

Chapter Seven

Torsion Theory

7.1 Simple torsion theory

When a uniform circular shaft is subjected to a torque, it can be shown that every section of the shaft is subjected to a state of pure shear (Fig. 7.1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary to make the following basic assumptions:

- (1) The material is homogeneous, i.e. of uniform elastic properties throughout.
- (2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.
- (3) The stress does not exceed the elastic limit or limit of proportionality.
- (4) Circular Sections remain circular.
- (5) Cross-sections remain plane. (This is certainly not the case with the torsion of non-circular sections.)
- (6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.

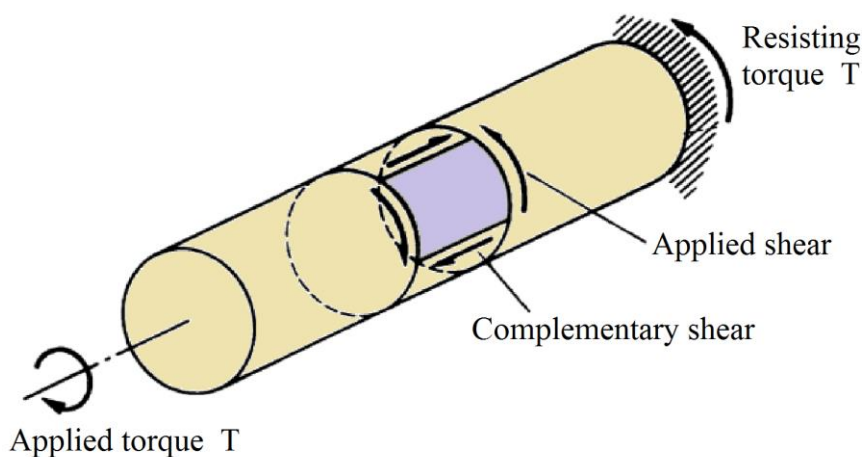


Fig. 7.1 Shear system set up on an element in the surface of a shaft subjected to torsion.

(a) Angle of twist

Consider the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed (Fig. 7.2). Under the action of this torque a radial line at the free end of the shaft twists through an angle θ , point A moves to B , and AB subtends an angle γ at the fixed end. This is then the angle of distortion of the shaft, i.e. **the shear strain**. Since

$$\text{angle in radians} = \text{arc} \div \text{radius}$$

$$\text{arc } AB = R \theta = L \gamma$$

$$\therefore \gamma = R \theta / L \quad (7.1)$$

From the definition of shear or rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

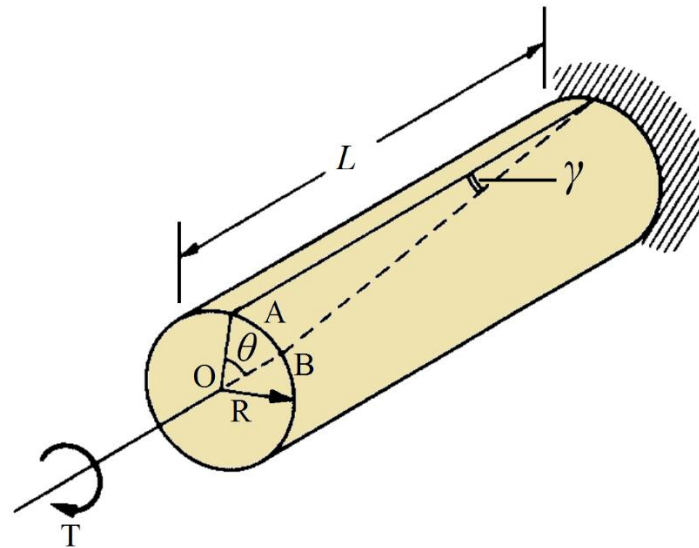


Fig. 7.2.

$$\therefore \gamma = \frac{\tau}{G} \quad (7.2)$$

where τ is the shear stress set up at radius R .

Therefore equating eqns. (7.1) and (7.2),

$$\begin{aligned} \frac{R \theta}{L} &= \frac{\tau}{G} \\ \frac{\tau}{R} &= \frac{G \theta}{L} = \frac{\tau'}{r} \end{aligned} \quad (7.3)$$

where τ' is the shear stress at any other radius r .

(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness dr as shown in Fig. 7.3 each subjected to a shear stress τ' .

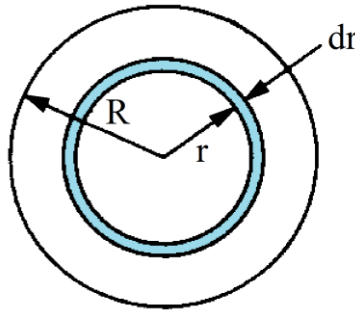


Fig. 7.3 Shaft cross-section.

The force set up on each element

$$\begin{aligned} &= \text{shear stress} \times \text{area} \\ &= \tau' \times 2 \pi r dr \text{ (approximately)} \end{aligned}$$

This force will produce a moment about the center axis of the shaft,

$$\begin{aligned} \text{Moment} &= \text{force} \times r \\ &= (\tau' \times 2 \pi r dr) \times r \\ &= 2 \pi \tau' r^2 dr \end{aligned}$$

The total torque on the section T ,

$$T = \int_0^R 2 \pi \tau' r^2 dr$$

Now the shear stress τ' will vary with the radius r , therefore must be replaced in terms of r before the integral is evaluated.

From eq. (7.3)

$$\begin{aligned} \tau' &= \frac{G \theta}{L} r \\ T &= \int_0^R 2 \pi \frac{G \theta}{L} r^3 dr \\ &= \frac{G \theta}{L} \int_0^R 2 \pi r^3 dr \end{aligned}$$

The integral $\int_0^R 2 \pi r^3 dr$ is called the **polar second moment of area J** , and may be evaluated as a standard form for solid and hollow shafts as shown below.

$$\begin{aligned} \therefore T &= \frac{G \theta}{L} J \\ \text{or } \frac{T}{J} &= \frac{G \theta}{L} \end{aligned} \quad (7.4)$$

Combining equations (7.3) and (7.4) produces the simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G \theta}{L} \quad (7.5)$$

7.2 Shear stress and shear strain in shafts

The shear stresses, which are developed in a shaft subjected to pure torsion, are given by the simple torsion theory as follow:

$$\tau = \frac{G \theta}{L} R$$

Now from the definition of the shear or rigidity modulus G ,

$$\tau = G \gamma$$

The two equations combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G \theta}{L} R = G \gamma \quad (7.6)$$

or, in terms of internal radius r ,

$$\tau' = \frac{G \theta}{L} r = G \gamma \quad (7.7)$$

7.3 Section modulus

It is convenient to re-write the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\begin{aligned} \frac{T}{J} &= \frac{\tau}{R} \\ \therefore \tau &= \frac{TR}{J} \end{aligned}$$

The above equation yields the greatest value possible of the shaft for τ ,

$$\tau_{max} = \frac{TR}{J}$$

$$\therefore \tau_{max} = \frac{T}{Z} \quad (7.8)$$

where Z is the *polar section modulus*.

For solid shafts,

$$Z = \frac{\pi D^3}{16} \quad (7.9)$$

and for hollow shafts,

$$Z = \frac{\pi (D^4 - d^4)}{16 D} \quad (7.10)$$

7.4 Series connection of the composite shafts

If two or more shafts of different material are connected together, each carries the same torque, then the shafts are said to be connected in series

$$T_1 = T_2 = T \quad T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \quad (7.11)$$

In some applications, the angles of twist in each shaft are equal.

$$\theta_1 = \theta_2$$

For similar materials in each shaft, $G_1 = G_2$.

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

or

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \quad (7.12)$$

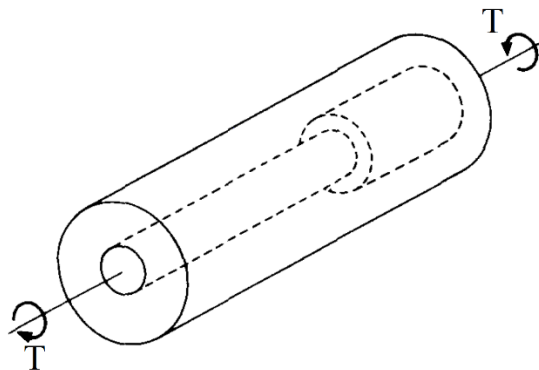


Fig. 7.4 "Series-connected shaft" common torque.

7.5 Parallel connection of the composite shafts

If two or more materials are rigidly fixed together, the applied torque is shared between them, then the composite shaft is said to be connected in parallel.

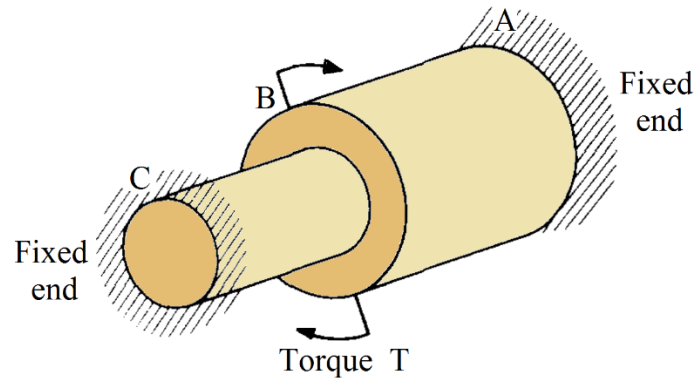


Fig. 7.5 "Parallel connected shaft" shared torque.

For parallel connection, the torque is sum of both values

$$\text{Total torque } T = T_1 + T_2 \quad (7.13)$$

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad (7.14)$$

7.6 Torsional rigidity

The quantity GJ is termed the **torsional rigidity** of the shaft and is given by

$$GJ = \frac{T}{\theta/L} \quad (7.15)$$

7.7 Variation of data along shaft length-torsion of tapered shafts

Consider, the tapered shaft shown in Fig. 7.6 with its diameter changing linearly from d_A to d_B over a length L .

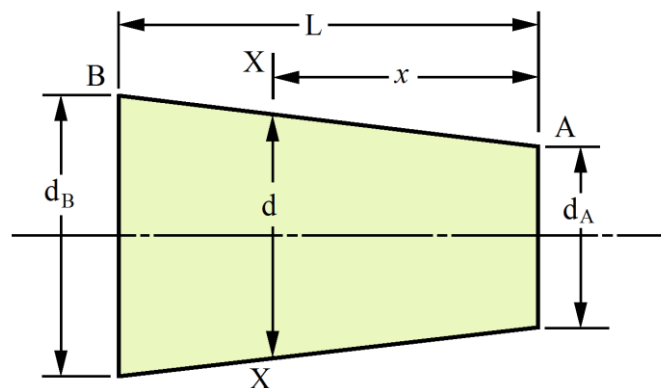


Fig. 7.6 Torsion of a tapered shaft.

The diameter at any section x from end A is given by

$$d = d_A + (d_B - d_A) \frac{x}{L}$$

The simple torsion theory applied to an element at section XX in order to determine the angle of twist of the shaft,

$$\frac{G d\theta}{dx} = \frac{T}{J_{XX}}$$

Therefore, the total angle of twist of the shaft is given by

$$\theta = \int_0^L \frac{T}{G J_{XX}} dx$$

$$J_{XX} = \frac{\pi d^4}{32} = \frac{\pi}{32} \left[d_A + (d_B - d_A) \frac{x}{L} \right]^4$$

Substituting and integrating,

$$\theta = \frac{32 T L}{3 \pi G} \left[\frac{1}{d_A^3} - \frac{1}{d_B^3} \right] \left[\frac{1}{d_B} - \frac{1}{d_A} \right] = \frac{32 T L}{3 \pi G} \left[\frac{d_A^2 + d_A d_B + d_B^2}{d_A^3 d_B^3} \right]$$

When $d_A = d_B = d$ this reduces to $\theta = \frac{32 T L}{\pi G d^4}$ the standard result for a parallel shaft.

7. 8 Power transmitted by shafts

If a shaft carries a torque T Newton meters and rotates at ω rad/s it will do work at the rate of

$$T\omega \text{ (Nm/s) (or joule/s)}$$

Now the rate at which a system works is defined as its power, the basic unit of power being the Watt (1 Watt = 1 Nm/s).

Thus, the power transmitted by the shaft:

$$\text{power} = T\omega \text{ (Watts)}$$

Also measured in kilowatts (kW) or megawatts (MW).

Example 7.1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per meter of the shaft length if $G = 80 \text{ GN/m}^2$.

(b) If the shaft were bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60 mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution:

(a) From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \tau_{max} = \frac{TR_{max}}{J}$$

$$\text{Power} = T\omega \quad \therefore \text{torque } T = \frac{\text{power}}{\omega}$$

$$\therefore T = \frac{75 \times 10^3}{150 \times 2\pi / 60} = 4.77 \text{ kN m}$$

and $J = \frac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6} \text{ m}^4$

$$\therefore \tau_{max} = \frac{TR_{max}}{J} = \frac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = \mathbf{24.3 \text{ MN/m}^2}$$

Also from the torsion theory

$$\begin{aligned} \theta &= \frac{TL}{GJ} = \frac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m} \\ &= 6.07 \times 10^{-3} \times \frac{360}{2\pi} = \mathbf{0.348 \text{ degrees/m}} \end{aligned}$$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}(100^4 - 60^4)10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$$

The torque carried by the modified shaft is then given by

$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = \mathbf{4.15 \times 10^3 \text{ Nm}}$$

Now, weight/meter of original shaft

$$W = \rho g V = \rho g AL = \frac{\pi}{4} (100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3} \rho g$$

where ρ is the density of the shaft material.

$$\begin{aligned} \text{Also, weight/meter of modified shaft} &= \frac{\pi}{4} (100^2 - 60^2) 10^{-6} \times 1 \times \rho g \\ &= 5.027 \times 10^{-3} \rho g \end{aligned}$$

$$\text{Power/weight ratio for original shaft} = \frac{T \omega}{\text{weight/meter}}$$

$$= \frac{4.77 \times 10^3 \omega}{7.854 \times 10^{-3} \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3 \omega}{5.027 \times 10^{-3} \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = \mathbf{36 \%}$$

Example 7.2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/m. The maximum shear stress in the shaft is limited to 70 MN/m² and the angle of twist to 3.8° in a length of 4 m. For the shaft material $G = 80 \text{ GN/m}^2$.

Solution:

Maximum shear stress condition

$$\text{Since Power} = T \omega \quad \text{and} \quad \omega = 200 \times \frac{2\pi}{60} = 20.94 \text{ rad/s}$$

$$\text{then} \quad T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3 \text{ Nm}$$

From the torsion theory

$$J = \frac{TR}{\tau}$$

$$\therefore \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

But $d/D = 0.75$

$$\therefore \frac{\pi}{32}D^4 (1 - 0.75^4) = 20.43 \times 10^{-6} D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$\therefore D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

and $d = 50.5 \text{ mm}$

Angle of twist condition

Again from the torsion theory

$$J = \frac{TL}{G\theta}$$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2 \pi}$$

$$\frac{\pi}{32}D^4 (1 - 0.75^4) = 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$\therefore D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

and $d = 56.5 \text{ mm}$

Thus the dimensions required for the shaft to satisfy both conditions are **outer diameter 75.3 mm and inner diameter 56.5 mm.**

Example 7.3

- (a)** A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m².
- (b)** If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

Solution:

(a) The shafts in series since each part is subjected to the same torque.

From the torsion theory

$$T = \frac{\tau J}{R}$$

The maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts, the torque allowable for a known value of shear stress is dependent only on the value of J .

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\therefore \text{least value of } J = \frac{\pi}{32}(50^4 - 38^4)10^{-12} = 0.41 \times 10^{-6} \text{ m}^4$$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m² (at 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3 \text{ Nm}$$

$$\begin{aligned} \text{Maximum power} &= T \omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60} \\ &= 25.2 \times 10^3 = \mathbf{25.2 \text{ kW}} \end{aligned}$$

(b) Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, eq. (7.12) applies,

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\therefore \frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32}(50^4 - 25^4)10^{-12}}{\frac{\pi}{32}(50^4 - 38^4)10^{-12}} = 1.43$$

$$\therefore L_2 = 1.43 L_1$$

$$\text{But } L_1 + L_2 = 510 \text{ mm}$$

$$\therefore L_1(1 + 1.43) = 510$$

$$L_1 = \frac{510}{2.43} = \mathbf{210 \text{ mm}} \quad L_2 = 510 - 210 = \mathbf{300 \text{ mm}}$$

Example 7.4

A circular bar ABC , 3 m long, is rigidly fixed at its ends A and C . The portion AB is 1.8 m long and of 50 mm diameter and BC is 1.2 m long and of 25 mm diameter. If a twisting moment of 680 Nm is applied at B , determine the values of the resisting moments at A and C and the maximum stress in each section of the shaft. What will be the angle of twist of each portion ? For the material of the shaft $G = 80 \text{ GN/m}^2$.

Solution:

In this case the two portions of the shaft are in parallel and the applied torque is shared between them. Let suffix 1 refer to portion AB and suffix 2 to portion BC . Since the angles of twist in each portion are equal and G is common to both sections,

$$\frac{T_1 L_1}{J_1} = \frac{T_2 L_2}{J_2}$$

$$\begin{aligned} T_1 &= \frac{J_1}{J_2} \times \frac{L_2}{L_1} \times T_2 = \frac{\frac{\pi}{32} \times 50^4}{\frac{\pi}{32} \times 25^4} \times \frac{1.2}{1.8} \times T_2 \\ &= \frac{16 \times 1.2}{1.8} T_2 = 10.67 T_2 \end{aligned}$$

$$\text{Total torque} = T_1 + T_2 = T_2 (10.67 + 1) = 680$$

$$\therefore T_2 = \frac{680}{11.67} = \mathbf{58.3 \text{ Nm}}$$

$$\text{and } T_1 = \mathbf{621.7 \text{ Nm}}$$

For portion AB,

$$\tau_{max} = \frac{T_2 R_2}{J_2} = \frac{58.3 \times 12.5 \times 10^{-3}}{\frac{\pi}{32} \times 25^4 \times 10^{-12}} = 19.0 \times 10^6 \text{ N/m}^2$$

$$\text{Angle of twist for each portion} = \frac{T_1 L_1}{J_1 G}$$

$$\theta = \frac{621.7 \times 1.8}{\frac{\pi}{32} \times 50^4 \times 10^{-12} \times 80 \times 10^9} = 0.0228 \text{ rad} \times \frac{360}{2\pi} = 1.3 \text{ degrees}$$

Problems

7.1 A solid steel shaft A of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m² in the steel.

It is proposed to replace A by a hollow shaft B, of the same external diameter but with a limiting shearing stress of 75 MN/m². Determine the internal diameter of B to transmit the same power at the same speed. [38.6 kW, 33.4 mm]

7.2 Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20 % and the greatest intensity of shear stress is limited to 75 MN/m². The internal diameter of the shaft is to be 80 % of the external diameter. (The mean torque is that derived from the horsepower equation.) [135.2, 108.2 mm]

7.3 A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the shaft are:

(a) the shaft must not twist more than 0.02 radian on a length of 10 diameters.

(b) the working stress must not exceed 60 MN/m².

If the modulus of rigidity of steel is 80 GN/m² what is

(i) the diameter of the shaft required.

(ii) the actual working stress.

(iii) the angle of twist of the 3 m length. [150 mm; 60 MN/m²; 0.030 rad]

7.4 A hollow shaft has to transmit 6 MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m² nor the angle of twist 0.3° per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. $G = 80 \text{ GN/m}^2$. [61.5 mm]

7.5 A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m² and 55 MN/m² respectively, and the maximum torque is 1.4 times the mean torque, calculate:

(a) the diameter of the shaft.

(b) the diameter of the bolts. [67.2, 13.8 mm]

7.6 A hollow low carbon steel shaft is subjected to a torque of 0.25 MN m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m² determine the required diameters and the angle of twist in degrees per metre length of shaft. $G = 80 \text{ GN/m}^2$.

[264, 88 mm; 0.38°]

7.8 Opposing axial torques are applied at the ends of a straight bar *ABCD*. Each of the parts *AB*, *BC* and *CD* is 500 mm long and has a hollow circular cross-section, the inside and outside diameters being respectively, *AB* 25 mm and 60 mm, *BC* 25 mm and 70 mm, *CD* 40 mm and 70 mm. The modulus of rigidity of the material is 80 GN/m² throughout. Calculate:

(a) the maximum torque which can be applied if the maximum shear stress is not to exceed 75 MN/m².

(b) the maximum torque if the twist of *D* relative to *A* is not to exceed 2°.

[3.085 kN m, 3.25 kN m]

7.9 A solid steel shaft of 200 mm diameter transmits 5 MW at 500 rev/min. It is proposed to alter the horsepower to 7 MW and the speed to 440 rev/min and to replace the solid shaft by a hollow shaft made of the same type of steel but having only 80 % of the weight of the solid shaft. The length of both shafts is the same and the hollow shaft is to have the same maximum shear stress as the solid shaft. Find

(a) the ratio between the torque per unit angle of twist per metre for the two shafts.

(b) the external and internal diameters for the hollow shaft.

[2.085; 261, 190 mm]

7.10 A shaft *ABC* rotates at 600 rev/min and is driven through a coupling at the end *A*. At *B* a pulley takes off two-thirds of the power, the remainder being absorbed at *C*. The part *AB* is 1.3 m long and of 100 mm diameter, *BC* is 1.7 m long and of 75 mm diameter. The maximum shear stress set up in *BC* is 40 MN/m². Determine the maximum stress in *AB* and the power transmitted by it, and calculate the total angle of twist in the length *AC*. Take $G = 80 \text{ GN/m}^2$. [16.9 MN/m²; 208 kW; 1.61°]

7.11 A composite shaft consists of a steel rod of 75 mm diameter surrounded by a closely fitting brass tube firmly fixed to it. Find the outside diameter of the tube such that when a torque is applied to the composite shaft it, will be shared equally by the two materials. $G_S = 80 \text{ GN/m}^2$; $G_B = 40 \text{ GN/m}^2$.

If the torque is 16 kN m, calculate the maximum shearing stress in each material and the angle of twist on a length of 4 m. [98.7 mm; 96.6, 63.5 MN/m²; 7.38°]

7.12 A circular bar 4 m long with an external radius of 25 mm is solid over half its length and bored to an internal radius of 12 mm over the other half. If a torque of 120 Nm is applied at the center of the shaft, the two ends being fixed, determine the maximum shear stress set up in the surface of the shaft and the work done by the torque in producing this stress. [2.51 MN/m²; 0.151 Nm]

7.13 The shaft of Problem 7.12 is now fixed at one end only and the torque applied at the free end. How will the values of maximum shear stress and work done change ?

[5.16 MN/m²; 0.603 Nm]