# Chapter Five

# Shear Stress Distribution

## 5.1 Introduction

Consider the case of two rectangular-sectioned beams lying one on top of the other and supported on simple supports as shown in Fig. 5.1. If some form of vertical loading is applied the beams will bend as shown in Fig. 5.2, i.e. if there is negligible friction between the mating surfaces of the beams each beam will bend independently of the other and as a result the lower surface of the top beam will slide relative to the upper surface of the lower beam.

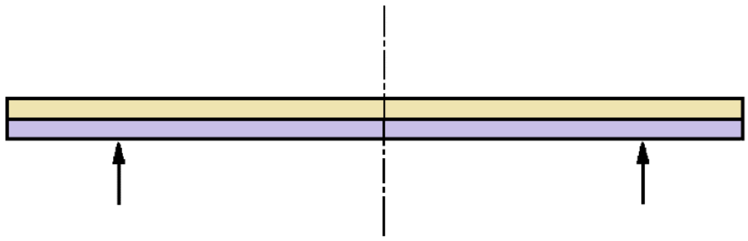


**Fig. 5.1** Two beams (unconnected) on simple supports prior to loading.

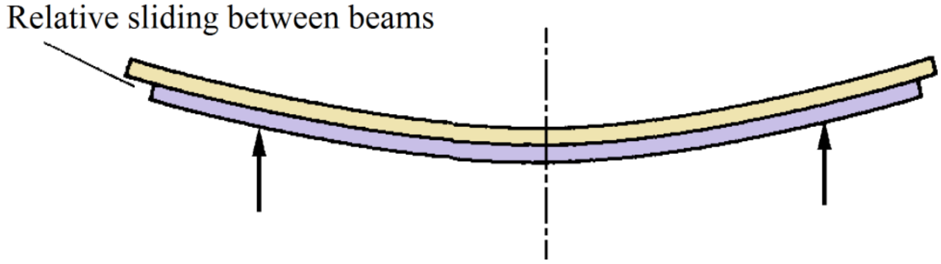



**Fig. 5.2** the presence of shear (relative sliding) between adjacent planes of a beam in bending.

## 5.2 Shear Stress Distribution due to Bending

Consider the portion of a beam of length *dx*, as shown in Fig. 5.3 (a), and an element *AB* distance *y* from the N.A. Under any loading system, the B.M. across the beam will change from *M* at *B* to (*M* + *dM*) at *A*. Now as a result of bending,

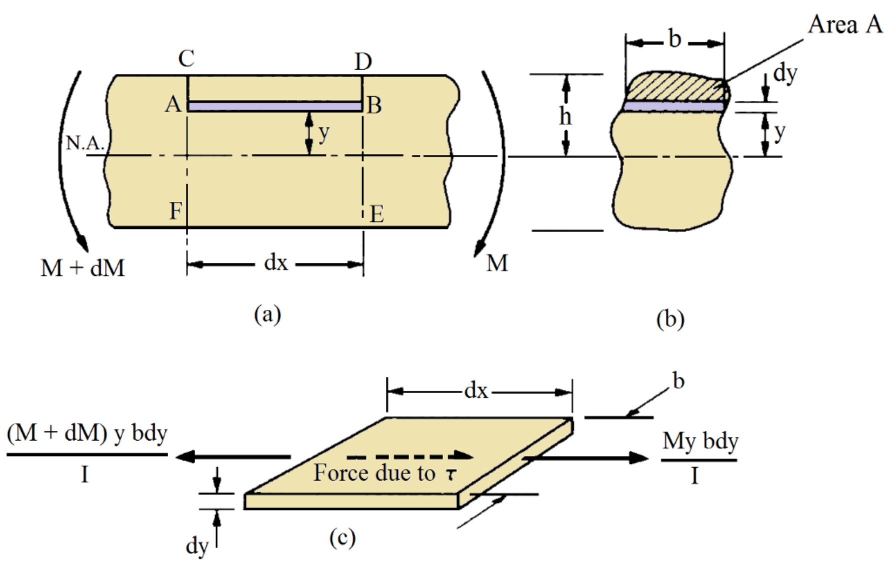



**Fig. 5.3.**

$$\text{longitudinal stress } \sigma = \frac{My}{I}$$

$$\text{longitudinal stress at } A = \frac{(M+dM)\,y}{I}$$

and

$$\text{longitudinal stress at } B = \frac{My}{I}$$

$\therefore$

$$\text{longitudinal force on the element at } A = \sigma A = \frac{(M+dM)\,y}{I} \times bdy$$

and

$$\text{longitudinal force on the element at } B = \frac{My}{I} \times bdy$$

The force system on the element is therefore as shown in Fig. 5.3 (c) with a net out-of-balance force to the left

$$= \frac{(M+dM)\,y}{I}\,bdy - \frac{My}{I}\,bdy$$

$$= \frac{dM}{I}\,y\,bdy$$

Therefore total out-of-balance force from all sections above height $y$

$$= \int_{y}^{h} \frac{dM}{I}\,y\,bdy$$

For equilibrium, this force is resisted by a shear force set up on the section of length *dx* and breadth *b*, as shown in Fig. 5.4.

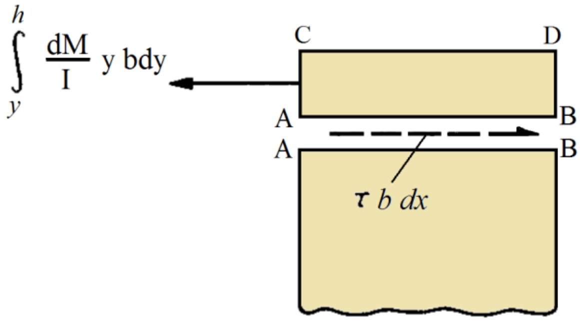



**Fig. 5.4.**

Thus if the shear stress is $\tau$, then

$$\tau\, bdx = \frac{dM}{I} \int_y^h y\, bdy \qquad (5.1)$$

But, $\int_y^h y\, bdy$ = first moment of area of shaded portion of Fig. 5.3 (b) about the N.A.

$$= A\, \bar{y}$$

Where, $A$ is the area of shaded portion, and $\bar{y}$ the distance of its centroid from the N.A.

Also $\frac{dM}{dx}$ = rate of change of the B.M.

= S.F. $Q$ at the section

$$\tau = \frac{Q\, A\, \bar{y}}{I\, b} \qquad (5.2)$$

or, alternatively,

$$\tau = \frac{Q}{I\, b} \int_y^h y\, dA \qquad \text{where } dA = bdy \qquad (5.3)$$

## 5.3 Application to rectangular sections

Consider the rectangular-sectioned beam of Fig. 5.5 subjected at a given transverse cross-section to a S.F. $Q$.

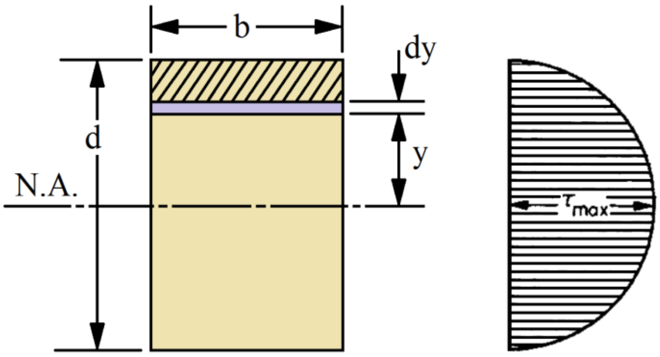



**Fig. 5.5** Shear stress distribution due to bending of a rectangular section beam.

$$\tau = \frac{Q\,A\,\bar{y}}{I\,b}$$

$$= \frac{Q}{I\,b} \times b\left(\frac{d}{2} - y\right) \times \left[y + \frac{\left(\frac{d}{2} - y\right)}{2}\right]$$

$$= \frac{Q \times 12}{bd^3 \times b} \times b\left(\frac{d}{2} - y\right)\frac{\left(\frac{d}{2} + y\right)}{2}$$

$$= \frac{6\,Q}{bd^3}\left[\frac{d^2}{4} - y^2\right] (i.e.\ a\ parabola) \qquad (5.4)$$

$$\tau_{max} = \frac{6\,Q}{bd^3} \times \frac{d^2}{4} = \frac{3\,Q}{2\,bd} \quad when\ y = 0 \qquad (5.5)$$

$$average\ \tau = \frac{Q}{bd}$$

$$\tau_{max} = \frac{3}{2}\ \tau_{average} \qquad (5.6)$$

## 5.4 Application to I-section beams

Consider the I-section beam shown in Fig. 5.6.

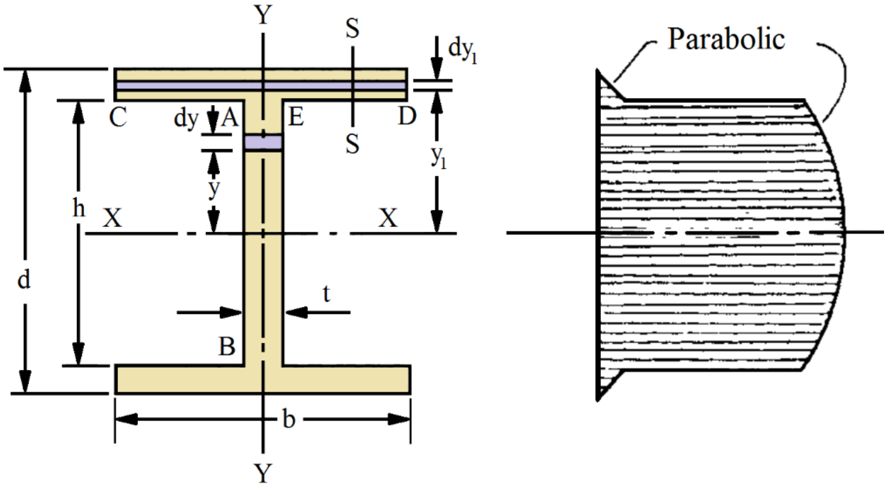



**Fig. 5.6** Shear stress distribution due to bending of an I-section beam.

**(a) Vertical shear in the web**

The shear stress distribution due to bending at any point in a given transverse cross-section is given by the eq. (5.3)

$$\tau = \frac{Q}{I\,b}\int_{y}^{d/2} y\,dA$$

In the case of the I-beam, the width of the section is not constant so that the quantity *dA* will be different in the web and the flange. Therefore, equation (5.3) must be modified to

$$\tau = \frac{Q}{It}\int_{y}^{h/2} t\,y\,dy + \frac{Q}{It}\int_{h/2}^{d/2} b\,y_1\,dy_1$$

$$= \frac{Q}{2\,I}\left[\frac{h^2}{4} - y^2\right] + \frac{Q\,b}{2\,It}\left[\frac{d^2}{4} - \frac{h^2}{4}\right] \qquad \text{where } y = h/2$$

$$\boldsymbol{\tau_A = \tau_B = \frac{Q\,b}{2\,It}\left[\frac{d^2}{4} - \frac{h^2}{4}\right]} \qquad (5.7)$$

The maximum shear occurs at the N.A., where $y = 0$,

$$\boldsymbol{\tau_{max} = \frac{Q\,h^2}{8\,I} + \frac{Q\,b}{2\,It}\left[\frac{d^2}{4} - \frac{h^2}{4}\right]} \qquad (5.8)$$

**(b) Vertical shear in the flanges**

The vertical shear in the flange, where the width of the section is b is given by the eq. (5.3) as

$$\tau = \frac{Q}{I\,b}\int_{y_1}^{d/2} y_1\, dA$$

$$= \frac{Q}{I\,b}\int_{y_1}^{d/2} y_1\, b\, dy_1 = \frac{\boldsymbol{Q}}{\boldsymbol{I}}\left[\frac{\boldsymbol{d^2}}{\boldsymbol{8}} - \frac{\boldsymbol{y_1^2}}{\boldsymbol{2}}\right] \qquad (5.9)$$

The maximum value is that at the bottom of the flange when $y_1 = h/2$,

$$\boldsymbol{\tau_{max}} = \frac{Q}{I}\left[\frac{d^2}{8} - \frac{h^2}{8}\right] = \frac{\boldsymbol{Q}}{\boldsymbol{8\,I}}\left[\boldsymbol{d^2 - h^2}\right] \qquad (5.10)$$

**5.5 Application to circular sections**

Consider the element of thickness *dz* and breadth *b* shown in Fig. 5.7.

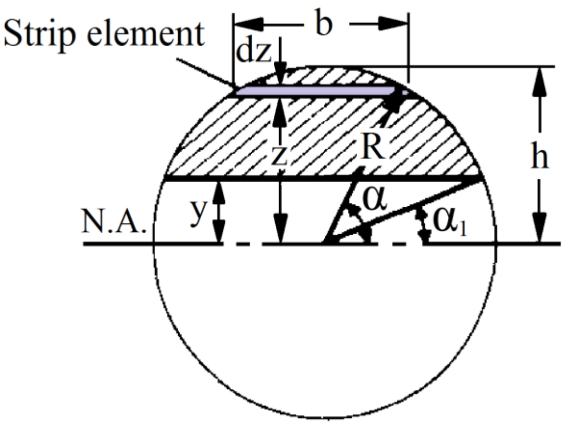



**Fig. 5.7**

$$\tau = \frac{Q}{Ib}\int_{z}^{h} b\,y\,dy$$

Now $b = 2R\cos\alpha$, $y = z = R\sin\alpha$ and $dz = R\cos\alpha\, d\alpha$ and, at section distance *y* from the N.A., $b = 2R\cos\alpha_1$,

$$\tau = \frac{Q}{I \times 2R\cos\alpha_1}\int_{\alpha_1}^{\pi/2} 2R\cos\alpha\, R\sin\alpha\, R\cos\alpha\, d\alpha$$

$$= \frac{Q \times 4}{2R\cos\alpha_1 \times \pi R^4} \int_{\alpha_1}^{\pi/2} 2R^3 \cos^2\alpha \sin\alpha \, d\alpha \qquad \text{since } I = \frac{\pi R^4}{4}$$

$$= \frac{4Q}{\pi R^2 \cos\alpha_1} \left[ -\frac{\cos^3\alpha}{3} \right]_{\alpha_1}^{\pi/2}$$

$$= \frac{4Q}{3\pi R^2 \cos\alpha_1} [\cos^3\alpha_1]_{\pi/2}^{\alpha_1} = \frac{4Q\cos^2\alpha_1}{3\pi R^2}$$

$$= \frac{4Q}{3\pi R^2}\left[1 - \sin^2\alpha_1\right] = \frac{4Q}{3\pi R^2}\left[1 - \left(\frac{y}{R}\right)^2\right] \qquad (5.11)$$

i.e. a parabola with its maximum value at $y = 0$.

$$\tau_{max} = \frac{4Q}{3\pi R^2}$$

$$\text{mean stress } = \frac{Q}{\pi R^2}$$

$$\frac{\textbf{maximum shear stress}}{\textbf{mean shear stress}} = \frac{\frac{4Q}{3\pi R^2}}{\frac{Q}{\pi R^2}} = \frac{4}{3} \qquad (5.12)$$

Alternative procedure

Using eq. (5.2), namely $\tau = \frac{Q A \bar{y}}{I b}$, and referring to Fig. 5.7,

$$\frac{b}{2} = \left(R^2 - z^2\right)^{1/2} = R\cos\alpha \qquad \text{and} \qquad R\sin\alpha = \frac{z}{R}$$

$$A\bar{y} \text{ for the shaded segment} = \int_{R\sin\alpha}^{R} A\bar{y} \text{ for strip element}$$

$$= \int_{R\sin\alpha}^{R} b\, dz\, z$$

$$= 2\int_{R\sin\alpha}^{R} \left(R^2 - z^2\right)^{1/2} z\, dz$$

$$= \frac{2}{3}\left[(R^2 - z^2)^{3/2}\right]_{R}^{R\sin\alpha}$$

$$= \frac{2}{3}\left[R^2\left(1-\sin^2\alpha\right)\right]^{3/2}$$

$$= \frac{2}{3}R^3(\cos^2\alpha)^{3/2} = \frac{2}{3}R^3\cos^3\alpha$$

$$\tau = \frac{QA\bar{y}}{Ib} = \frac{Q\times\frac{2}{3}R^3\cos^3\alpha}{\frac{\pi R^4}{4}\times 2R\cos\alpha}$$

since $$I = \frac{\pi R^4}{4}$$

$$\tau = \frac{4Q}{3\pi R^2}\cos^2\alpha = \frac{4Q}{3\pi R^2}(1-\sin^2\alpha)$$

$$= \frac{4Q}{3\pi R^2}\left[1-\left(\frac{y}{R}\right)^2\right] \qquad \text{same as} \qquad (5.13)$$

### *Example 5.1*

At a given position on a beam of uniform I-section, the beam is subjected to a shear force of 100 kN. Plot a curve to show the variation of shear stress across the section and hence determine the ratio of the maximum shear stress to the mean shear stress.

### *Solution:*

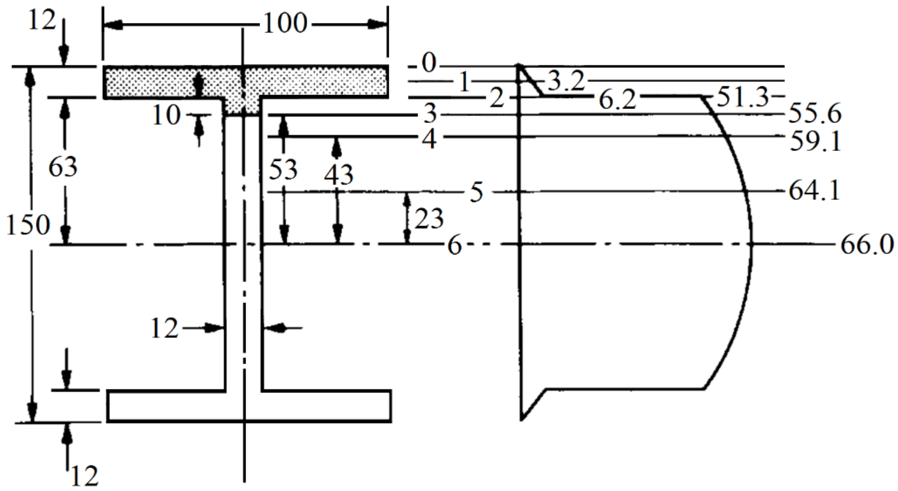
Consider the I-section shown in Fig. 5.8. By symmetry, the centroid of the section is at mid-height and the neutral axis passes through this position. The second moment of area of the section is then given by

$$I = \frac{100\times150^3\times10^{-12}}{12} - \frac{88\times126^3\times10^{-12}}{12}$$

$$= (28.125-14.67)10^{-6} = 13.46\times10^{-6}\ \text{m}^4$$

The distribution of shear stress across the section is

$$\tau = \frac{QA\bar{y}}{Ib} = \frac{100\times10^3 A\bar{y}}{13.46\times10^{-6}b} = 7.43\times10^9\frac{A\bar{y}}{b}$$




(a) Beam cross-section (all dimension in mm) (b) Shear stress distribution ($MN/m^2$)

**Fig. 5.8.**

In this case, because of symmetry, only sections above the N.A. need be considered, since a similar distribution will be obtained below the N.A.

| **Section** | $A \times 10^{-6}$ ($m^2$) | $\bar{y} \times 10^{-3}$ (m) | $b \times 10^{-3}$ (m) | $\tau = \frac{Q A \bar{y}}{I b}$ ($MN/m^2$) |
|---|---|---|---|---|
| 0 | 0 | - | - | - |
| 1 | 100 × 6 = 600 | 72 | 100 | 3.2 |
| 2 | 100 × 12 = 1200 | 69 | 100 | 6.2 |
| 2 | 1200 | 69 | 12 | 51.3 |
| 3 | 1320 | 68 | 12 | 55.6 |
| 4 | 1440 | 66.3 | 12 | 59.1 |
| 5 | 1680 | 61.6 | 12 | 64.1 |
| 6 | 1956 | 54.5 | 12 | 66.0 |

The centroid of the shaded T-section from the top edge Fig. 5.8 (a),

$$h = \frac{\sum yA}{\sum A}, \qquad \therefore \; h = \frac{(100 \times 12 \times 6)10^{-9} + (10 \times 12 \times 17)10^{-9}}{[(100 \times 12) + (10 \times 12)]10^{-6}}$$

$$h = \frac{9240}{1320} = 7 \text{ mm}$$

$$\bar{y}_3 = 75 - 7 = 68 \text{ mm}$$

The shear stress distribution due to bending is shown in Fig. 5.8 (b), giving a maximum shear stress of $\boldsymbol{\tau_{max} = 66\ MN/m^2}$.

Now the mean shear stress across the section is:

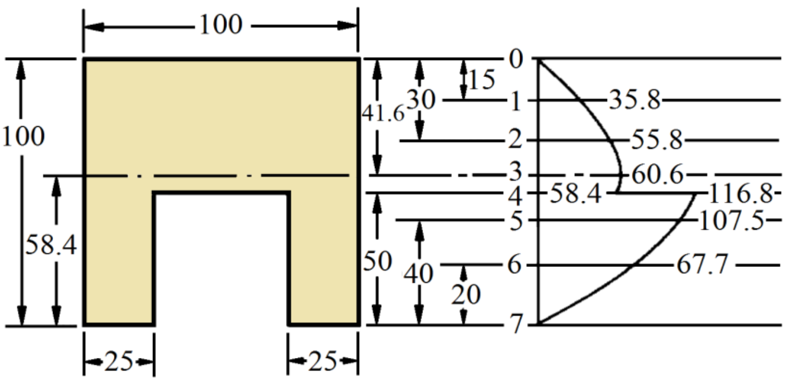
$$\tau_{mean} = \frac{\text{shear force}}{\text{area}} = \frac{100 \times 10^3}{3.912 \times 10^{-3}}$$

$$= 25.6\ \ MN/m^2$$

$$\frac{\text{max. shear stress}}{\text{mean shear stress}} = \frac{66}{25.6} = \mathbf{2.58}$$

## *Example 5.2*

At a certain section, a beam has the cross-section shown in Fig. 5.9. The beam is simply supported at its ends and carries a central concentrated load of 500 kN together with a load of 300 kN/m uniformly distributed across the complete span of 3 m. Draw the shear stress distribution diagram for a section 1 m from the left-hand support.




(a) Beam cross - section (mm) (b) Shear stress distribution ($MN/m^2$)

**Fig. 5.9.**

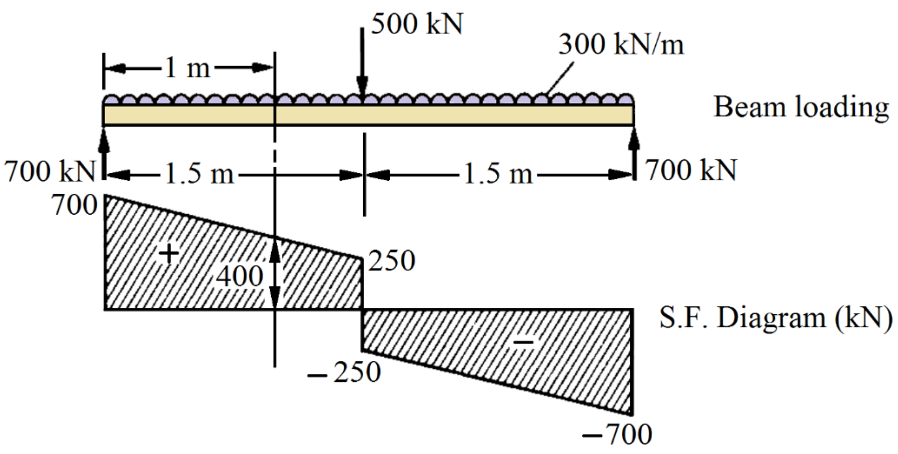



**Fig. 5.10.**

***Solution:***

From the S.F. diagram for the beam Fig. 5.10, it is evident that the S.F. at the section 1 m from the left-hand support is 400 kN,

$$Q = 400 \ \text{kN}$$

To find the position of the N.A. of the beam section of Fig. 5.9 (a), take moments of area about the base.

$$\bar{y} = \frac{\sum yA}{\sum A}$$

$$\bar{y} = \frac{(100 \times 100 \times 50)10^{-9} - (50 \times 50 \times 25)10^{-9}}{[(100 \times 100) - (50 \times 50)]10^{-6}}$$

$$\bar{y} = 0.0584 \text{ m} = 58.4 \text{ mm}$$

Then $$I_{\text{N.A.}} = \left[\frac{100 \times 41.6^3}{3} + 2\left(\frac{25 \times 58.4^3}{3}\right) + \left(\frac{50 \times 8.4^3}{3}\right)\right]10^{-12}$$

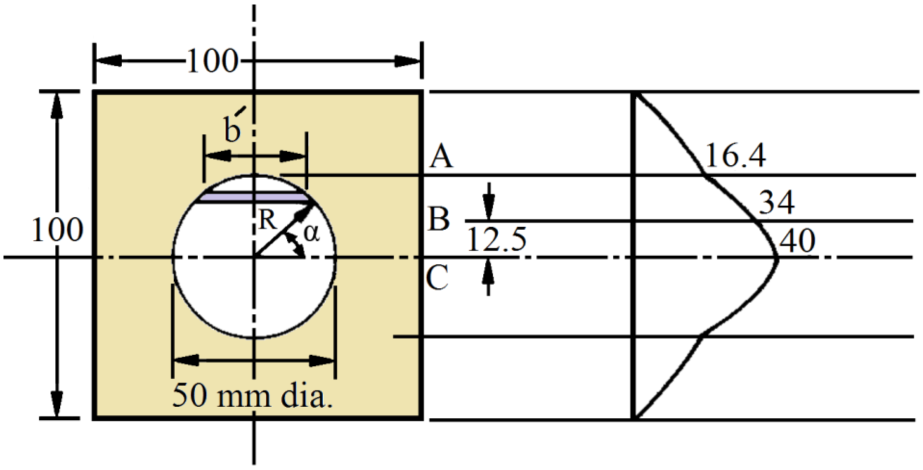
$$= (2.41 + 3.3 + 0.0099)10^{-6} = 5.72 \times 10^{-6} \text{ m}^4$$

$$\tau = \frac{Q A \bar{y}}{I b} = \frac{400 \times 10^3 A \bar{y}}{5.72 \times 10^{-6} b} = 7 \times 10^{10} \frac{A \bar{y}}{b}$$

| **Section** | $A \times 10^{-6}$ **(m²)** | $\bar{y} \times 10^{-3}$ **(m)** | $b \times 10^{-3}$ **(m)** | $\tau = 7 \times 10^{10} \frac{A\bar{y}}{b}$ **(MN/m²)** |
|---|---|---|---|---|
| 0 | 0 | - | - | 0 |
| 1 | 1500 | 34.1 | 100 | 35.8 |
| 2 | 3000 | 26.6 | 100 | 55.8 |
| 3 | 4160 | 20.8 | 100 | 60.6 |
| 4 | 2500 | 33.4 | 100 | 58.4 |
| 4 | 2500 | 33.4 | 50 | 116.8 |
| 5 | 2000 | 38.4 | 50 | 107.5 |
| 6 | 1000 | 48.4 | 50 | 67.7 |
| 7 | 0 | - | - | 0 |

## *Example 5.3*

Determine the values of the shear stress owing to bending at points *A*, *B* and *C* in the beam cross-section shown in Fig. 5.11, when subjected to a shear force of $Q$ = 140 kN. Hence, sketch the shear stress distribution diagram.




(a) Beam cross-section (mm) (b) Shear stress distribution (MN/m²)

**Fig. 5.11.**

## *Solution:*

By symmetry the centroid will be at the center of the section and

$$I_{N.A.} = \frac{100 \times 100^3}{12} \times 10^{-12} - \frac{\pi \times 50^4}{64} \times 10^{-12}$$

$$= (8.33 - 0.31)10^{-6} = 8.02 \times 10^{-6} \text{ m}^4$$

*At A*:

$$\tau_A = \frac{Q A \bar{y}}{I b} = \frac{140 \times 10^3 \times (100 \times 25 \times 37.5)10^{-9}}{8.02 \times 10^{-6} \times 100 \times 10^{-3}} = \mathbf{16.4\ MN/m^2}$$

*At B*:

Here the required $A\bar{y}$ is obtained by subtracting $A\bar{y}$ for the portion of the circle above *B* from that of the rectangle above *B*. Now for the circle

$$A\bar{y} = \int_{12.5}^{25} b' y \, dy \qquad \text{(Fig 5.11 (a))}$$

But, when $y = 12.5$, $\sin\alpha = \frac{12.5}{25} = \frac{1}{2}$ $\therefore\ \alpha = \pi/6$

and when $y = 25$, $\sin\alpha = \frac{25}{25} = 1$ $\therefore\ \alpha = \pi/2$

Also $b' = 2R\cos\alpha$, $y = R\sin\alpha$ and $dy = R\cos\alpha\, d\alpha$

$\therefore$ for circle portion,
$$A\bar{y} = \int_{\pi/6}^{\pi/2} 2R\cos\alpha \; R\sin\alpha \; R\cos\alpha\, d\alpha$$

$$= \int_{\pi/6}^{\pi/2} 2R^3 \cos^2\alpha \, \sin\alpha \, d\alpha$$

$$= 2R^3 \left[ -\frac{\cos^3\alpha}{3} \right]_{\pi/6}^{\pi/2}$$

$$= \frac{2 \times 25^3 \times 10^{-9}}{3} \left[ \left( \frac{\sqrt{3}}{2} \right)^3 \right] = 6.75 \times 10^{-6} \text{ m}^3$$

for complete section above $B$

$$A\bar{y} = 100 \times 37.5 \times 31.25 \times 10^{-9} - 6.75 \times 10^{-6}$$
$$= 110.25 \times 10^{-6} \text{ m}^3$$

and
$$b' = 2R\cos\frac{\pi}{6} = 2 \times 25 \times 10^{-3} \times \frac{\sqrt{3}}{2} = 43.3 \times 10^{-3} \text{ m}$$
$$b = (100 - 43.3)10^{-3} = 56.7 \times 10^{-3} \text{ m}$$

$$\tau_B = \frac{Q\,A\bar{y}}{I\,b} = \frac{140 \times 10^3 \times 110.25 \times 10^{-6}}{8.02 \times 10^{-6} \times 56.7 \times 10^{-3}} = \mathbf{34\ MN/m^2}$$

*At C*:

$$A\bar{y} \text{ for semicircle } = 2R^3 \left[ -\frac{\cos^3\alpha}{3} \right]_0^{\pi/2}$$

$$= \frac{2 \times 25^3 \times 10^{-9}}{3} [-(0-1)] = 10.41 \times 10^{-6} \text{ m}^3$$

$A\bar{y}$ for section above $C$

$$= (100 \times 50 \times 25)10^{-9} - 10.41 \times 10^{-6} = 114.59 \times 10^{-6} \text{ m}^3$$

and $b = (100 - 50)10^{-3} = 50 \times 10^{-3}$ m

$$\tau_C = \frac{Q\,A\bar{y}}{I\,b} = \frac{140 \times 10^3 \times 114.59 \times 10^{-6}}{8.02 \times 10^{-6} \times 50 \times 10^{-3}} = \mathbf{40\ MN/m^2}$$

## Problems

**5.1** A uniform I-section beam has flanges 150 mm wide by 8 mm thick and a web 180 mm wide and 8 mm thick. At a certain section, there is a shearing force of 120 kN. Draw a diagram to illustrate the distribution of shear stress across the section as a result of bending. What is the maximum shear stress? [86.7 MN/m2.]

**5.2** A girder has a uniform T cross-section with flange 250 mm × 50 mm and web 220 mm × 50 mm. At a certain section of the girder, there is a shear force of 360 kN.
Plot neatly to scale the shear-stress distribution across the section, stating the values:
(a) where the web and the flange of the section meet.
(b) at the neutral axis. [7.47, 37.4, 39.6 MN/$m^2$]

**5.3** A beam having an inverted T cross-section has an overall width of 150 mm and overall depth of 200 mm. The thickness of the crosspiece is 50 mm and of the vertical web 25 mm. At a certain section along the beam, the vertical shear force is found to be 120 kN. Draw neatly to scale, using 20 mm spacing except where closer intervals are required, and a shear-stress distribution diagram across this section. If the mean stress is calculated over the whole of the cross-sectional area, determine the ratio of the maximum shear stress to the mean shear stress. [3.37]

**5.4** The channel section shown in Fig. 5.12 is simply supported over a span of 5 m and carries a uniformly distributed load of 15 kN/m, run over its whole length. Sketch the shearing-stress distribution diagram at the point of maximum shearing force and mark important values. Determine the ratio of the maximum shearing stress to the average shearing stress. [3, 9.2, 9.3 MN/$m^2$; 2.42]

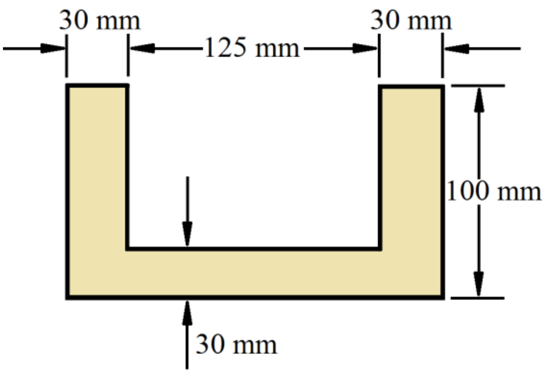



**Fig. 5.12.**

**5.5** The cross-section of a beam, which carries a shear force of 20 kN as shown in Fig. 5.13. Plot a graph to scale, which shows the distribution of shear stress due to bending across the cross-section. [21.7, 5.2, 5.23 MN/$m^2$]

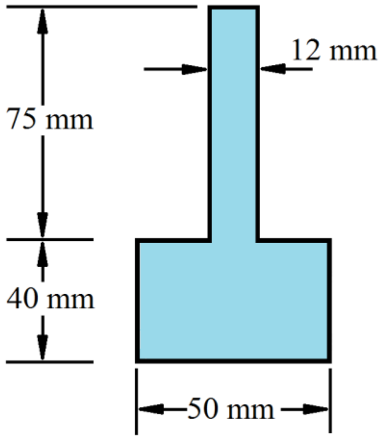



**Fig. 5.13.**

**5.7** Deduce an expression for the shearing stress at any point in a section of a beam owing to the shearing force at that section. State the assumptions made.

A simply supported beam carries a central load *W*. The cross-section of the beam is rectangular of depth *d*. At what distance from the neutral axis will the shearing stress be equal to the mean shearing stress on the section? [ $d/\sqrt{12}$ ]

**5.9** Using customary notation, show that the shear stress over the cross-section of a loaded beam is given by $\tau = \dfrac{Q A \bar{y}}{I b}$.

The cross-section of a beam is an isosceles triangle of base *B* and height *H*, the base being arranged in a horizontal plane. Find the shear stress at the neutral axis owing to a shear force *Q* acting on the cross-section and express it in terms of the mean shear stress. (The second moment of area of a triangle about its base is $BH^3/12$)

[ $\dfrac{8Q}{3BH}$ , $\dfrac{4}{3}\tau_{mean}$ ]