

## Chapter Nine

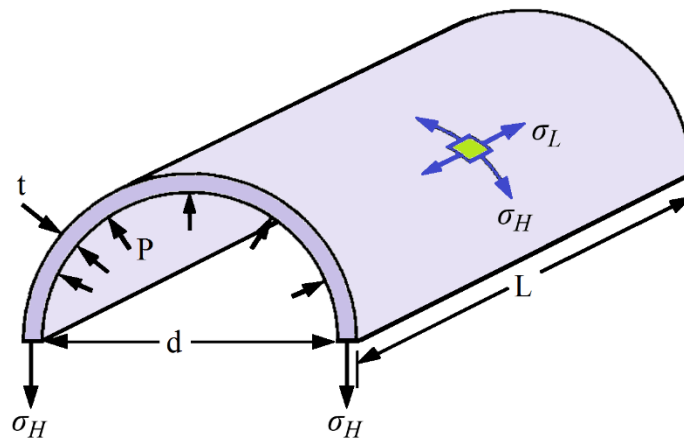
### Thin Cylinders and Shells

#### 9.1 Thin cylinders under internal pressure

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder material, namely the **circumferential** or **hoop stress**, the **radial stress** and the **longitudinal stress**. Provided the ratio of **thickness** to **inside diameter (t/d)** of the cylinder is **less than 1/20**. It is reasonable to assume the hoop and longitudinal stresses are constant across the wall thickness, and the magnitude of the radial stress set up is small in comparison with the hoop and longitudinal stresses that it can be neglected.

#### 9.2 Hoop or circumferential stress

Hoop stress is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in Fig. 9.1.



**Fig. 9.1** Half of a thin cylinder subjected to internal pressure.

Total force on half-cylinder owing to internal pressure =  $p \times \text{projected area} = p \times dL$

Total resisting force owing to hoop stress  $\sigma_H$  set up in the cylinder walls

$$= 2\sigma_H \times Lt$$

$$\therefore 2\sigma_H Lt = pdL$$

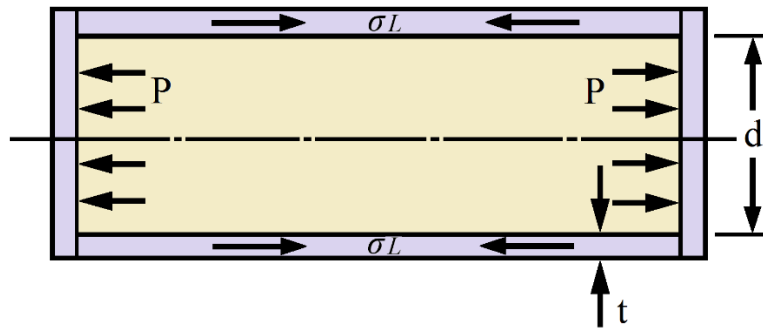
$$\therefore \text{circumferential or hoop stress } \sigma_H = \frac{pd}{2t} \quad (9.1)$$

### 9.3 Longitudinal stress

Consider the cylinder shown in Fig. 9.2.

Total force on the end of the cylinder owing to internal pressure

$$= \text{pressure} \times \text{area} = p \times \frac{\pi d^2}{4}$$



**Fig. 9.2** Cross-section of a thin cylinder.

Area of metal resisting this force =  $\pi d t$  (approximately)

$$\therefore \text{stress set up} = \frac{\text{force}}{\text{area}} = p \times \frac{\pi d^2/4}{\pi d t} = \frac{pd}{4t}$$

$$\text{longitudinal stress } \sigma_L = \frac{pd}{4t} \quad (9.2)$$

### 9.4 Changes in dimensions

#### (a) Change in length

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{longitudinal strain} = \frac{1}{E} [\sigma_L - \nu \sigma_H]$$

and

$$\text{change in length} = \text{longitudinal strain} \times \text{original length}$$

$$= \frac{1}{E} [\sigma_L - \nu \sigma_H] L$$

$$\therefore \text{change in length} = \frac{pd}{4t E} [1 - 2\nu] L \quad (9.3)$$

**(b) Change in diameter**

The change in diameter may be determined from the strain on a diameter.

$$\text{change in diameter} = \frac{pd^2}{4tE} [2 - \nu] \quad (9.4)$$

**(c) Change in internal volume (For Cylinders)**

$$\text{Change in internal volume} = \frac{pd}{4tE} [5 - 4\nu] V \quad (9.5)$$

**9.5 Thin spherical shell under internal pressure**

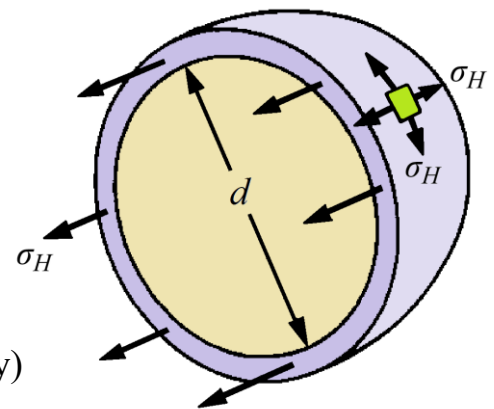
Because of the symmetry of the sphere the stresses set up to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress. The thin sphere having thickness to diameter ratios ( $t/d$ ) less than  $1/20$ . The stress system is one of equal biaxial hoop stresses.

Force on half-sphere owing to internal pressure

$$= \text{pressure} \times \text{projected area}$$

$$= p \times \frac{\pi d^2}{4}$$

$$\text{Resisting force} = \sigma_H \times \pi dt \quad (\text{approximately})$$



**Fig. 9.4** Half of a thin sphere.

$$\therefore p \times \frac{\pi d^2}{4} = \sigma_H \times \pi dt$$

$$\text{or} \quad \sigma_H = \frac{pd}{4t}$$

$$\text{circumferential or hoop stress} = \frac{pd}{4t} \quad (9.6)$$

**9.6 Change in internal volume (For spheres)**

$$\therefore \text{change in internal volume} = \frac{3pd}{4tE} [1 - \nu] V \quad (9.7)$$

### 9.7 Vessels subjected to fluid pressure

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure.

Now the **bulk modulus** of a fluid is defined as follows:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

where, in this case, volumetric stress = pressure  $p$

and  $\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V}$

$$\therefore K = \frac{p}{\delta V/V} = \frac{pV}{\delta V}$$

$$\text{change in volume of fluid under pressure} = \frac{pV}{K} \quad (9.8)$$

Extra fluid required to raise cylinder pressure by  $p$

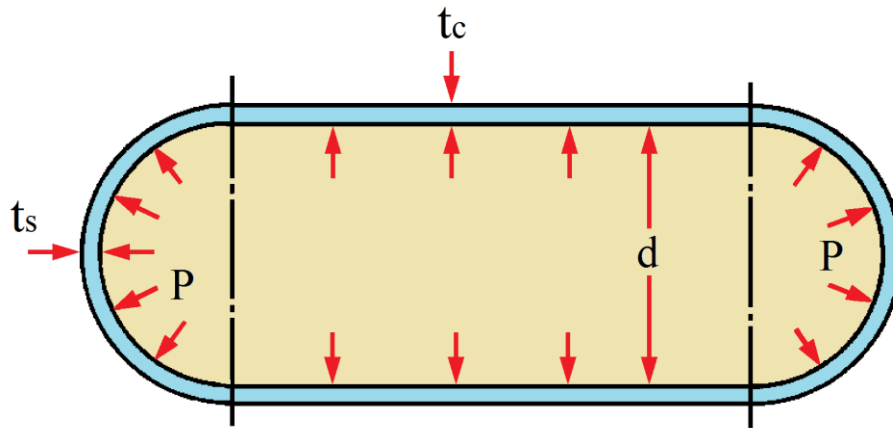
$$= \frac{pd}{4tE} [5 - 4\nu]V + \frac{PV}{K} \quad (9.9)$$

Similarly, for spheres, the extra fluid required is

$$= \frac{3pd}{4tE} [1 - \nu]V + \frac{PV}{K} \quad (9.10)$$

### 9.8 Cylindrical vessel with hemispherical ends

Consider the vessel shown in Fig. 9.5 in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness).



**Fig. 9.5** Cross-section of a thin cylinder with hemispherical ends.

#### (a) For the cylindrical portion

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\begin{aligned} \therefore \text{hoop or circumferential strain} &= \frac{1}{E} [\sigma_{H_c} - \nu \sigma_{L_c}] \\ &= \frac{pd}{4t_c E} [2 - \nu] \end{aligned}$$

#### (b) For the hemispherical ends

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\begin{aligned} \text{hoop strain} &= \frac{1}{E} [\sigma_{H_s} - \nu \sigma_{H_s}] \\ &= \frac{pd}{4t_s E} [1 - \nu] \end{aligned}$$

Thus, equating the two strains yields,

$$\frac{pd}{4t_c E} [2 - \nu] = \frac{pd}{4t_s E} [1 - \nu]$$

$$\frac{t_s}{t_c} = \frac{(1 - \nu)}{(2 - \nu)} \quad (9.11)$$

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

The thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur.

### 9.10 Effects of end plates and joints

The preceding sections have all assumed uniform material properties throughout the components and have neglected the effects of endplates and joints which are necessary requirements for their production. In general, the strength of the components will be reduced by the presence of, for example, riveted joints, and this should be taken into account by the introduction of a **joint efficiency factor  $\eta$**  into the equations previously derived.

**For thin cylinders:**

$$\text{hoop stress } \sigma_H = \frac{pd}{2t \eta_L}$$

where  $\eta_L$  is the efficiency of the **longitudinal joints**,

$$\text{longitudinal stress } \sigma_L = \frac{pd}{4t \eta_C}$$

where  $\eta_C$  is the efficiency of the **circumferential joints**.

**For thin spheres:**

$$\text{hoop stress } \sigma_H = \frac{pd}{4t \eta}$$

**Example 9.1**

A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of 7 MN/m<sup>2</sup>. Determine the change in internal diameter and the change in length. In addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder.  $E = 200 \text{ GN/m}^2$ .  $\nu = 0.3$ .

**Solution:**

$$\begin{aligned}
 \text{(a) From eqn. (9.4), change in diameter} &= \frac{pd^2}{4tE} (2 - \nu) \\
 &= \frac{7 \times 10^6 \times 75^2 \times 10^{-6}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9} (2 - 0.3) \\
 &= 33.4 \times 10^{-6} \text{ m} \\
 &= \mathbf{33.4 \text{ } \mu\text{m}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) From eqn. (9.3), change in length} &= \frac{pdL}{4tE} (1 - 2\nu) \\
 &= \frac{7 \times 10^6 \times 75 \times 10^{-3} \times 250 \times 10^{-3}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9} (1 - 0.6) \\
 &= \mathbf{26.2 \text{ } \mu\text{m}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) Hoop stress } \sigma_H &= \frac{pd}{2t} = \frac{7 \times 10^6 \times 75 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}} \\
 &= \mathbf{105 \text{ MN/m}^2} \\
 \text{longitudinal stress } \sigma_L &= \frac{pd}{4t} = \frac{7 \times 10^6 \times 75 \times 10^{-3}}{4 \times 2.5 \times 10^{-3}} \\
 &= \mathbf{52.5 \text{ MN/m}^2}
 \end{aligned}$$

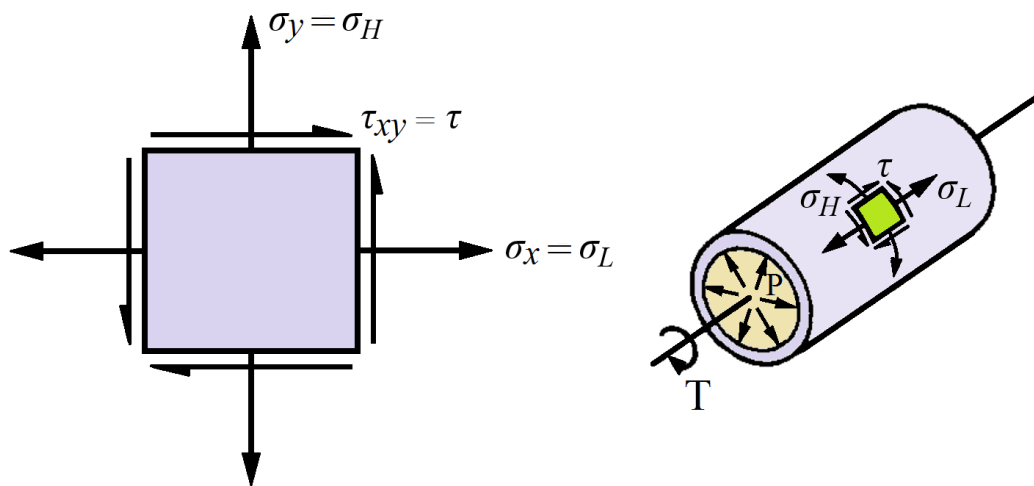
In addition to these stresses a shear stress  $\tau$  is set up.

From the torsion theory,

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \quad \tau = \frac{TR}{J}$$

Now 
$$J = \frac{\pi (80^4 - 75^4) \times 10^{-12}}{32} = 0.92 \times 10^{-6} \text{ m}^4$$

Then 
$$\text{shear stress } \tau = \frac{200 \times 40 \times 10^{-3}}{0.92 \times 10^{-6}} = \mathbf{8.695 \text{ MN/m}^2}$$



**Fig. 9.7** Enlarged view of the stresses acting on an element in the surface of a thin cylinder subjected to torque and internal pressure.

The principal stresses are given by

$$\begin{aligned} \sigma_1 \text{ and } \sigma_2 &= \frac{1}{2} (\sigma_x + \sigma_y) \pm \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4 \tau_{xy}^2} \\ &= \frac{1}{2} (105 + 52.5) \pm \frac{1}{2} \sqrt{(105 - 52.5)^2 + 4 (8.695)^2} \\ &= \frac{1}{2} \times 157.5 \pm \frac{1}{2} \sqrt{(2756.25 + 302.45)} \\ &= 78.75 \pm 27.652 \end{aligned}$$

Then 
$$\sigma_1 = 106.4 \text{ MN/m}^2 \quad \text{and} \quad \sigma_2 = 51.1 \text{ MN/m}^2$$

The principal stresses are

**106.4 MN/m<sup>2</sup> and 51.1 MN/m<sup>2</sup> both tensile.**



**Example 9.2**

A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by  $12.0 \times 10^{-6} \text{ m}^3$  when filled with a liquid at a pressure  $p$ . If  $E = 200 \text{ GN/m}^2$  and  $\nu = 0.25$ , and assuming rigid end plates, determine:

(a) the values of hoop and longitudinal stresses.

(b) the modifications to these values if joint efficiencies of 45 % (hoop) and 85 % (longitudinal) are assumed.

(c) the necessary change in pressure  $p$  to produce a further increase in internal volume of 15 %. The liquid may be assumed incompressible.

**Solution:**

(a) From eq. (9.5)

$$\text{change in internal volume} = \frac{pd}{4tE} (5 - 4\nu)V$$

$$\text{original volume } V = \frac{\pi}{4} \times 230^2 \times 10^{-6} \times 1 = 41.6 \times 10^{-3} \text{ m}^3$$

$$\text{Then} \quad \text{change in volume} = 12 \times 10^{-6} = \frac{P \times 230 \times 10^{-3} \times 41.6 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 200 \times 10^9} (5 - 1)$$

$$\begin{aligned} \text{Thus} \quad P &= \frac{12 \times 10^{-6} \times 4 \times 5 \times 10^{-3} \times 200 \times 10^9}{230 \times 10^{-3} \times 41.6 \times 10^{-3} \times 4} \\ &= 1.25 \text{ MN/m}^2 \end{aligned}$$

$$\begin{aligned} \text{Hence,} \quad \text{hoop stress} &= \frac{pd}{2t} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3}} \\ &= 28.8 \text{ MN/m}^2 \end{aligned}$$

$$\begin{aligned} \text{longitudinal stress} &= \frac{pd}{4t} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{4 \times 5 \times 10^{-3}} \\ &= 14.4 \text{ MN/m}^2 \end{aligned}$$

**(b)** Hoop stress (acting on the longitudinal joints)

$$\begin{aligned}\text{Hoop stress} &= \frac{pd}{2t \eta_L} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3} \times 0.85} \\ &= \mathbf{33.9 \text{ MN/m}^2}\end{aligned}$$

Longitudinal stress (acting on the circumferential joints)

$$\begin{aligned}&= \frac{pd}{4t \eta_c} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 0.45} \\ &= \mathbf{32 \text{ MN/m}^2}\end{aligned}$$

**(c)** Since the change in volume is directly proportional to the pressure, the necessary 15 % increase in volume is achieved by increasing the pressure also by 15 %.

$$\begin{aligned}\text{Necessary increase in } p &= 0.15 \times 1.25 \times 10^6 \\ &= \mathbf{1.86 \text{ MN/m}^2}\end{aligned}$$

### Example 9.3

(a) A sphere, 1 m internal diameter and 6 mm wall thickness, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of 3 MN/m<sup>2</sup> gauge? For water,  $K = 2.1 \text{ GN/m}^2$ .

(b) The sphere is placed in service and filled with gas until there is a volume change of  $72 \times 10^{-6} \text{ m}^3$ . Determine the pressure exerted by the gas on the walls of the sphere.

(c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure?

For the material of the sphere  $E = 200 \text{ GN/m}^2$ ,  $\nu = 0.3$  and the yield stress  $\sigma_y$  in simple tension = 280 MN/m<sup>2</sup>.

### Solution:

(a) Bulk modulus  $K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$

Now volumetric stress = pressure  $p = 3 \text{ MN/m}^2$

and volumetric strain = change in volume  $\div$  original volume

$$K = \frac{p}{\delta V/V}$$

$$\begin{aligned} \therefore \text{change in volume of water} &= \frac{pV}{K} = \frac{3 \times 10^6}{2.1 \times 10^9} \times \frac{4\pi}{3} (0.5)^3 \\ &= 0.748 \times 10^{-3} \text{ m}^3 \end{aligned}$$

(b) From eqn. (9.7) the **change in volume** is given by

$$\text{change in internal volume } \delta V = \frac{3pd}{4tE} (1 - \nu)V$$

$$\therefore 72 \times 10^{-6} = \frac{3p \times 1 \times \frac{4\pi}{3} (0.5)^3 (1 - 0.3)}{4 \times 6 \times 10^{-3} \times 200 \times 10^9}$$

$$\begin{aligned} \therefore p &= \frac{72 \times 10^{-6} \times 4 \times 6 \times 200 \times 10^6 \times 3}{3 \times 4\pi (0.5)^3 \times 0.7} \\ &= 314 \times 10^3 \text{ N/m}^2 = 314 \text{ kN/m}^2 \end{aligned}$$

(c) The maximum stress set up in the sphere will be the hoop stress,

$$\sigma_1 = \sigma_H = \frac{pd}{4t}$$

Now, according to the maximum principal stress theory, failure will occur when the maximum principal stress equals the value of the yield stress of a specimen subjected to simple tension,

when  $\sigma_1 = \sigma_y = 280 \text{ MN/m}^2$

$$280 \times 10^6 = \frac{pd}{4t}$$

$$p = \frac{280 \times 10^6 \times 4 \times 6 \times 10^{-3}}{1}$$

$$= 6.72 \times 10^6 \text{ N/m}^2 = \mathbf{6.72 \text{ MN/m}^2}$$

The sphere would therefore yield at a pressure of 6.7 MN/m<sup>2</sup>.

### Problems

**9.1** Determine the hoop and longitudinal stresses set up in a thin boiler shell of circular cross-section, 5m long and of 1.3 m internal diameter when the internal pressure reaches a value of 2.4 bar (240 kN/m<sup>2</sup>). What will then be its change in diameter? The wall thickness of the boiler is 25 mm.  $E = 210 \text{ GN/m}^2$ ,  $\nu = 0.3$ .

[6.24, 3.12 MN/m<sup>2</sup>; 0.033 mm]

**9.2** Determine the change in volume of a thin cylinder of original volume  $65.5 \times 10^{-3} \text{ m}^3$  and length 1.3 m if its wall thickness is 6 mm and the internal pressure 14 bar (1.4 MN/m<sup>2</sup>). For the cylinder material  $E = 210 \text{ GN/m}^2$ ,  $\nu = 0.3$ .

[ $17.5 \times 10^{-6} \text{ m}^3$ ]

**9.3** What must be the wall thickness of a thin spherical vessel of diameter 1 m if it is to withstand an internal pressure of 70 bar (7 MN/m<sup>2</sup>) and the hoop stresses are limited to 270 MN/m<sup>2</sup>?

[12.96 mm]

**9.4** A steel cylinder 1 m long, of 150 mm internal diameter and plate thickness 5 mm, is subjected to an internal pressure of 70 bar (7 MN/m<sup>2</sup>), the increase in volume owing to the pressure is  $16.8 \times 10^{-6} \text{ m}^3$ . Find the values of Poisson's ratio and the modulus of rigidity. Assume  $E = 210 \text{ GN/m}^2$ .

[0.299; 80.8 GN/m<sup>2</sup>]

**9.5** Define bulk modulus  $K$ , and show that the decrease in volume of a fluid under pressure  $p$  is  $pV/K$ . Hence derive a formula to find the extra fluid which must be pumped into a thin cylinder to raise its pressure by an amount  $p$ .

How much fluid is required to raise the pressure in a thin cylinder of length 3 m, internal diameter 0.7 m, and wall thickness 12 mm by 0.7 bar (70 kN/m<sup>2</sup>)?  $E = 210 \text{ GN/m}^2$  and  $\nu = 0.3$  for the material of the cylinder and  $K = 2.1 \text{ GN/m}^2$  for the fluid.

[ $5.981 \times 10^{-3} \text{ m}^3$ ]

**9.6** A spherical vessel of 1.7 m diameter is made from 12 mm thick plate, and it is to be subjected to a hydraulic test. Determine the additional volume of water which it is necessary to pump into the vessel, when the vessel is initially just filled with water, in order to raise the pressure to the proof pressure of 116 bar ( $11.6 \text{ MN/m}^2$ ).

The bulk modulus of water is  $2.9 \text{ GN/m}^2$ . For the material of the vessel,  $E = 200 \text{ GN/m}^2$ ,  $\nu = 0.3$ .  $[26.14 \times 10^{-3} \text{ m}^3]$

**9.7** A thin-walled steel cylinder is subjected to an internal fluid pressure of 21 bar ( $2.1 \text{ MN/m}^2$ ). The boiler is of 1 m inside diameter and 3 m long and has a wall thickness of 33 mm. Calculate the hoop and longitudinal stresses present in the cylinder and determine what torque may be applied to the cylinder if the principal stress is limited to  $150 \text{ MN/m}^2$ .  $[35, 17.5 \text{ MN/m}^2; 6 \text{ MNm}]$

**9.8** A thin cylinder of 300 mm internal diameter and 12 mm thickness is subjected to an internal pressure  $p$ , while the ends are subjected to an external pressure of  $1/2 p$ . Determine the value of  $p$  at which elastic failure will occur according to (a) the maximum shear stress theory, and (b) the maximum shear strain energy theory, if the limit of proportionality of the material in simple tension is  $270 \text{ MN/m}^2$ . What will be the volumetric strain at this pressure?  $E = 210 \text{ GN/m}^2$ ;  $\nu = 0.3$ .

$[21.6, 23.6 \text{ MN/m}^2, 2.289 \times 10^{-3}, 2.5 \times 10^{-3}]$

**9.9** A brass pipe has an internal diameter of 400 mm and a metal thickness of 6 mm. A single layer of high tensile wire of diameter 3 mm is wound closely round it at a tension of 500 N. Find (a) the stress in the pipe when there is no internal pressure; (b) the maximum permissible internal pressure in the pipe if the working tensile stress in the brass is  $60 \text{ MN/m}^2$ ; (c) the stress in the steel wire under condition (b). Treat the pipe as a thin cylinder and neglect longitudinal stresses and strains.  $E_s = 200 \text{ GN/m}^2$ ;  $E_B = 100 \text{ GN/m}^2$ .  $[27.8, 3.04 \text{ MN/m}^2; 104.8 \text{ MN/m}^2]$

**9.10** A cylindrical vessel of 1 m diameter and 3 m long is made of steel 12 mm thick and filled with water at 16°C. The temperature is then raised to 50°C. Find the stresses induced in the material of the vessel given that over this range of temperature water increases 0.006 per unit volume. (Bulk modulus of water = 2.9 GN/m<sup>2</sup>;  $E_{\text{steel}} = 210$  GN/m<sup>2</sup> and  $\nu = 0.3$ .) Neglect the expansion of the steel owing to temperature rise.

[663, 331.5 MN/m<sup>2</sup>]

**9.11** A 3 m long aluminium-alloy tube, of 150 mm outside diameter and 5 mm wall thickness, is closely wound with a single layer of 2.5 mm diameter steel wire at a tension of 400 N. It is then subjected to an internal pressure of 70 bar (7 MN/m<sup>2</sup>).

(a) Find the stress in the tube before the pressure is applied.

(b) Find the final stress in the tube.

$E_A = 70$  GN/m<sup>2</sup>;  $\nu_A = 0.28$ ;  $E_s = 200$  GN/m<sup>2</sup>

[ - 32, 20.5 MN/m<sup>2</sup>]

**9.12** (a) Derive the equations for the circumferential and longitudinal stresses in a thin cylindrical shell.

(b) A thin cylinder of 300 mm internal diameter, 3 m long and made from 3 mm thick metal, has its ends blanked off. Working from first principles, except that you may use the equations derived above, find the change in capacity of this cylinder when an internal fluid pressure of 20 bar is applied.  $E = 200$  GN/m<sup>2</sup>;  $\nu = 0.3$ . [201 × 10<sup>-6</sup> m<sup>3</sup>]

**9.13** Show that the tensile hoop stress set up in a thin rotating ring or cylinder is given by:

$$\sigma_H = \rho \omega^2 r^2$$

Hence determine the maximum angular velocity at which the disc can be rotated if the hoop stress is limited to 20 MN/m<sup>2</sup>. The ring has a mean diameter of 260 mm.

[3800 rev/min]