Chapter Six

Slope and Deflection of Beams

6.1 Relationship between loading, S.F., B.M., slope and deflection

Consider a beam *AB*, which is initially horizontal when unloaded. If this deflects to a new position *A′ B′* under load, the slope at any point *C* is
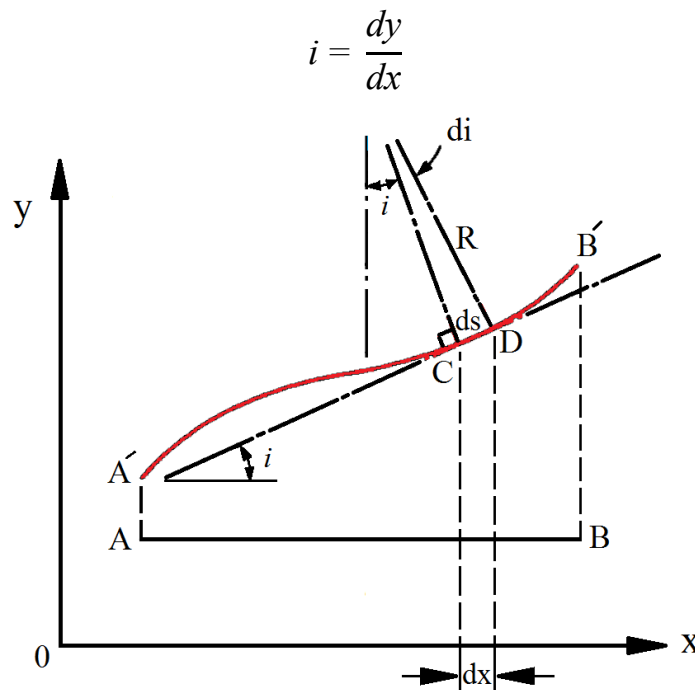
$$i = \frac{dy}{dx}$$

Fig. 6.1 Unloaded beam *AB* deflected to *A′ B′* under load.

This is usually very small in practice, and for small curvatures (Fig. 6.1)

$$ds = dx = Rdi$$

$$\frac{di}{dx} = \frac{1}{R}$$

$$i = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{1}{R} \tag{6.1}$$

Now from the simple bending theory

$$\frac{M}{I} = \frac{E}{R}$$

$$\frac{1}{R} = \frac{M}{EI}$$

Therefore, substituting in eq. (6.1)

$$M = EI \, \frac{d^2y}{dx^2} \tag{6.2}$$

The adopted sign convention illustrated in Fig. 6.2.

(a)   Deflection $y = \delta$ positive upwards

(b)   Slope
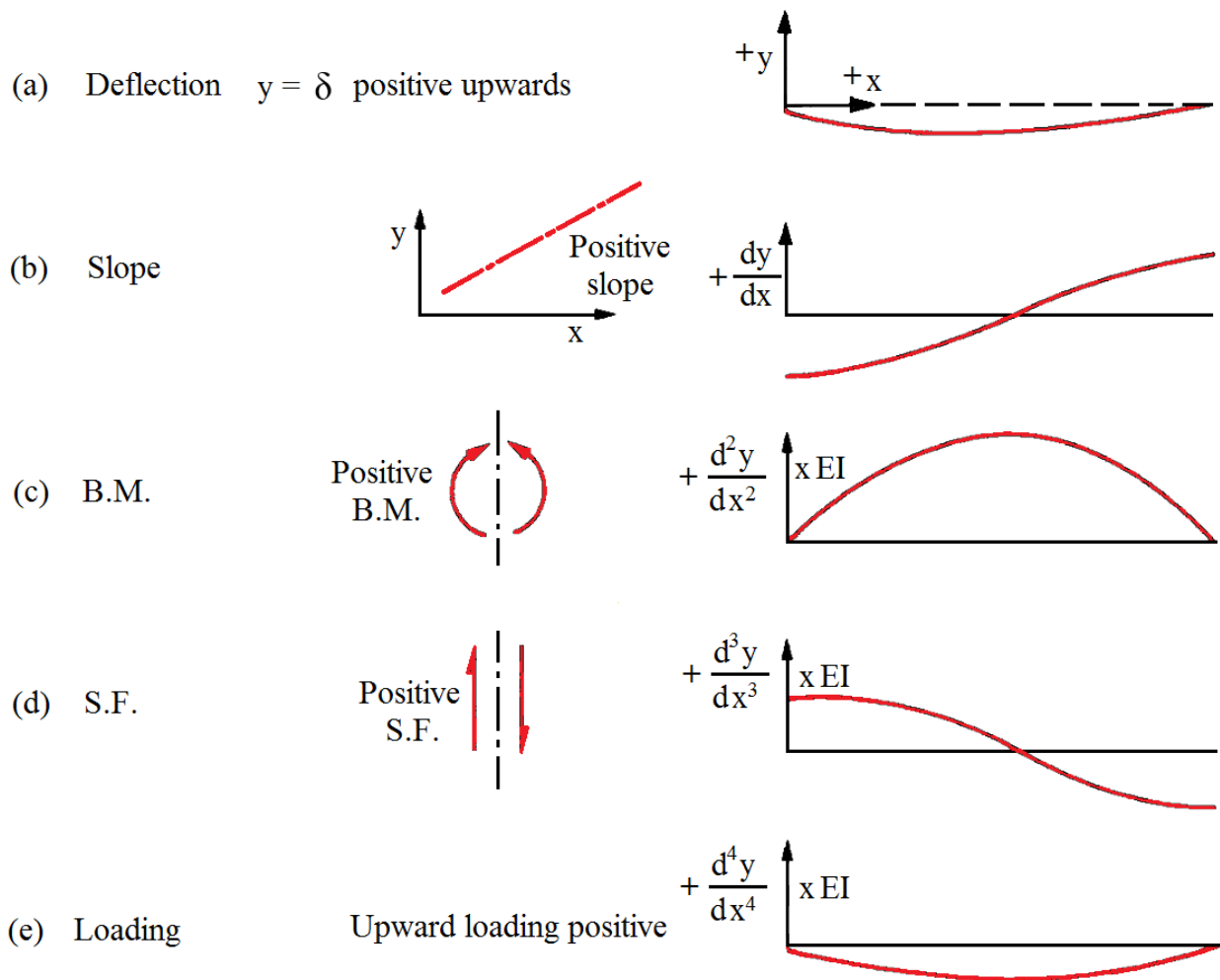
(c)   B.M.

(d)   S.F.

(e)   Loading

Fig. 6.2 Sign conventions for load, S.F., B.M., slope and deflection.

6.2 Direct integration method

If the value of the *B.M.* at any point on a beam is known in terms of *x* is provided the equation applies along the complete beam, then integration of equation will yield slopes and deflections at any point,

$$M = EI \frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI} \, dx + A$$

$$y = \iint \left(\frac{M}{EI} dx \right) dx + Ax + B \tag{6.4}$$

where *A* and *B* are constants of integration evaluated from known conditions of slope and deflection for particular values of *x*.

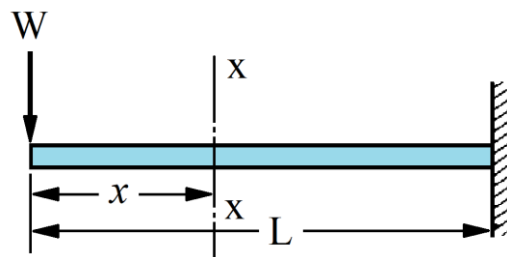### (a) Cantilever with concentrated load at the end

Fig. 6.3.

$$M_{xx} = EI \frac{d^2y}{dx^2} = - \, W x$$

$$EI \frac{dy}{dx} = - \, \frac{Wx^2}{2} + A$$

assuming *EI* is constant.

$$EI \, y = - \, \frac{Wx^3}{6} + Ax + B$$

Now when $x = L, \quad \dfrac{dy}{dx} = 0 \quad \therefore \quad A = \dfrac{WL^2}{2}$

and when $x = L, \quad y = 0 \quad \therefore \quad B = \dfrac{WL^3}{6} - \dfrac{WL^2}{2}L = \dfrac{-WL^3}{3}$

$$y = \frac{1}{EI} \left[- \frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3} \right] \tag{6.5}$$

This equation gives the deflection at all values of x and produces a maximum value at the tip of the cantilever, therefore to find a maximum deflection substitute $x = 0$,

$$\textbf{Maximum deflection} = y_{max} = -\frac{WL^3}{3\,EI} \tag{6.6}$$

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

Similarly
$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{W x^2}{2} + \frac{W L^2}{2}\right] \tag{6.7}$$

and produces a maximum value again when $x = 0$.

$$\textbf{Maximum slope} = \left(\frac{dy}{dx}\right)_{max} = \frac{WL^2}{2\,EI} \qquad \textbf{(positive)} \tag{6.8}$$

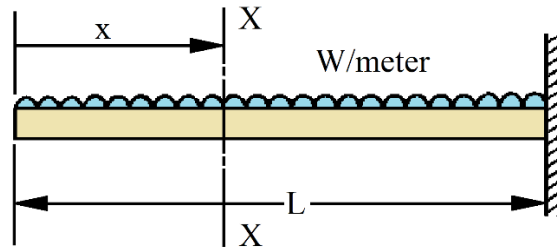**(b) Cantilever with uniformly distributed load**

Fig. 6.4.

$$M_{xx} = EI\,\frac{d^2 y}{dx^2} = -\frac{w x^2}{2}$$
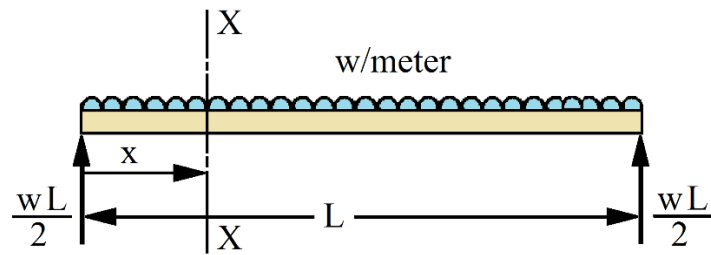
$$EI\,\frac{dy}{dx} = -\frac{w x^3}{6} + A$$

$$EI\,y = -\frac{w x^4}{24} + Ax + B$$

Again, when $\quad x = L, \quad \dfrac{dy}{dx} = 0 \quad$ and $\quad A = \dfrac{w L^3}{6}$

and when $\quad x = L, \quad y = 0 \quad$ and $\quad B = \dfrac{w L^4}{24} - \dfrac{w L^4}{6}L = -\dfrac{w L^4}{8}$

$$\therefore \qquad y = \frac{1}{EI}\left[-\frac{w x^4}{24} + \frac{w L^3 x}{6} - \frac{w L^4}{8}\right] \tag{6.9}$$

At $x = 0$, $\qquad y_{max} = -\dfrac{w L^4}{8\,EI} \quad$ and $\quad \left(\dfrac{dy}{dx}\right)_{max} = \dfrac{w L^3}{6\,EI} \tag{6.10}$

(c) Simply-supported beam with uniformly distributed load

Fig. 6.5.

$$M_{xx} = EI \frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI \frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$$

$$EI\, y = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$$

At $\qquad x = 0, \quad y = 0 \qquad \therefore \qquad B = 0$

At $\qquad x = L, \quad y = 0 \qquad \therefore \qquad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL$
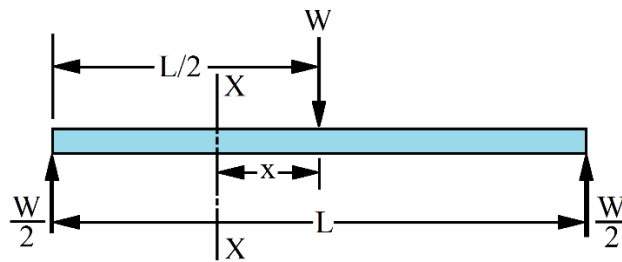
$\therefore \qquad\qquad A = -\dfrac{wL^3}{24}$

$\therefore \qquad\qquad y = \dfrac{1}{EI}\left[\dfrac{wLx^3}{12} - \dfrac{wx^4}{24} - \dfrac{wL^3x}{24}\right]$ $\qquad\qquad$ (6.11)

In this case the maximum deflection will occur at the center of the beam where $x = L/2$.

$\therefore \qquad y_{max} = \dfrac{1}{EI}\left[\dfrac{wL}{12}\left(\dfrac{L^3}{8}\right) - \dfrac{w}{24}\left(\dfrac{L^4}{16}\right) - \dfrac{wL^3}{24}\left(\dfrac{L}{2}\right)\right]$

$$y_{max} = -\frac{5\,wL^4}{384\,EI} \qquad\qquad (6.12)$$

Similarly $\qquad \left(\dfrac{dy}{dx}\right)_{max} = \pm\, \dfrac{wL^3}{24\,EI}$ $\quad$ at the ends of the beam. $\qquad$ (6.13)

(d) Simply supported beam with central concentrated load

Fig. 6.6.

In order to obtain a single expression for *B.M.* which will apply across the complete beam in this case it is convenient to take the origin for *x* at the center, then:

$$M_{xx} = EI \frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) = \frac{WL}{4} - \frac{Wx}{2}$$

$$EI \frac{dy}{dx} = \frac{WL}{4}x - \frac{Wx^2}{4} + A$$

$$EI\,y = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

At $\qquad\qquad x = 0, \qquad \dfrac{dy}{dx} = 0 \qquad \therefore \qquad A = 0$

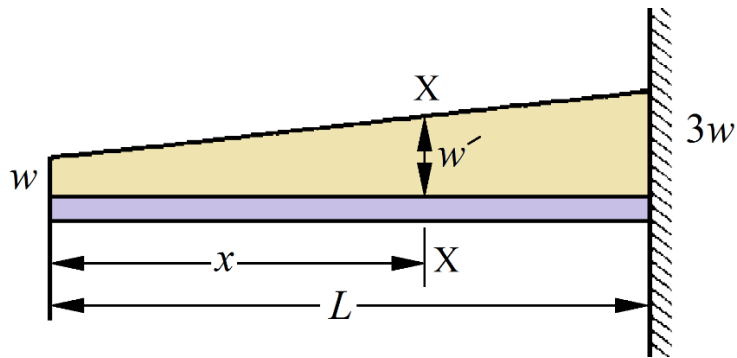At $\qquad\qquad x = \dfrac{L}{2}, \qquad y = 0 \qquad \therefore \qquad 0 = \dfrac{WL^3}{32} - \dfrac{WL^3}{96} + B$

$$\therefore \qquad\qquad B = -\frac{WL^3}{48}$$

$$\therefore \qquad\qquad y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right] \qquad\qquad (6.14)$$

$$\therefore \qquad\qquad y_{max} = -\frac{WL^3}{48\,EI} \qquad \text{at the center} \qquad\qquad (6.15)$$

and $\qquad\qquad \left(\dfrac{dy}{dx}\right)_{max} = \pm\,\dfrac{WL^2}{16\,EI} \qquad \text{at the ends} \qquad\qquad (6.16)$

(e) Cantilever subjected to non-uniform distributed load

Fig. 6.7.

The loading at section *XX* is

$$w' = EI\ \frac{d^4y}{dx^4} = -\left[w + (3w - w)\frac{x}{L}\right] = -w\left(1 + \frac{2x}{L}\right)$$

Integrating,

$$EI\ \frac{d^3y}{dx^3} = -w\left(x + \frac{x^2}{L}\right) + A \qquad\qquad (1)$$

$$EI\ \frac{d^2y}{dx^2} = -w\left(\frac{x^2}{2} + \frac{x^3}{3L}\right) + Ax + B \qquad\qquad (2)$$

$$EI\ \frac{dy}{dx} = -w\left(\frac{x^3}{6} + \frac{x^4}{12L}\right) + \frac{Ax^2}{2} + Bx + C \qquad\qquad (3)$$

$$EI\ y = -w\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \qquad\qquad (4)$$

Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At $x = 0$,   S.F. is zero

∴   from (1) $A = 0$

At $x = 0$,   B.M. is zero

∴   from (2) $B = 0$

At $x = L$,   slope $dy/dx = 0$ (slope normally assumed zero at a built-in support)

∴   from (3)

$$0 = -w\left(\frac{L^3}{6} + \frac{L^3}{12}\right) + C, \qquad\qquad C = \frac{wL^3}{4}$$

At $x = L$, $y = 0$

$\therefore$ from (4)

$$0 = -w\left(\frac{L^4}{24} + \frac{L^4}{60}\right) + \frac{w L^4}{4} + D$$

$\therefore$

$$D = -\frac{23\,w\,L^4}{120}$$

$$EI\,y = -\frac{w\,x^4}{24} - \frac{w\,x^5}{60\,L} + \frac{w\,L^3 x}{4} - \frac{23\,w\,L^4}{120}$$

Then, for example, the deflection at the tip of the cantilever, where $x = 0$, is

$$y = -\frac{23\,w\,L^4}{120\,EI}$$

6.3 Macaulay's method

The simple integration method used in the previous examples can only be used when a single expression for B.M. applies along the complete length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions.

In this method

1. Write the equation to the whole beam (to the end span).

2. We will apply that the term $W(x - a)$ is integrated with respect to $(x - a)$ and not x.

$$W\frac{(x-a)^2}{2} \quad \text{and} \quad W\frac{(x-a)^3}{6}$$

3. In addition, since the term $W(x - a)$ applies only after the discontinuity, i.e. when $x > a$, it should be considered only when $x > a$ or when $(x - a)$ is positive. For these reasons such terms are conventionally put into square or curly brackets and called *Macaulay terms*.

4. Macaulay's terms should be neglected if equal to zero or negative value.

For the whole beam, therefore,

$$EI\frac{dy}{dx} = M + Q\,x - W\,[(x - a)]$$
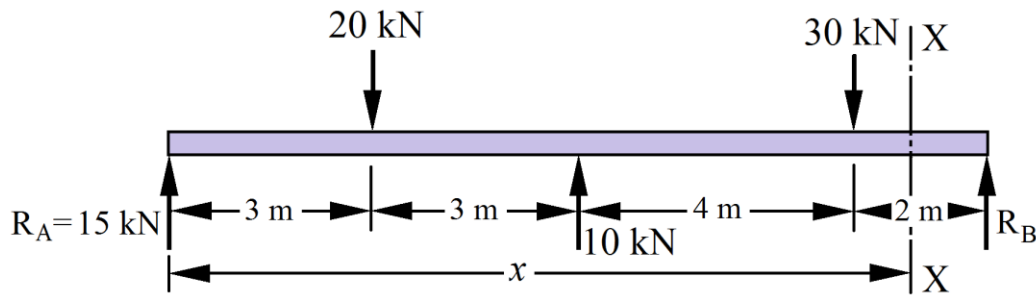
Fig. 6.8.

Using the Macaulay method, the equation for the B.M. at any general section XX is then given by

$$\text{B.M. } xx = 15\,x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

by integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 15\,\frac{x^2}{2} - 20\left[\frac{(x-3)^2}{2}\right] + 10\left[\frac{(x-6)^2}{2}\right] - 30\left[\frac{(x-10)^2}{2}\right] + A$$

and

$$\frac{EI}{10^3}y = 15\,\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] + Ax + B$$

where A and B are two constants of integration.

Now when $x = 0$, $y = 0$ $\quad \therefore \quad B = 0$

and when $x = 12$, $y = 0$

$$0 = \frac{15 \times 12^3}{6} - 20\left[\frac{9^3}{6}\right] + 10\left[\frac{6^3}{6}\right] - 30\left[\frac{2^3}{6}\right] + 12\,A$$

$$0 = 4320 - 2430 + 360 - 40 + 12A$$

$\therefore \qquad 12\,A = -4680 + 2470 = -2210$

$\therefore \qquad A = -184.2$

The deflection at any point is given by

$$\frac{EI}{10^3}y = 15\,\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2\,x$$

The deflection at mid-span is found by substituting $x = 6$ in the above equation.

$$\therefore \quad \text{central deflection} = \frac{10^3}{EI}\left[\frac{15 \times 6^3}{6} - \frac{20 \times 3^3}{6} - 184.2 \times 6\right]$$

$$= -\frac{655.2 \times 10^3}{EI}$$

With typical values of $E = 208$ GN/m^2 and $I = 82 \times 10^{-6}$ m^4

central deflection $= 38.4 \times 10^{-3}$ m $=$ **38.4 mm**

6.4 Macaulay's method for uniform distributed loads (u.d.1.s)

If a beam carries a uniformly distributed load over the complete span as shown in Fig. 6.9 (a) the B.M. equation is

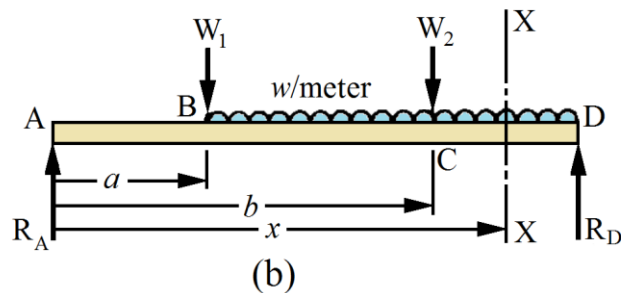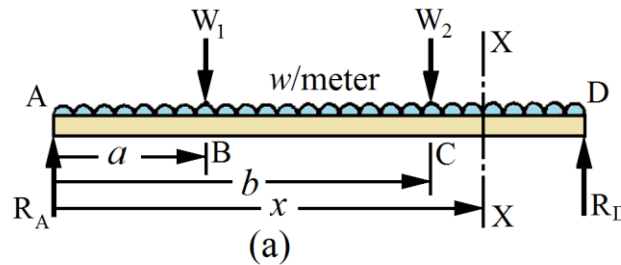$$\text{B.M. }_{XX} = EI\frac{d^2y}{dx^2} = R_A x - \frac{w x^2}{2} - W_1[(x-a)] - W_2[(x-b)]$$

(a)

(b)

Fig. 6.9.

If the u.d.1. starts at B as shown in Fig. 6.9 (b) the B.M. equation is modified and the Macaulay term is written inside square brackets.

$$\text{B.M. }_{XX} = EI\frac{d^2y}{dx^2} = R_A x - W_1[(x-a)] - w\left[\frac{(x-a)^2}{2}\right] - W_2[(x-b)]$$

Integrating,

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - W_1\left[\frac{(x-a)^2}{2}\right] - w\left[\frac{(x-a)^3}{6}\right] - W_2\left[\frac{(x-b)^2}{2}\right] + A$$

$$EI\,y = R_A \frac{x^3}{6} - W_1 \left[\frac{(x-a)^3}{6}\right] - w\left[\frac{(x-a)^4}{24}\right] - W_2\left[\frac{(x-b)^3}{6}\right] + Ax + B$$

Note: the Macaulay terms are integrated with respect to $(x - a)$ and **they must be ignored when negative**. Substitution of end conditions will then yield the values of the constants A and B in the normal way and hence the required values of slope or deflection.

6.5 Macaulay's method for beams with u.d.1. applied over part of the beam

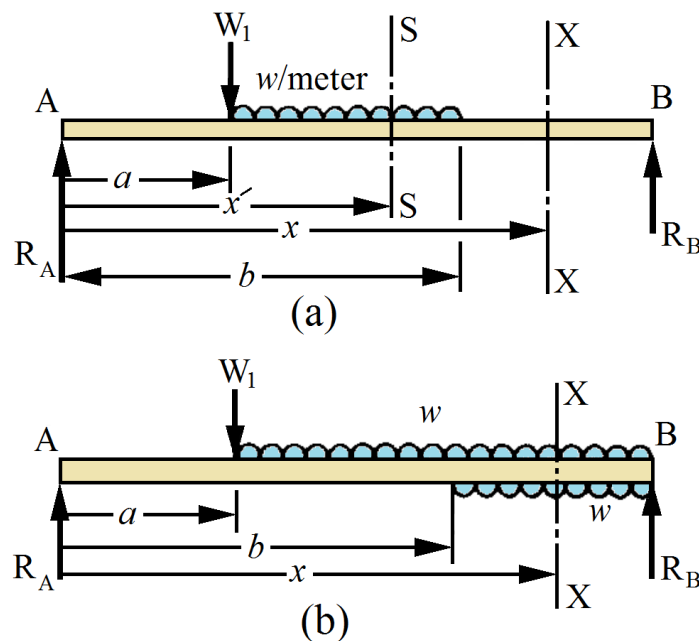Consider the beam loading as shown in Fig. 6.10 (a).

Fig. 6.10.

The B.M. at the section *SS* is given by

$$\text{B.M.}_{SS} = EI\frac{d^2y}{dx^2} = R_A\,x' - W_1[(x' - a)] - W\left[\frac{(x' - a)^2}{2}\right]$$

The B.M. equation for any general section *XX* is given by

$$\text{B.M.}_{XX} = EI\frac{d^2y}{dx^2} = R_A\,x - W_1[(x - a)] - w\left[\frac{(x - a)^2}{2}\right] + w\left[\frac{(x - b)^2}{2}\right]$$

6.6. Macaulay's method for couple applied at a point

Consider the beam *AB* shown in Fig. 6.11 with a moment or couple *M* applied at some point *C*. Considering the equilibrium of moments about each end in turn produces reactions of

$$R_A = \frac{M}{L} \quad \text{upwards,} \quad \text{and} \quad R_B = \frac{M}{L} \quad \text{downwards}$$

Fig. 6.11.

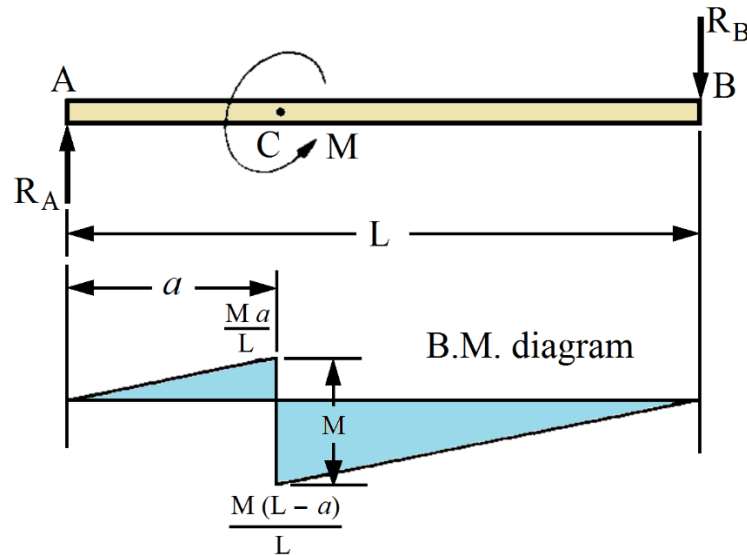For sections between *A* and *C* the B.M. is $\dfrac{M}{L}x$.

For sections between *C* and *B* the B.M. is $\dfrac{M}{L}x - M$.

the Macaulay method is written in the form

$$M\,[(x - a)^0]$$

when *x* is less than *a*, the term (*x* − *a*) is neglected.

$$M_{xx} = EI\,\frac{d^2 y}{dx^2} = \frac{M\,x}{L} - M\,[(x - a)^0] \qquad (6.17)$$

Then $\qquad EI\,\dfrac{dy}{dx} = \dfrac{M\,x^2}{2\,L} - M\,[(x - a)] + A$

Example 6.1

Determine the slope and deflection under the 50 kN load for the beam loading system shown in Fig. 6.12. Find also the position and magnitude of the maximum deflection. $E = 200$ GN/m²; $I = 83 \times 10^{-6}$ m⁴.
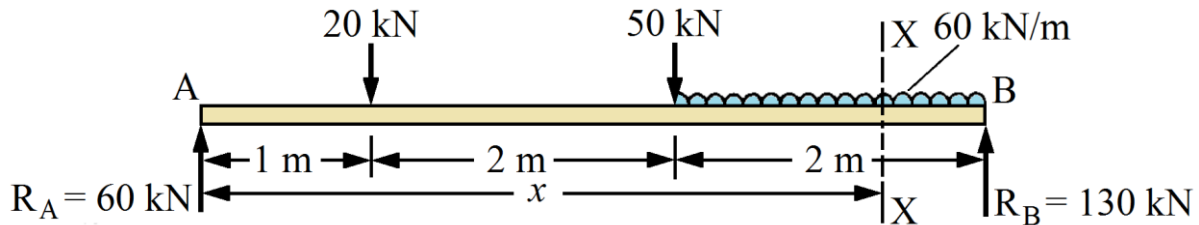

Fig. 6.12.

Solution:

Taking moments about either end of the beam gives

$$R_A = 60 \text{ kN and } R_B = 130 \text{ kN}$$

Applying Macaulay's method,

$$\text{B.M. }_{XX} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = 60\,x - 20[(x-1)] - 50[(x-3)] - 60\left[\frac{(x-3)^2}{2}\right] \tag{1}$$

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60\,x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] + A \tag{2}$$

$$\frac{EI}{10^3}\,y = \frac{60\,x^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] + A\,x + B \tag{3}$$

Now, when $x = 0,$ $y = 0$ $\therefore$ $B = 0$

when $x = 5,$ $y = 0$ substituting in Eq. (3)

$$0 = \frac{60 \times 5^3}{6} - \frac{20 \times 4^3}{6} - \frac{50 \times 2^3}{6} - \frac{60 \times 2^4}{24} + 5A$$

$$0 = 1250 - 213.3 - 66.7 - 40 + 5A$$

$\therefore$ $5A = -930 ,$ $A = -186$

Substituting in (2),

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60\,x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] - 186$$

$\therefore$ slope at $x = 3$ m (i.e. under the 50 kN load)

$$= \frac{10^3}{EI}\left[\frac{60 \times 3^2}{2} - \frac{20 \times 2^2}{2} - 186\right] = \frac{10^3 \times 44}{200 \times 10^9 \times 83 \times 10^{-6}}$$

$$= \mathbf{0.00265 \ rad}$$

And, substituting in (3),

$$\frac{EI}{10^3}\,y = \frac{60 \times 3^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] - 186\,x$$

$\therefore$ deflection at $x = 3$ m

$$y = \frac{10^3}{EI}\left[\frac{60 \times 3^3}{6} - \frac{20 \times 2^3}{6} - 186 \times 3\right]$$

$$= \frac{10^3}{EI}[270 - 26.67 - 558] = -\frac{10^3 \times 314.7}{200 \times 10^9 \times 83 \times 10^{-6}}$$

$$= -0.01896 \text{ m} = -\mathbf{19 \ mm}$$

In order to determine the maximum deflection, assume the maximum deflection point will occur somewhere between the 20 kN and 50 kN loads. From Eq. (2),

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60\,x^2}{2} - 20\frac{(x-1)^2}{2} - 186$$

$$= 30\,x^2 - 10\,x^2 + 20\,x - 10 - 186$$

$$= 20\,x^2 + 20\,x - 196$$

But, where the deflection is a maximum, the slope is zero

$\therefore$ $\qquad\qquad\qquad 0 = 20\,x^2 + 20\,x - 196$

$\therefore$ $\qquad\qquad\qquad x = \dfrac{-20 \pm (400 + 15680)^{1/2}}{40} = \dfrac{-20 \pm 126.8}{40}$
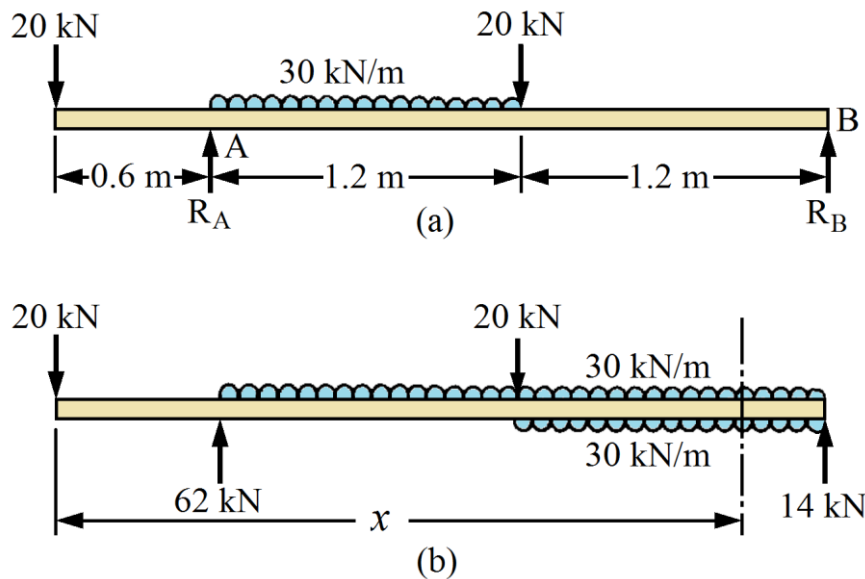
$$x = \mathbf{2.67 \ m}$$

Then, from Eq. (3), the maximum deflection is given by

$$\delta_{max} = -\frac{10^3}{EI}\left[\frac{60 \times 2.67^3}{6} - \frac{20 \times 1.67^3}{6} - 186 \times 2.67\right]$$

$$= -\frac{10^3 \times 321.78}{200 \times 10^9 \times 83 \times 10^{-6}} = -0.00194 = -\mathbf{19.4 \ mm}$$

Example 6.2

Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in Fig. 6.13 (a) using Macaulay's method. $EI = 0.65$ MN m².

Fig. 6.13.

Solution:

Taking moments about B

$$(3 \times 20) + (30 \times 1.2 \times 1.8) + (1.2 \times 20) = 2.4\ R_A$$

$$R_A = 62 \text{ kN} \quad \text{and} \quad R_B = 20 + (30 \times 1.2) + 20 - 62 = 14 \text{ kN}$$

Using the modified Macaulay approach for distributed loads over part of a beam introduced in Fig. 6.13 (b),

$$M_{XX} = \frac{EI}{10^3}\frac{d^2 y}{dx^2} = -20\,x + 62[(x - 0.6)] - 30\left[\frac{(x - 0.6)^2}{2}\right]$$

$$+ 30\left[\frac{(x - 1.8)^2}{2}\right] - 20[(x - 1.8)]$$

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{-20\,x^2}{2} + 62\left[\frac{(x - 0.6)^2}{2}\right] - 30\left[\frac{(x - 0.6)^3}{6}\right] + 30\left[\frac{(x - 1.8)^3}{6}\right]$$

$$- 20\left[\frac{(x - 1.8)^2}{2}\right] + A$$

$$\frac{EI}{10^3} y = \frac{-20 x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right]$$

$$-20\left[\frac{(x-1.8)^3}{6}\right] + Ax + B$$

Now, when $x = 0.6$, $y = 0$,

∴

$$0 = -\frac{20 \times 0.6^3}{6} + 0.6\, A + B$$

$$0.72 = 0.6\, A + B \tag{1}$$

and when $x = 3$, $y = 0$,

∴

$$0 = -\frac{20 \times 3^3}{6} + \frac{62 \times 2.4^3}{6} - \frac{30 \times 2.4^4}{24} + \frac{30 \times 1.2^4}{24} - \frac{20 \times 1.2^3}{6} + 3A + B$$

$$= -90 + 142.848 - 41.472 + 2.592 - 5.76 + 3A + B$$

$$-8.208 = 3A + B \tag{2}$$

Subtracting Eq. (1) from Eq. (2)

$$-8.928 = 2.4\, A \qquad ∴ \quad A = -3.72$$

Substituting in (1),

$$B = 0.72 - 0.6\,(-3.72), \qquad ∴ \quad B = 2.952$$

Substituting into the Macaulay deflection equation,

$$\frac{EI}{10^3} y = -\frac{20 x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right]$$

$$-20\left[\frac{(x-1.8)^3}{6}\right] - 3.72\, x + 2.952$$

At $x = 1$

$$y = \frac{10^3}{EI}\left[-\frac{20}{6} + \frac{62}{6} \times 0.4^3 - \frac{30 \times 0.4^4}{24} - 3.72 \times 1 + 2.952\right]$$

$$= \frac{10^3}{EI}[-3.33 + 0.661 - 0.032 - 3.72 + 2.952]$$

$$= -\frac{10^3 \times 3.472}{0.65 \times 10^6} = -5.34 \times 10^{-3}\ \text{m} = -\mathbf{5.34\ mm}\ \text{ deflected downwards}$$

Problems

6.1 A beam of length 10 m is symmetrically placed on two supports 7 m apart. The loading is 15 kN/m between the supports and 20 kN at each end. What is the central deflection of the beam? $E = 210$ GN/m^2; $I = 200 \times 10^{-6}$ m^4.   [6.8 mm]

6.2 Derive the expression for the maximum deflection of a simply supported beam of negligible weight carrying a point load at its mid-span position. The distance between the supports is L, the second moment of area of the cross-section is I and the modulus of elasticity of the beam material is E.

The maximum deflection of such a simply supported beam of length 3 m is 4.3 mm when carrying a load of 200 kN at its mid-span position. What would be the deflection at the free end of a cantilever of the same material, length and cross-section if it carries a load of l00 kN at a point 1.3 m from the free end?   [13.4 mm]

6.3 A horizontal beam, simply supported at its ends, carries a load which varies uniformly from 15 kN/m at one end to 60 kN/m at the other. Estimate the central deflection if the span is 7 m, the section 450 mm deep and the maximum bending stress 100 MN/m^2. $E = 210$ GN/m^2.   [21.9 mm]

6.4 A beam *AB*, 8 m long, is freely supported at its ends and carries loads of 30 kN and 50 kN at points 1 m and 5 m respectively from *A*. Find the position and magnitude of the maximum deflection. $E = 210$ GN/m^2; $I = 200 \times 10^{-6}$ m^4.   [14.4 mm]

6.5 A beam 7 m long is simply supported at its ends and loaded as follows: 120 kN at 1 m from one end *A*, 20 kN at 4 m from *A* and 60 kN at 5 m from *A*. Calculate the position and magnitude of the maximum deflection. The second moment of area of the beam section is 400×10^{-6} m^4 and E for the beam material is 210 GN/m^2.

[9.8 mm at 3.474 m]

6.6 A beam *ABCD*, 6 m long, is simply-supported at the right-hand end *D* and at a point *B* 1 m from the left-hand end *A*. It carries a vertical load of 10 kN at *A*, a second concentrated load of 20 kN at *C*, 3 m from *D*, and a uniformly distributed load of 10 kN/m between *C* and *D*. Determine the position and magnitude of the maximum deflection if E = 208 GN/m² and $I = 35 \times 10^{-6}$ m⁴.   [3.553 m from *A*, 11.95 mm]

6.7 A 3 m long cantilever *ABC* is built-in at *A*, partially supported at *B*, 2 m from *A*, with a force of 10 kN and carries a vertical load of 20 kN at *C*. A uniformly distributed load of 5 kN/m is also applied between *A* and *B*. Determine (a) the values of the vertical reaction and built-in moment at *A* and (b) the deflection of the free end *C* of the cantilever.

Develop an expression for the slope of the beam at any position and hence plot a slope diagram. E = 208 GN/m² and $I = 24 \times 10^{-6}$ m⁴.   [20 kN, 50 kNm, - 15 mm]

6.8 Develop a general expression for the slope of the beam of question 6.6 and hence plot a slope diagram for the beam. Use the slope diagram to confirm the answer given in question 6.6 for the position of the maximum deflection of the beam.

6.9 What would be the effect on the end deflection for question 6.7, if the built-in end *A* were replaced by a simple support at the same position and point *B* becomes a full simple support position (i.e. the force at *B* is no longer 10 kN). What general observation can you make about the effect of built-in constraints on the stiffness of beams?   [5.7 mm]

6.10 A beam *AB* is simply supported at *A* and *B* over a span of 3 m. It carries loads of 50 kN and 40 kN at 0.6 m and 2 m respectively from *A*, together with a uniformly distributed load of 60 kN/m between the 50 kN and 40 kN concentrated loads. If the cross-section of the beam is such that $I = 60 \times 10^{-6}$ m⁴ determine the value of the deflection of the beam under the 50 kN load. E = 210 GN/m². Sketch the S.F. and B.M. diagrams for the beam.   [3.7 mm]

6.11 Obtain the relationship between the B.M., S.F., and intensity of loading of a laterally loaded beam. A simply supported beam of span L carries a distributed load of intensity $k\,x^2/L^2$ where x is measured from one support towards the other. Find:

(a) the location and magnitude of the greatest bending moment.

(b) the support reactions.   [0.63L, 0.0393 kL^2, $kL/12$, $kL/4$]

6.12 A uniform beam 4 m long is simply supported at its ends, where couples are applied, each 3 kN m in magnitude but opposite in sense. If $E = 210$ GN/m^2 and $I = 90 \times 10^{-6}$ m^4 determine the magnitude of the deflection at mid-span.

What load must be applied at mid-span to reduce the deflection by half ?

[0.317 mm, 2.25 kN]

6.13 A 500 mm × 175 mm steel beam of length 8 m is supported at the left-hand end and at a point 1.6 m from the right-hand end. The beam carries a uniformly distributed load of 12 kN/m on its whole length, an additional uniformiy distributed load of 18 kN/m on the length between the supports and a point load of 30 kN at the right-hand end. Determine the slope and deflection of the beam at the section midway between the supports and also at the right-hand end. *EI* for the beam is 1.5×10^8 Nm^2.

[1.13×10^{-4}, 3.29 mm, 9.7×10^{-4}, 1.71 mm]