Chapter Four

Bending Theory

4.1 Assumptions used in Bending Theory

Assumptions used in derivation of the simple bending theory;

1- The beam is initially straight and unstressed.

2- The material of the beam is perfectly homogeneous and isotropic, the same density and elastic properties throughout.

3- The elastic limit is nowhere exceeded.

4- Young's modulus for the material is the same in tension and compression.

5- Plane cross-sections remain plane before and after bending.

6- Every cross-section of the beam is symmetrical about the plane of bending, about an axis perpendicular to the N.A.

7- There is no resultant force perpendicular to any cross-section.

4.2 Simple Bending Theory

If we now consider a beam initially unstressed and subjected to a constant B.M. along its length, pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius R.

- There must be a one surface in tension and the other in compression.
- There is axis at which the stress is zero and called neutral axis N.A.

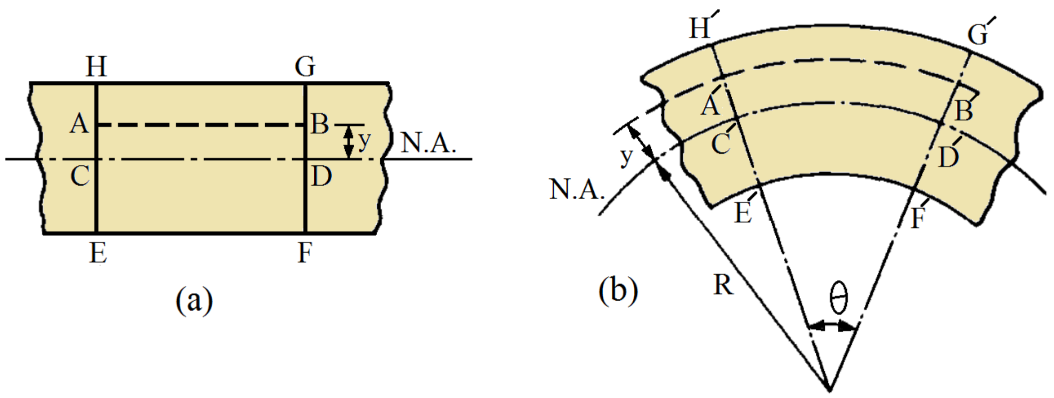

Fig. 4.1 Beam subjected to pure bending **(a)** before, **(b)** after, the moment is applied.

Consider the two cross-sections of a beam, *HE* and *GF*, originally parallel Fig. 4.1(a). When the beam is bent, Fig. 4.1(b), it is assumed that these sections remain plane; *H' E'* and *G' F'*, the final positions of the sections, are still straight lines. They will then subtend some angle θ.

Consider some fiber *AB* in the material, distance *y* from the N.A. When the beam is bent this will stretch to *A'B'*.

$$\text{Strain in fiber } AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB}$$

But *AB* = *CD*, and, the N.A. is unstressed, *CD* = *C'D'*.

$$\text{Strain} = \frac{A'B' - C'D'}{C'D'} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\frac{\text{stress}}{\text{strain}} = \text{Young's Modulus } E$$

$$\text{Strain} = \frac{\sigma}{E}$$

Equating the two equations for strain,

$$\frac{\sigma}{E} = \frac{y}{R}$$

Or $$\frac{\sigma}{y} = \frac{E}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.1)$$

Consider the cross-section of the beam (Fig. 4.2). From eqn. (4.1) the stress on a fiber at distance *y* from the N.A. is

$$\sigma = \frac{E}{R} y$$

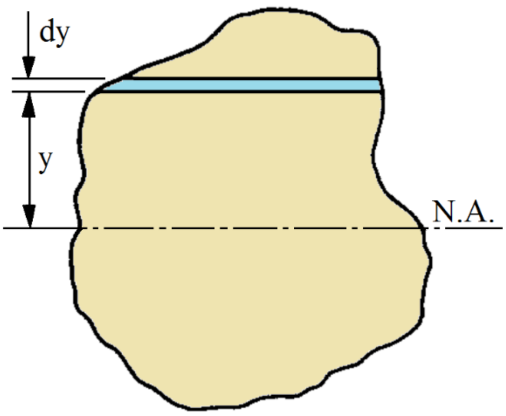

Fig. 4.2 Beam cross-section.

If the strip of area ***δA***, therefore, the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R}\, y\, \delta A$$

This has a moment about the N.A. of

$$F\, y = \frac{E}{R}\, y^2\, \delta A$$

The total moment for the whole cross-section is

$$M = \sum \frac{E}{R}\, y^2\, \delta A$$

$$= \frac{E}{R} \sum y^2\, \delta A$$

Since E and R are assumed constant.

The term $\sum y^2\, \delta A$ is called the ***second moment*** of ***area*** of the cross-section and given the symbol ***I***.

$$M = \frac{E}{R}\, I \quad \text{and} \quad \frac{M}{I} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2)$$

Combining eqns. (4.1) and (4.2) we have the bending theory equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.3)$$

4.3 Neutral axis

As stated above, it is clear that if, in bending, one surface of the beam is subjected to tension and the opposite surface to compression there must be a region within the beam cross-section at which the stress changes sign, where the stress is zero and this is termed the ***neutral axis***.

Further, eqn. (4.3) may be re-written in the form

$$\sigma = \frac{M}{I}\, y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.4)$$

Eqn. (4.4) shows, at any section the stress is directly proportional to ***y***, the distance from the N.A., $\boldsymbol{\sigma}$ varies linearly with ***y***, the **maximum stress** values occurring in the **outside surface** of the beam where ***y*** is a **maximum**.

Typical stress distributions in bending are shown in Fig. 4.3. It is evident that **the material near the N.A.** is always subjected to relatively **low stresses** compared with the areas most removed from the axis.

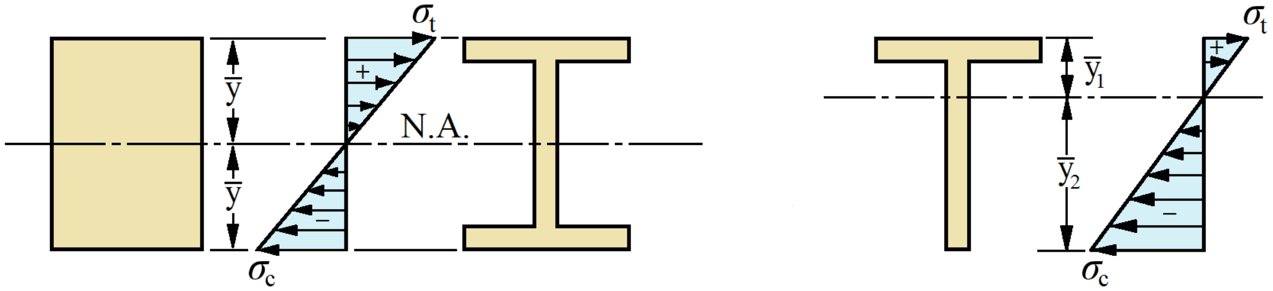

Fig. 4.3 Typical bending stress distributions.

4.4 Second moment of area

Consider the rectangular beam cross-section shown in Fig. 4.4 and an element of area *dA*, thickness *dy*, breadth *B* and distance *y* from the N.A. which by symmetry passes through the center of the section.

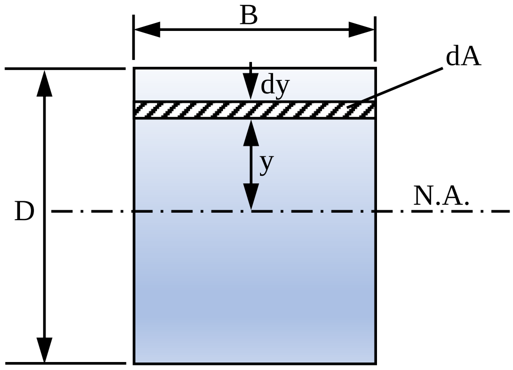

Fig. 4.4 symmetrical rectangular beam

The ***second moment of area I*** about the center of the rectangle has been defined earlier as

$$I = \int y^2 \, dA$$

For the rectangular section the second moment of area, an axis through the center about the N.A. is given by

$$I_{N.A} = \int_{-D/2}^{D/2} y^2 \, B \, dy = B \int_{-D/2}^{D/2} y^2 dy$$

$$= B\left[\frac{y^3}{3}\right]_{-D/2}^{D/2} = \frac{\boldsymbol{B\,D^3}}{\mathbf{12}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.5)$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of 0 to *D*.

$$I = B\left[\frac{y^3}{3}\right]_{0}^{D} = \frac{\boldsymbol{B\,D^3}}{\mathbf{3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.6)$$

For symmetrical sections, for instance, the I-section shown in Fig. 4.5,

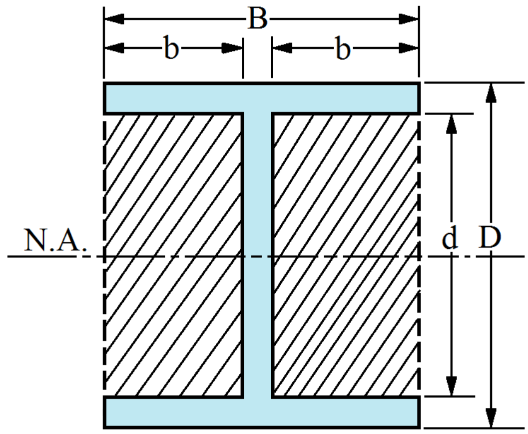

Fig. 4.5 symmetrical I-section.

$$\boldsymbol{I_{N.A.} = I \text{ of dotted rectangle} - I \text{ of shaded portions}}$$

$$= \frac{B\,D^3}{12} - 2\left(\frac{b\,d^3}{12}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.7)$$

For ***unsymmetrical sections*** such as the T-section as shown in Fig. 4.6

$$\boldsymbol{I_{N.A} = I_{ABCD} - I_{\text{shaded areas}} + I_{EFGH}}$$

(about *DC*) (about *DC*) (about *HG*)

(Each of these quantities may be written in the form $BD^3/3$).

Note: It can also be calculated using *parallel axis theorem*.

As an alternative procedure it is possible to determine the second moment of area of each rectangle about an axis through its own centroid ($I_G = BD^3/12$) and to "shift" this value to the equivalent value about the N.A. by means of the ***parallel* axis *theorem.***

$$\boldsymbol{I_{N.A.} = I_G + Ah^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.8)$$

where ***A*** is the area of the rectangle and ***h*** the distance of its centroid ***G*** from the N.A.

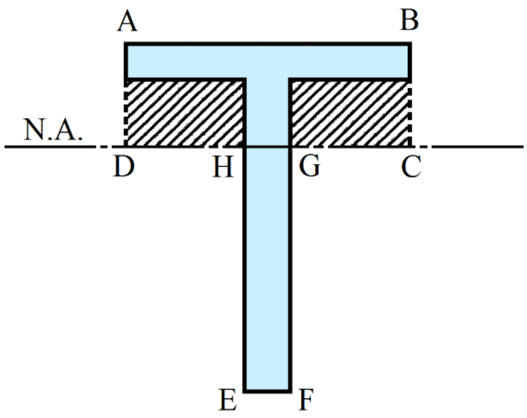

Fig. 4.6 unsymmetrical T-section.

4.5 Bending of composite beams

A composite beam is one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together, two timber joists and a reinforcing steel plate, then it is termed a ***flitched beam.***

The method of solution in such a case is to replace one of the materials by an ***equivalent section*** of the other.

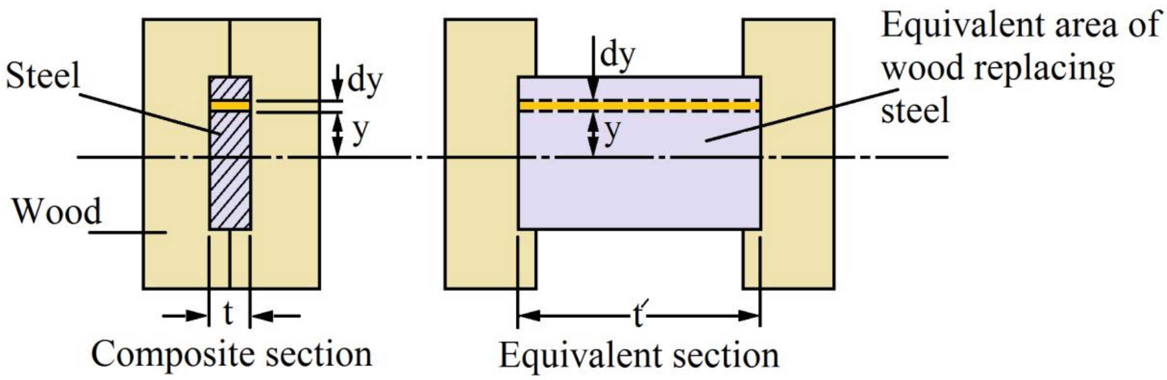

Fig. 4.7 Bending of composite or flitched beams: original beam cross-section and equivalent of uniform material (wood) properties.

The moment at any section must be the same in the equivalent section as in the original so that the **force** at any given ***dy*** in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma\, t\, dy = \sigma' t' dy$$

$$\boldsymbol{\sigma\, t = \sigma' t'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.9)$$

$$\epsilon\, E\, t = \epsilon' E' t'$$

Since $$\frac{\sigma}{\epsilon} = E$$

Similarity the **strains** must be equal,

$$\epsilon = \epsilon'$$

$$E\,t = E'\,t' \quad \text{or} \quad \frac{t'}{t} = \frac{E}{E'} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.10)$$

$$\boldsymbol{t' = \frac{E}{E'}\,t} \quad \ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.11)$$

Thus to replace the steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows:

from equ. (4.9) $$\frac{\sigma}{\sigma'} = \frac{t'}{t}$$

and from equ. (4.10) $$\frac{\sigma}{\sigma'} = \frac{E}{E'} \quad or \quad \boldsymbol{\sigma = \frac{E}{E'}\,\sigma'} \quad \ldots\ldots\ldots. \quad (4.12)$$

stress in steel** = **modular ratio** × **stress in equivalent wood

Example 4.1

An I-section girder, **200** mm wide by **300** mm deep, with flange and web of thickness **20** mm is used as a simply supported beam over a span of **7** m. The girder carries a distributed load of **5** kN/m and a concentrated load of **20** kN at mid-span. Determine: **(a)** the second moment of area of the cross-section of the girder, **(b)** the maximum stress set-up.

Solution:

(a) The second moment of area of the cross-section may be found in two ways.

- ***Method 1 -Use of standard forms***

For sections with symmetry about the N.A., the standard *I* value for a rectangle about an axis through its centroid is $bd^3/12$, (as shown in Fig. 4.8).

$$I_{\text{girder}} = I_{\text{rectangle}} - I_{\text{shaded portions}}$$

$$= \left[\frac{200 \times 300^3}{12}\right]10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right]10^{-12}$$

$$= (4.5 - 2.64)10^{-4} = \mathbf{1.86 \times 10^{-4}\ m^4}$$

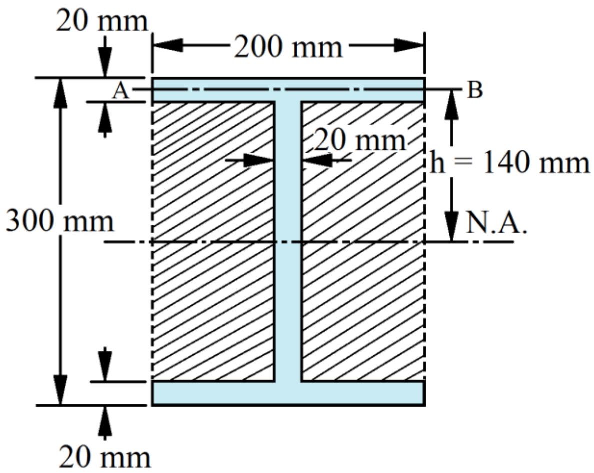

Fig. 4.8 symmetrical I-section.

- ***Method 2 - Parallel axis theorem***

Consider the section divided into three parts - the web and the two flanges.

$$I_{\text{N.A.}} \text{ for the web} = \frac{bd^3}{12} = \left[\frac{20 \times 260^3}{12}\right]10^{-12}$$

$$I \text{ of flange about } AB = \frac{bd^3}{12} = \left[\frac{200 \times 20^3}{12}\right]10^{-12}$$

Therefore using the parallel axis theorem

$$I_{\text{N.A.}} \text{ for flange} = I_{AB} + Ah^2$$

where h is the distance between the N.A. and AB,

$$I_{\text{N.A.}} \text{ for flange} = \left[\frac{200 \times 20^3}{12}\right]10^{-12} + \left[(200 \times 20)140^2\right]10^{-12}$$

Therefore, the total $I_{\text{N.A.}}$ of girder

$$= 10^{-12}\left\{\left[\frac{20 \times 260^3}{12}\right] + 2\left[\frac{200 \times 20^3}{12}\right] + (200 \times 20 \times 140^2)\right\}$$

$$= 10^{-6}(29.3 + 0.267 + 156.8)$$

$$\mathbf{= 1.86 \times 10^{-4}\ m^4}$$

(b) The maximum stress may be found from the simple bending theory of eqn. (4.4),

$$\sigma_{max} = \frac{M_{max}\, y_{max}}{I}$$

Now the maximum B.M. for a beam carrying a u.d.1. is at the center and given by ***wL²/8*** **.** Similarly, the value for the central concentrated load is ***WL/4*** also at the center.

$$M_{\max} = \frac{WL}{4} + \frac{wL^2}{8} = \left[\frac{20 \times 10^3 \times 7}{4}\right] + \left[\frac{5 \times 10^3 \times 7^2}{8}\right] \text{N m}$$

$$= (35.0 + 30.63)10^3 = 65.63 \text{ kN m}$$

$$\sigma_{\max} = \frac{65.63 \times 10^3 \times 150 \times 10^{-3}}{1.9 \times 10^{-4}} = \mathbf{51.8\ MN/m^2}$$

The maximum stress in the girder is 52 MN/m^2, this value being compressive on the upper surface and tensile on the lower surface.

Example 4.2

A uniform T-section beam is 100 mm wide and 150 mm deep with a flange thickness of 25 mm and a web thickness of 12 mm. If the limiting bending stresses for the material of the beam are 80 MN/m^2 in compression and 160 MN/m^2 in tension, find the maximum u.d.1. that the beam can carry over a simply supported span of 5 m.

Solution:

The second moment of area in the simple bending theory is taken about the N.A. this always passes through the centroid of the section we can take moments of area about the base to determine the position of the centroid and hence the N.A.

$$\bar{y} = \frac{\sum yA}{\sum A}$$

$$\bar{y} = \frac{(100 \times 25 \times 137.5)10^{-9} + (125 \times 12 \times 62.5)10^{-9}}{[(100 \times 25) + (125 \times 12)]10^{-6}}$$

$$\bar{y} = \frac{437.5 \times 10^{-6}}{4000 \times 10^{-6}} = 109.4 \times 10^{-3} = 109.4 \text{ mm}$$

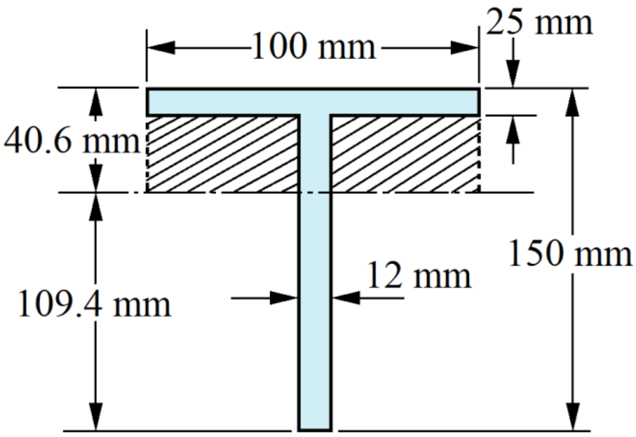

Fig. 4.9 unsymmetrical T-section.

The second moment of area *I* dividing the section into convenient rectangles with their edges.

$$I = \frac{1}{3}\left[(100 \times 40.6^3) - \left(88 \times 15.6^3\right) + (12 \times 109.4^3)\right]10^{-12}$$

$$= \frac{1}{3}(6.69 - 0.33 + 15.71)10^{-6} = 7.36 \times 10^{-6}\ \text{m}^4$$

Now the maximum compressive stress will occur on the upper surface where $y = 40.6$ mm, and, using the limiting compressive stress value quoted,

$$M = \frac{\sigma I}{y} = \frac{80 \times 10^6 \times 7.36 \times 10^{-6}}{40.6 \times 10^{-3}} = 14.5\ \text{kN m}$$

This suggests a maximum allowable B.M. of 14.5 kN m. It is now necessary to check the tensile stress criterion, which must apply on the lower surface,

$$M = \frac{\sigma I}{y} = \frac{160 \times 10^6 \times 7.36 \times 10^{-6}}{109.4 \times 10^{-3}} = 10.76\ \text{kN m}$$

The greatest moment that can be applied to retain stresses within both conditions quoted is $M = 10.76$ kN m.

But for a simply supported beam with u.d.l.,

$$M_{max} = \frac{w L^2}{8}$$

$$w = \frac{8\,M}{L^2} = \frac{8 \times 10.76 \times 10^3}{5^2} = \mathbf{3.4\ kN/m}$$

The u.d.1. must be limited to 3.4 kN/m.

Example 4.3

A flitched beam consists of two 50 mm × 200 mm wooden beams and a 12 mm × 80 mm steel plate. The plate is placed centrally between the wooden beams and recessed into each, so that, when rigidly joined, the three units form a 100 mm × 200 mm section as shown in Fig. 4.10. Determine the moment of resistance of the flitched beam when the maximum bending stress in the timber is 12 MN/m^2. What will be the maximum bending stress in the steel?

For steel E = 200 GN/m^2; for wood E = 10 GN/m^2.

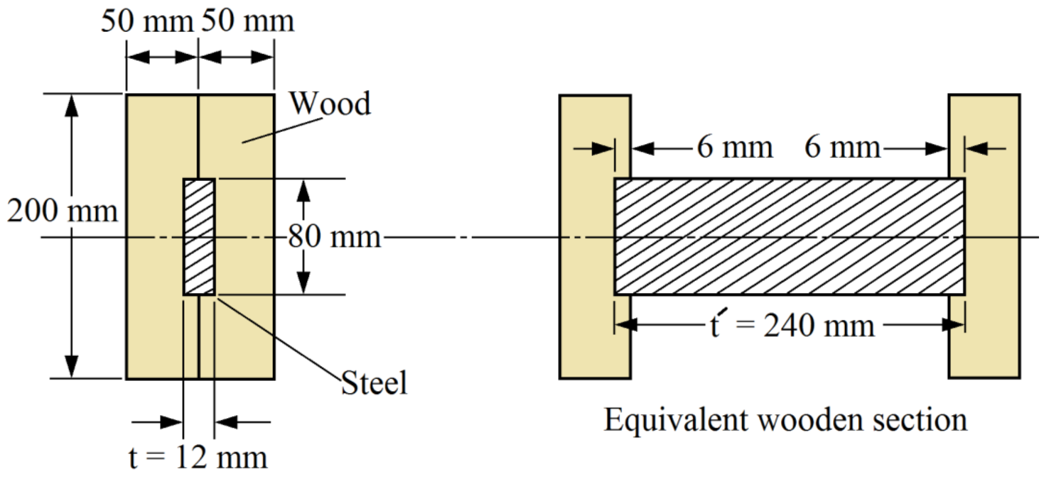

Fig. 4.10 flitched beam.

The thickness ***t′*** of the wood equivalent to the steel which it replaces is given by eqn. (4.11),

$$t' = \frac{E}{E'}\, t = \frac{200 \times 10^9}{10 \times 10^9} \times 12 = 240 \text{ mm}$$

Then, for the equivalent section

$$I_{\text{N.A.}} = 2\left[\frac{50 \times 200^3}{12}\right] - 2\left[\frac{6 \times 80^3}{12}\right] + \left[\frac{240 \times 80^3}{12}\right] 10^{-12}$$

$$= (66.67 - 0.51 + 10.2)\, 10^{-6} = 76.36 \times 10^{-6} \text{ m}^4$$

Now, the maximum stress in the timber is 12 MN/m^2, and this will occur at y = 100 mm; thus, from the bending theory,

$$M = \frac{\sigma I}{y} = \frac{12 \times 10^6 \times 76.36 \times 10^{-6}}{100 \times 10^{-3}} = \mathbf{9.2\ kN\ m}$$

The maximum stress in the steel with this moment applied is then determined by finding first the maximum stress in the equivalent wood at the same position at y = 40 mm. Therefore, maximum stress in equivalent wood;

$$\sigma'_{max} = \frac{M\,y}{I} = \frac{9.2 \times 10^3 \times 40 \times 10^{-3}}{76.36 \times 10^{-6}} = 4.82 \times 10^6 \ \text{N/m}^2$$

The maximum stress in the steel is given by

$$\sigma_{max} = \frac{E}{E'}\,\sigma'_{max} = \frac{200 \times 10^9}{10 \times 10^9} \times 4.82 \times 10^6$$

$$= 96 \times 10^6 = \mathbf{96\ \ MN/m^2}$$

Problems

4.1 Determine the second moments of area about the axes XX for the sections shown in Fig. 4.11. [15.69, 7.88, 41.15, 24; all × 10^{-6} m^4]

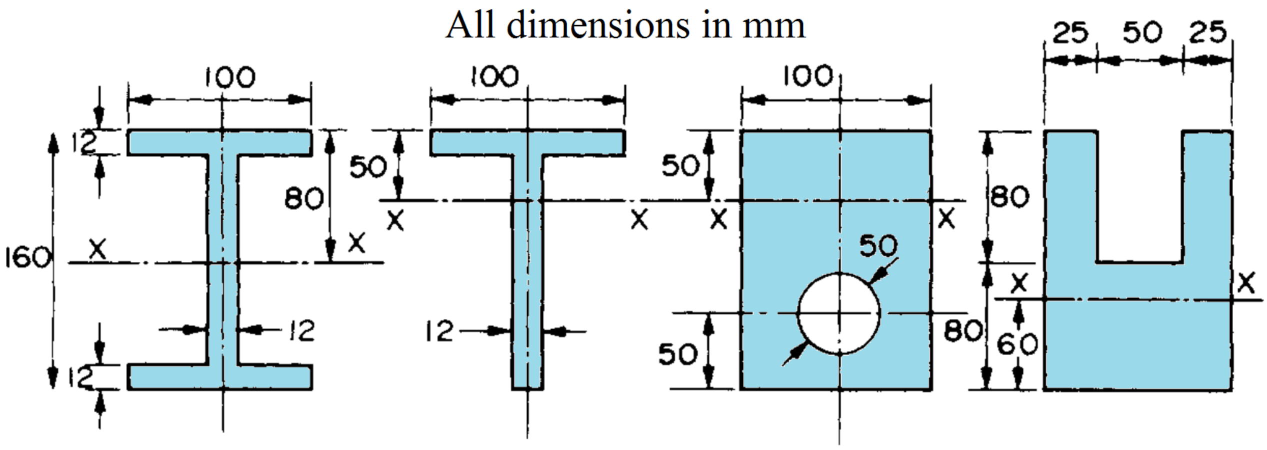

Fig. 4.11.

4.2 A rectangular section beam has a depth equal to twice its width. It is the same material and mass per unit length as an I-section beam 300 mm deep with flanges 25 mm thick and 150 mm wide and a web 12 mm thick. Compare the flexural strengths of the two beams. [8.59: 1]

4.3 A conveyor beam has the cross-section shown in Fig. 4.12 and it is subjected to a bending moment in the plane *YY*. Determine the maximum permissible bending moment, which can be applied to the beam (a) for bottom flange in tension, and (b) for bottom flange in compression, if the safe stresses for the and compression are 30 MN/m^2 and 150 MN/m^2 respectively. [32.3, 84.8 kN m]

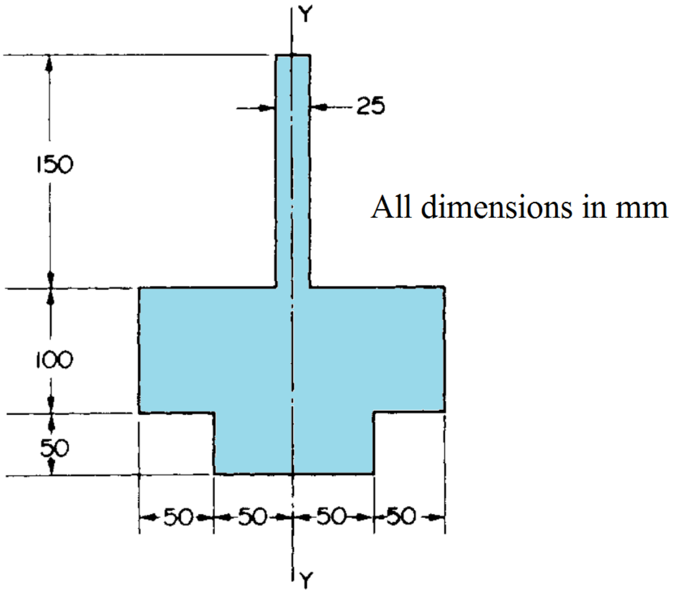

Fig. 4.12.

4.4 A horizontal steel girder has a span of 3 m and is built-in at the left-hand end and freely supported at the other end. It carries a uniformly distributed load of 30 kN/m over the whole span, together with a single concentrated load of 20 kN at a point 2 m from the left-hand end. The supporting conditions are such that the reaction at the left-hand end is 65 kN.

(a) Determine the bending moment at the left-hand end and draw the B.M. diagram.

(b) Give the value of the maximum bending moment.

(c) If the girder is 200 mm deep, and has a second moment of area of 40 × 10^{-6} m^4, determine the maximum stress resulting from bending. [40 kN m; 100 MN/m^2]

4.5 Figure 4.13 represents the cross-section of an extruded alloy member, which acts as a simply supported beam with the 75 mm wide flange at the bottom. Determine the moment of resistance of the section if the maximum permissible stresses in tension and compression are respectively 60 MN/m^2 and 45 MN/m^2. [2.62 kN m]

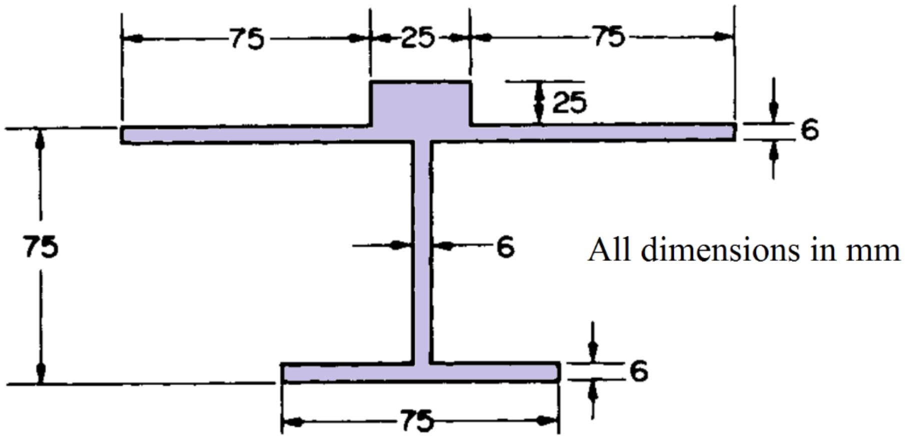

Figure 4.13.

4.6 A trolley consists of a pressed steel section as shown in Fig. 4.14. At each end, there are rollers at 350 mm centers. If the trolley supports a mass of 50 kg evenly distributed over the 350 mm length of the trolley calculate, using the data given in Fig. 4.14, the maximum compressive and tensile stress due to bending in the pressed steel section. State clearly your assumptions. [14.8, 42.6 MN/m^2]

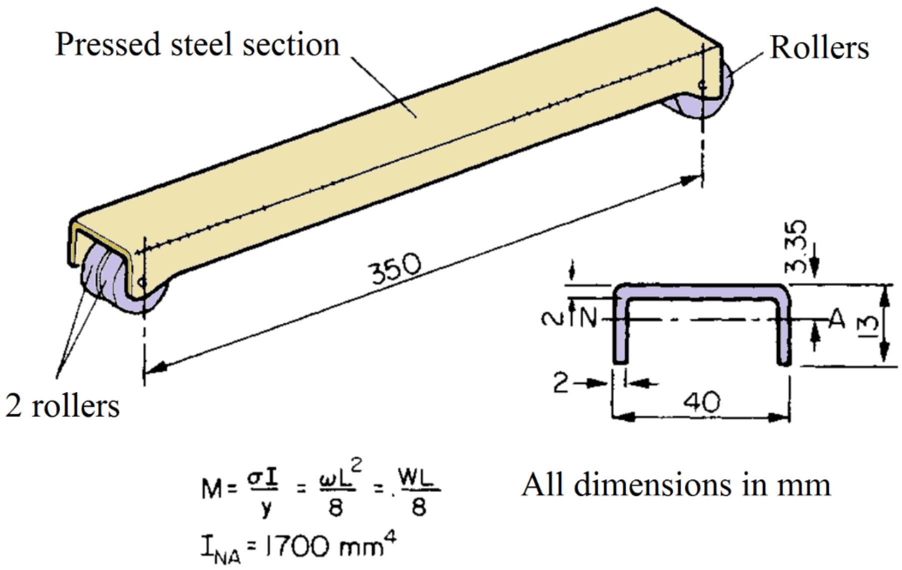

Fig. 4.14.