2}\right)}{\left(0.16^{2}-0.08^{2}\right)} \\
=-\frac{3.8}{1.92}=\mathbf{1 . 9 8} \mathbf{M N} / \mathbf{m}^{2}(\text { compressive }) \\
\therefore \quad \Delta D= \\
\frac{0.32}{207 \times 10^{9}}[6-0.29(-10)-0.29(-1.98)] 10^{6} \\
\Delta D=\frac{0.32}{207 \times 10^{9}}(6+2.9+0.575) 10^{6} \\
\Delta \boldsymbol{D}=\mathbf{1 4 . 7} \mathbf{\mu m}
\end{gathered}
$$

## Example 10.3

A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is $10 \mathrm{MN} / \mathrm{m}^{2}$. The compound tube is then subjected to an internal pressure of $80 \mathrm{MN} / \mathrm{m}^{2}$. Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

## Solution:

Shrinkage only - outer tube
At $\quad r=0.15 \mathrm{~m}, \quad \sigma_{r}=0 \quad$ and $\quad$ at $\quad r=0.125 \mathrm{~m}, \quad \sigma_{r}=-10 \mathrm{MN} / \mathrm{m}^{2}$

$$
\therefore \quad \begin{align*}
0 & =A-\frac{B}{0.15^{2}}=A-44.5 B  \tag{1}\\
-10 & =A-\frac{B}{0.125^{2}}=A-64 B \tag{2}
\end{align*}
$$

Subtracting (1)-(2), $10=19.5 B \quad \therefore \quad B=0.514$
Substituting in (1), $\quad A=44.5 B \quad \therefore A=22.85$

$$
\begin{array}{ll}
\therefore \quad \text { hoop stress at } 0.15 \mathrm{~m} \text { radius }=A+44.5 B=45.7 \mathrm{MN} / \mathrm{m}^{2} \\
& \text { hoop stress at } 0.125 \mathrm{~m} \text { radius }=A+64 B=55.75 \mathrm{MN} / \mathrm{m}^{2}
\end{array}
$$

## Shrinkage only- inner tube

At $\quad r=0.10 \mathrm{~m}, \quad \sigma_{r}=0 \quad$ and $\quad$ at $\quad r=0.125 \mathrm{~m}, \quad \sigma_{r}=-10 \mathrm{MN} / \mathrm{m}^{2}$

$$
\begin{align*}
\therefore & =A-\frac{B}{0.1^{2}}=A-100 B  \tag{3}\\
-10 & =A-\frac{B}{0.125^{2}}=A-64 B \tag{4}
\end{align*}
$$

Subtracting (3) - (4),

$$
10=-36 B
$$

$$
\therefore \quad B=-0.278
$$

Substituting in (3),

$$
A=100 B
$$

$$
\therefore \quad A=-27.8
$$

$\therefore \quad$ hoop stress at 0.125 m radius $=A+64 B=-45.6 \mathrm{MN} / \mathrm{m}^{2}$ hoop stress at 0.10 m radius $=A+100 B=-55.6 \mathrm{MN} / \mathrm{m}^{2}$

Considering internal pressure only (on complete cylinder)
At $r=0.15 \mathrm{~m}, \quad \sigma_{r}=0 \quad$ and $\quad$ at $r=0.10 \mathrm{~m}, \quad \sigma_{r}=-80 \mathrm{MN} / \mathrm{m}^{2}$

$$
\begin{array}{r}
0=A-44.5 B \\
-80=A-100 B \tag{6}
\end{array}
$$

Subtracting (5) - (6), $80=55.5 B \quad \therefore \quad B=1.44$

From (5),

$$
A=44.5 B \quad \therefore \quad A=64.2
$$

$$
\begin{aligned}
\therefore \quad \text { At } \quad r & =0.15 \mathrm{~m}, & & \sigma_{H}=A+44.5 B=128.4 \mathrm{MN} / \mathrm{m}^{2} \\
r & =0.125 \mathrm{~m}, & & \sigma_{H}=A+64 B=156.4 \mathrm{MN} / \mathrm{m}^{2} \\
r & =0.10 \mathrm{~m}, & & \sigma_{H}=A+100 B=208.2 \mathrm{MN} / \mathrm{m}^{2}
\end{aligned}
$$

The resultant stresses for combined shrinkage and internal pressure are then:
outer tube: $r=0.15 \mathrm{~m}$,

$$
\sigma_{H}=128.4+45.7=174.1 \quad \mathrm{MN} / \mathrm{m}^{2}
$$

$$
r=0.125 \mathrm{~m}, \quad \sigma_{H}=156.4+55.75=212.15 \mathrm{MN} / \mathrm{m}^{2}
$$

inner tube: $r=0.125 \mathrm{~m}$, $\sigma_{H}=156.4-45.6=110.8 \mathrm{MN} / \mathrm{m}^{2}$ $r=0.10 \mathrm{~m}$,
$\sigma_{H}=208.2-55.6=152.6 \mathrm{MN} / \mathrm{m}^{2}$


## Example 10.4

A compound tube is made by shrinking one tube of 100 mm internal diameter and 25 mm wall thickness on to another tube of 100 mm external diameter and 25 mm wall thickness. The shrinkage allowance, based on radius, is 0.01 mm . If both tubes are of steel (with $E=208 \mathrm{GN} / \mathrm{m}^{2}$ ), calculate the radial pressure set up at the junction owing to shrinkage.

## Solution:

Let $P$ be the required shrinkage pressure, then for the inner tube:
At $\quad r=0.025 \mathrm{~m}, \quad \sigma_{r}=0 \quad$ and $\quad$ at $\quad r=0.05 \mathrm{~m}, \quad \sigma_{r}=-P$

$$
\begin{align*}
0 & =A-\frac{B}{0.025^{2}}=A-1600 B  \tag{1}\\
-P & =A-\frac{B}{0.05^{2}}=A-400 B \tag{2}
\end{align*}
$$

Subtracting (2)-(1), $\quad-P=1200 B \quad \therefore \quad B=-P / 1200$
From (1),

$$
A=1600 B \quad \therefore \quad A=-\frac{1600 P}{1200}=-\frac{4 P}{3}
$$

Therefore at the common radius the hoop stress is given by eqn. (10.4),

$$
\begin{gathered}
\sigma_{H_{i}}=A+\frac{B}{0.05^{2}} \\
=-\frac{4 P}{3}+400\left(-\frac{P}{1200}\right)=-\frac{5 P}{3}=-1.67 P
\end{gathered}
$$

For the outer tube:
At $\quad r=0.05 \mathrm{~m}, \quad \sigma_{r}=-P \quad$ and $\quad$ at $\quad r=0.075 \mathrm{~m}, \quad \sigma_{r}=0$

$$
\begin{align*}
-P & =A-\frac{B}{0.05^{2}}=A-400 B  \tag{3}\\
0 & =A-\frac{B}{0.075^{2}}=A-178 B \tag{4}
\end{align*}
$$

Subtracting (4) - (3), $\quad P=222 B \quad \therefore \quad B=P / 222$

From (4),

$$
A=178 B \quad \therefore \quad A=\frac{178 P}{222}
$$

Therefore at the common radius the hoop stress is given by

$$
\begin{gathered}
\sigma_{H_{o}}=A+\frac{B}{0.05^{2}} \\
=\frac{178 P}{222}+\frac{P}{222} \times 400=\frac{578 P}{222}=2.6 P
\end{gathered}
$$

Now from eqn. (10.9) the shrinkage allowance is

$$
\begin{array}{cc} 
& \text { Shrinkage allowance }=\frac{r}{E}\left[\sigma_{H_{o}}-\sigma_{H_{i}}\right] \\
\therefore & 0.01 \times 10^{-3}=\frac{50 \times 10^{-3}}{208 \times 10^{9}}[2.6 P-(-1.67 P)] 10^{6}
\end{array}
$$

where $P$ has units of $\mathrm{MN} / \mathrm{m}^{2}$

$$
\begin{array}{ll}
\therefore & 4.27 P=\frac{0.01 \times 208 \times 10^{3}}{50}=41.6 \\
\therefore & \boldsymbol{P}=\mathbf{9 . 7 4} \mathbf{~ M N} / \mathbf{m}^{2}
\end{array}
$$

## Problems

10.1 A thick cylinder of 150 mm inside diameter and 200 mm outside diameter is subjected to an internal pressure of $15 \mathrm{MN} / \mathrm{m}^{2}$. Determine the value of the maximum hoop stress set up in the cylinder walls.
[53.4 MN/m²]
10.2 A cylinder of 100 mm internal radius and 125 mm external radius is subjected to an external pressure of $14 \mathrm{bar}\left(1.4 \mathrm{MN} / \mathrm{m}^{2}\right)$. What will be the maximum stress set up in the cylinder?
[-7.8 MN/m²]
10.3 The cylinder of Problem 10.2 is subjected to an additional internal pressure of 200 bar ( $20 \mathrm{MN} / \mathrm{m}^{2}$ ). What will be the value of the maximum stress?
[84.7 MN/m²]
10.4 A steel thick cylinder of external diameter 150 mm has two strain gauges fixed externally, one along the longitudinal axis and the other at right angles to read the hoop strain. The cylinder is subjected to an internal pressure of $75 \mathrm{MN} / \mathrm{m}^{2}$ and this causes the following strains:
(a) hoop gauge: $455 \times 10^{-6}$ tensile.
(b) longitudinal gauge: $124 \times 10^{-6}$ tensile.

Find the internal diameter of the cylinder assuming that Young's modulus for steel is $208 \mathrm{GN} / \mathrm{m}^{2}$ and Poisson's ratio is 0.283 .
10.5 A compound tube of 300 mm external and 100 mm internal diameter is formed by shrinking one cylinder on to another, the common diameter being 200 mm . If the maximum hoop tensile stress induced in the outer cylinder is $90 \mathrm{MN} / \mathrm{m}^{2}$ find the hoop stresses at the inner and outer diameters of both cylinders and show by means of a sketch how these stresses vary with the radius.

$$
\left[90,55.35 ;-92.4,57.8 \mathrm{MN} / \mathrm{m}^{2}\right]
$$

10.6 A compound thick cylinder has a bore of 100 mm diameter, a common diameter of 200 mm and an outside diameter of 300 mm . The outer tube is shrunk on to the inner tube, and the radial stress at the common surface owing to shrinkage is $30 \mathrm{MN} / \mathrm{m}^{2}$.

Find the maximum internal pressure the cylinder can receive if the maximum circumferential stress in the outer tube is limited to $110 \mathrm{MN} / \mathrm{m}^{2}$. Determine also the resulting circumferential stress at the outer radius of the inner tube. [79, - $18 \mathrm{MN} / \mathrm{m}^{2}$ ]
10.7 Working from first principles find the interference fit per metre of diameter if the radial pressure owing to this at the common surface of a compound tube is $90 \mathrm{MN} / \mathrm{m}^{2}$, the inner and outer diameters of the tube being 100 mm and 250 mm respectively and the common diameter being 200 mm . The two tubes are made of the same material, for which $E=200 \mathrm{GN} / \mathrm{m}^{2}$. If the outside diameter of the inner tube is originally 200 mm , what will be the original inside diameter of the outer tube for the above conditions to apply when compound?
[199.44 mm]
10.8 A compound cylinder is formed by shrinking a tube of 200 mm outside and 150 mm inside diameter on to one of 150 mm outside and 100 mm inside diameter. Owing to shrinkage the radial stress at the common surface is $20 \mathrm{MN} / \mathrm{m}^{2}$. If this cylinder is now subjected to an internal pressure of $100 \mathrm{MN} / \mathrm{m}^{2}$ (1000 bar), what is the magnitude and position of the maximum hoop stress? [164 MN/ $\mathrm{m}^{2}$ at inside of outer cylinder]
10.9 A thick cylinder has an internal diameter of 75 mm and an external diameter of 125 mm . The ends are closed and it cames an internal pressure of $60 \mathrm{MN} / \mathrm{m}^{2}$. Neglecting end effects, calculate the hoop stress and radial stress at radii of 37.5 mm , $40 \mathrm{~mm}, 50 \mathrm{~mm}, 60 \mathrm{~mm}$ and 62.5 mm . Plot the values on a diagram to show the variation of these stresses through the cylinder wall. What is the value of the longitudinal stress in the cylinder?
[Hoop: 128, 116, 86.5, 70.2, 67.5 MN/m². Radial: - 60, - 48.7, - 19, - 2.9, $0 \mathrm{MN} / \mathrm{m}^{2}$; 33.8 MN/m²]
10.10 A compound tube is formed by shrinking together two tubes with common radius 150 mm and thickness 25 mm . The shrinkage allowance is to be such that when an internal pressure of $30 \mathrm{MN} / \mathrm{m}^{2}$ ( 300 bar ) is applied the final maximum stress in each tube is to be the same. Determine the value of this stress. What must have been the difference in diameters of the tubes before shrinkage? $E=210 \mathrm{GN} / \mathrm{m}^{2}$.
[83.1 MN/m²; 0.025 mm ]
10.11 A steel shaft of 75 mm diameter is pressed into a steel hub of 100 mm outside diameter and 200 mm long in such a manner that under an applied torque of 6 kN m relative slip is just avoided. Find the interference fit, assuming a 75 mm common diameter, and the maximum circumferential stress in the hub. p $=0.3 . E=210 \mathrm{GN} / \mathrm{m}^{2}$

$$
\left[0.0183 \mathrm{~mm} ; 40.4 \mathrm{MN} / \mathrm{m}^{2}\right]
$$

10.12 A steel plug of 75 mm diameter is forced into a steel ring of 125 mm external diameter and 50 mm width. From a reading taken by fixing in a circumferential direction an electric resistance strain gauge on the external surface of the ring, the strain is found to be $1.49 \times 10^{-4}$. Assuming $\mu=0.2$ for the mating surfaces, find the force required to push the plug out of the ring. Also estimate the greatest hoop stress in the ring. $E=210 \mathrm{GN} / \mathrm{m}^{2}$.
[65.6 kN; $59 \mathrm{MN} / \mathrm{m}^{2}$ ]
10.13 A steel cylindrical plug of 125 mm diameter is forced into a steel sleeve of 200 mm external diameter and 100 mm long. If the greatest circumferential stress in the sleeve is $90 \mathrm{MN} / \mathrm{m}^{2}$, find the torque required to turn the sleeve, assuming $\mu=0.2$ at the mating surfaces.
[19.4 kNm]
10.14 A solid steel shaft of 0.2 m diameter has a bronze bush of 0.3 m outer diameter shrunk on to it. In order to remove the bush the whole assembly is raised in temperature uniformly. After a rise of $100^{\circ} \mathrm{C}$ the bush can just be moved along the shaft. Neglecting any effect of temperature in the axial direction, calculate the original interface pressure between the bush and the shaft.

$$
\begin{array}{llll}
\text { For steel: } & E=208 \mathrm{GN} / \mathrm{m}^{2}, & v=0.29, & \alpha=12 \times 10^{-6} \text { per }{ }^{\circ} \mathrm{C} \\
\text { For bronze: } & E=112 \mathrm{GN} / \mathrm{m}^{2}, & v=0.33, & \alpha=18 \times 10^{-6} \operatorname{per}^{\circ} \mathrm{C} .
\end{array}
$$

[20.2 MN/m²]
10.15 (a) State the Lamé equations for the hoop and radial stresses in a thick cylinder subjected to an internal pressure and show how these may be expressed in graphical form. Hence, show that the hoop stress at the outside surface of such a cylinder subjected to an internal pressure $P$ is given by

$$
\sigma_{H}=\frac{2 P R_{1}^{2}}{\left(R_{2}^{2}-R_{1}^{2}\right)}
$$

where $\mathrm{R}_{1}$ and $\mathrm{R}_{2}$ are the internal and external radii of the cylinder respectively.
(b) A steel tube is shrunk on to another steel tube to form acompound cylinder 60 mm internal diameter, 180 mm external diameter. The initial radial compressive stress at the 120 mm common diameter is $30 \mathrm{MN} / \mathrm{m}^{2}$. Calculate the shrinkage allowance. $E=200 \mathrm{GN} / \mathrm{m}^{2}$.
(c) If the compound cylinder is now subjected to an internal pressure of $25 \mathrm{MN} / \mathrm{m}^{2}$ calculate the resultant hoop stresses at the internal and external surfaces of the compound cylinder. $\quad\left[0.0768 \mathrm{~mm} ;-48.75,+54.25 \mathrm{MN} / \mathrm{m}^{2}\right]$
10.16 A bronze tube, 60 mm external diameter and 50 mm bore, fits closely inside a steel tube of external diameter 100 mm . When the assembly is at a uniform temperature of $15^{\circ} \mathrm{C}$ the bronze tube is a sliding fit inside the steel tube, that is, the two tubes are free from stress. The assembly is now heated uniformly to a temperature of $115^{\circ} \mathrm{C}$.
(a) Calculate the radial pressure induced between the mating surfaces and the thermal circumferential stresses, in magnitude and nature, induced at the inside and outside surfaces of each tube.
$\left[10.9 \mathrm{MN} / \mathrm{m}^{2} ; 23.3,12.3,-71.2,-60.3 \mathrm{MN} / \mathrm{m}^{2}\right.$ ]
(b) Sketch the radial and circumferential stress distribution across the combined wall thickness of the assembly when the temperature is $115^{\circ} \mathrm{C}$ and insert the numerical values. Use the tabulated data given below.

|  | Younge's Modulus <br> $(\mathrm{E})$ | Poisson's Ratio <br> $(v)$ | Coefficient of <br> linear expansion <br> $(\alpha)$ |
| :--- | :---: | :---: | :---: |
| Steel | $200 \mathrm{GN} / \mathrm{m}^{2}$ | 0.3 | $12 \times 10^{-6} /{ }^{\circ} \mathrm{C}$ |
| Bronze | $100 \mathrm{GN} / \mathrm{m}^{2}$ | 0.33 | $19 \times 10^{-6} /{ }^{\circ} \mathrm{C}$ |

10.17 A steel cylinder, 150 mm external diameter and 100 mm internal diameter, just fits over a brass cylinder of external diameter 100 mm and bore 50 mm . The compound cylinder is to withstand an internal pressure of such a magnitude that the pressure set up between the common junction surfaces is $30 \mathrm{MN} / \mathrm{m}^{2}$ when the internal pressure is applied. The external pressure is zero. Determine:
(a) the value of the internal pressure.
(b) the hoop stress induced in the material of both tubes at the inside and outside surfaces.

Lamé's equations for thick cylinders may be assumed without proof, and neglect any longtudinal stress and strain.

For steel, $\quad \mathrm{E}=207 \mathrm{GN} / \mathrm{m}^{2}(2.07 \mathrm{Mbar})$ and $\quad v=0.28$.
For brass, $\mathrm{E}=100 \mathrm{GN} / \mathrm{m}^{2}(1.00 \mathrm{Mbar})$ and $v=0.33$.
Sketch the hoop and radial stress distribution diagrams across the combined wall thickness, inserting the peak values.
[123 MN/m²; 125.4, 32.2 MN/m²; 78.2, 48.2 MN/m²]
10.18 Assuming the Lamé equations for stresses in a thick cylinder, show that the radial and circumferential stresses in a solid shaft owing to the application of external pressure are equal at all radii.

A solid steel shaft having a diameter of 100 mm has a steel sleeve shrunk on to it. The maximum tensile stress in the sleeve is not to exceed twice the compressive stress in the shaft. Determine
(a) the least thickness of the sleeve.
(b) the maximum tensile stress in the sleeve after shrinkage if the shrinkage allowance, based on diameter, is $0.015 \mathrm{~mm} . E=210 \mathrm{GN} / \mathrm{m}^{2}$.
[36.6 mm; 21 MN/m²]
10.19 A steel tube of internal radius 25 mm and external radius 40 mm is wound with wire of 0.75 mm diameter until the external diameter of the tube and wire is 92 mm . Find the maximum hoop stress set up within the walls of the tube if the wire is wound with a tension of $15 \mathrm{MN} / \mathrm{m}^{2}$ and an internal pressure of $30 \mathrm{MN} / \mathrm{m}^{2}$ ( 300 bar ) acts within the tube.
[49 MN/m ${ }^{2}$ ]
10.20 A thick cylinder of 100 mm internal diameter and 125 mm external diameter is wound with wire until the external diameter is increased by $30 \%$. If the initial tensile stress in the wire when being wound on the cylinder is $135 \mathrm{MN} / \mathrm{m}^{2}$, calculate the maximum stress set up in the cylinder walls.
$\left[-144.5 \mathrm{MN} / \mathrm{m}^{2}\right]$

