Chapter Three

Shearing Force and Bending Moment Diagrams

3.1 Shearing force (S.F.) sign convention

The shearing force (S.F.) at the section is defined as the algebraic sum of the forces taken on one side of the section forces upwards to the left of a section or downwards to the right of the section are positive. Fig. 3.1 (a) shows a positive S.F. system at X-X and Fig. 3.1 (b) shows a negative S.F. system.



Fig. 3.1 S.F. sign convention.

3.2 Bending moment (B.M.) sign convention

The bending moment (B.M.) is defined as the algebraic sum of the moments of the forces about the section, taken on either side of the section. As for S.F., a convenient sign convention must be adopted. Fig. 3.2(a) shows a positive bending moment system, and Fig. 3.2(b) illustrates a negative B.M. system.



Fig. 3.2 B.M. sign convention.

These diagrams, which illustrate the variation in the B.M. and S.F. values along the length of a beam or structure for any fixed loading condition, are termed B.M. and S.F. diagrams.

In the case of a cantilever carrying a concentrated load *W* at the end (Fig. 3.3), the S.F. at any section *X*-*X*, distance *x* from the free end, is S.F. = -W. This will be true whatever the value of *x*, and so the S.F. diagram becomes a rectangle. The B.M. at the same section *X*-*X* is (-Wx) and this will increase linearly with *x*. Therefore the B.M. diagram is a triangle. If the cantilever carries a uniformly distributed load, the S.F. at *X*-*X* is the net load to one side of *X*-*X*, i.e. (-wx). Therefore, the S.F. diagram becomes triangular, increasing to a maximum value of (-wL) at the support. The B.M. at *X*-*X* is obtained by treating the load to the left of *X*-*X* as a concentrated load of the same value acting at the centre of gravity.

B.M. at
$$X-X = -wx \frac{x}{2} = \frac{-wx^2}{2}$$

Plotted against *x* this produces the parabolic B.M. diagram as shown.



Fig. 3.3. S.F., B.M. diagrams for standard cases.

3.3. S.F. and B.M. diagrams for beams carrying concentrated loads only

Consider the simply supported beam carrying concentrated loads as shown in Fig. 3.4.



Fig. 3.4 simply supported beam carrying concentrated loads.

The values of the reactions at the ends of the beam may be calculated by taking moments about F.

$$\sum M_F = 0$$

 $R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$
 $R_A = 10 \text{ kN}$
 $\sum F_y = 0$
 $R_A + R_F = 10 + 20 + 30 - 20 = 40$
 $R_F = 30 \text{ kN}$

Therefore, the shear force (S.F.) at X-X is +20 kN, the resultant force at X-X tending to shear the beam is 20 kN.



The summation of the moments of the forces at *X*-*X*, the resultant of the B.M. being 40 kN.m, as shown in Fig. 3.6.



Fig. 3.6 Total B.M. at X-X.

The Bending Moment at the different points on the beam;





(a) Between *B*, and *C* the S.F. is zero and the B.M. remains constant.

(b) Between A, and B the S.F. is positive and the slope of the B.M. diagram is positive, and vice versa between E and F.

(c) The difference in B.M. between A and B = 20 kN m = area of S.F. diagram between A and B.

3.4. S.F. and B.M. diagrams for uniformly distributed loads

Consider the simply supported beam carrying a uniformly distributed load (u.d.1.) w = 25 kN/m across the complete span as shown in Fig. 3.8.





The beam is symmetrical therefore, it is necessary to evaluate the reactions and each reaction will take half the applied load.

$$R_A = R_B = \frac{25 \times 12}{2} = 150 \text{ kN}$$

Therefore, the S.F. at A equals + 150 kN.

Consider the beam divided into six equal parts 2 m long. The S.F. at any point is,

= 150 - 10 downwards between A and C

$$= 150 - (25 \times 2) = +100 \text{ kN}$$

From the Figure above, the S.F. at *A* is + 150 kN and that between *A* and *B*. The S.F. decreases uniformly producing the required sloping straight line, as shown in Fig. 3.8. Alternatively, this decreases gradually by the amount of the applied load by $25 \times 12 = 300$ kN to -150 kN at *B*.

For evaluating the B.M. it is assumed that a u.d.1. can be replaced by a concentrated load of equal value acting in the middle of its extended. When taking moments about *C*, the u.d.1. between *A* and *C* has an effect equivalent to that of a concentrated load of $25 \times 2 = 50$ kN acting in the center of *AC* at 1 m from *C*.

B.M. at $C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250$ kN m

Similarly, for moments at *D* the u.d.1. on *AD* can be replaced by a concentrated load of

 $25 \times 4 = 100$ kN at the center of *AD*, at *C*.

B.M. at
$$D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400$$
 kN m

Similarly,

B.M. at $E = (R_A \times 6) - (25 \times 6) \times 3 = 900 - 450 = 450$ kN m

The B.M. diagram will be symmetrical about the beam centerline; therefore, the values of B.M. at F and G will be the same as those at D and C respectively. The final diagram is parabolic as shown in Fig. 3.8. Since the B.M. is a maximum when the S.F. is zero.

3.5. S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads

Consider the beam loaded with a combination of concentrated loads and u.d.1.s. as shown in Fig. 3.9.



Fig. 3.9 Beam loaded concentrated loads and uniformly distributed loads.

Taking moments about E

$$(R_A \times 8) + (40 \times 2) = (10 \times 2 \times 7) + (20 \times 6) + (20 \times 3) + (10 \times 1) + (20 \times 3 \times 1.5)$$

$$8 R_A + 80 = 420$$

$$R_A = 42.5 \text{ kN} (= \text{S.F. at } A)$$

Now

$$R_A + R_E = (10 \times 2) + 20 + 20 + 10 + (20 \times 3) + 40 = 170$$

$$R_E = 127.5 \text{ kN}$$

In order to plot the B.M. diagram the following values must be calculated,

B.M. at A = 0 B.M. at $B = (42.5 \times 2) - (10 \times 2 \times 1) = 85 - 20$ = 65 kN mB.M. at $C = (42.5 \times 5) - (10 \times 2 \times 4) - (20 \times 3) = 212.5 - 80 - 60$ = 72.5 kN m B.M. at $D = (42.5 \times 7) - (10 \times 2 \times 6) - (20 \times 5) - (20 \times 2) - (20 \times 2 \times 1)$ = 297.5 - 120 - 100 - 40 - 40 = 297.5 - 300= -2.5 kN mB.M. at $E = (-40 \times 2)$ working from r.h.s. = -80 kN mB.M. at F= 0 B.M. midway between A and $B = (42.5 \times 1) - (10 \times 1 \times 0.5)$ = 42.5 - 5 = 37.5 kN m Similarly, B.M. midway between C and D = 45 kN m

B.M. midway between *D* and E = -39 kN m

The B.M. and S.F. diagrams are shown in Fig. 3.9.

3.6. Points of contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a **point of inflexion** and will be shown later to occur at the point or points on the beam where the B.M. is zero.

For the beam shown in Fig. 3.9, it is evident from the B.M. diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance x from C then at this point

B.M. =
$$(42.5)(5 + x) - (10 \times 2)(4 + x) - 20(3 + x) - 20x - \frac{20x^2}{2}$$

= $212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2$

Dr. Qahtan A. Jawad

Mechanical Engineering Department

 $= 72.5 - 17.5 x - 10 x^{2}$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5 x - 10 x^{2}$$

x = 1.96 or - 3.7

where

Since the last answer can be ignored (being outside the beam), the point of contraflexure must be situated at 1.96 m to the right of *C*.

3.7. Relationship between shear force Q, bending moment M and intensity of loading w

Consider the beam *AB* carrying a uniform loading intensity of *w* kN/m (uniformly distributed load) as shown in Fig. 3.10. By symmetry, each reaction takes half the total load, (wL/2).



Fig. 3.10 the beam carrying a uniform loading intensity.

The B.M. at any point *C*, distance *x* from *A*, is given by

$$M = \frac{wL}{2}x - (wx)\frac{x}{2}$$
$$M = \frac{1}{2}wLx - \frac{1}{2}wx^{2}$$

Differentiating,

$$\frac{dM}{dx} = \frac{1}{2}wL - wx$$

S.F. at $C = \frac{1}{2}wL - wx = Q$ (3.1)
 dM

$$\frac{dM}{dx} = Q \tag{3.2}$$

Differentiating eqn. (3.1),

$$\frac{dQ}{dx} = -w \tag{3.3}$$

Note:

(a) The maximum or minimum B.M. occurs when dM/dx = 0, but dM/dx = Q

When S.F. is zero B.M. is a maximum or minimum.

(**b**) The slope of the B.M. diagram = dM/dx = Q.

When Q = 0, the slope of the B.M. diagram is zero, therefore the B.M. is constant.

(c) Q represents the slope of the B.M. diagram it follows that

When the S.F. is positive, the slope of the B.M. diagram is positive, and where the

S.F. is negative, the slope of the B.M. diagram is negative.

(d) The point of inflexion or contraflexure occurs when the B.M. is zero.

3.8. S.F. and B.M. diagrams for an applied couple or moment

In general, the couple or moment can be applied in two ways:

(a) Couple or moment applied with horizontal loads

Consider the moment F.d is applied on the beam AB by the horizontal loads at a point C, and the distance a from A, as shown in Fig. 3.11.



Fig. 3.11 the moment is applied on the beam AB by horizontal loads.

Since this will tend to lift the beam at A, R_A acts downwards.

Moments about *B*: $R_A \times L = Fd$

$$R_A = \frac{Fd}{L}$$

and for vertical equilibrium

$$R_B = R_A = \frac{Fd}{L}$$

The S.F. diagram can be drawn as the horizontal loads have no effect on the vertical shear.

The B.M. at any section between A and C is

$$M_{XX} = -R_A x = \frac{-Fd}{L} x$$

Thus the value of the B.M. increases linearly from zero at A to $\left(\frac{-Fd}{I}a\right)$ at C.

Similarly, the B.M. at any section between C and B is

$$M_{XX} = -R_A x + Fd = R_B x' = \frac{Fd}{L} x'$$

The value of the B.M. increases linearly from zero at *B* to $(\frac{Fd}{L}b)$ at *C*, as shown in Fig. 3.11.

(b) moment applied with vertical loads

Consider the moment is applied with vertical loads on the beam *AB* as shown in Fig. 3.12; taking moments about *B*:

$$R_{A} L = F (d + b)$$
$$R_{A} = \frac{F (d + b)}{L}$$
$$R_{B} = \frac{F (a - d)}{L}$$

For the B.M. diagram an equivalent system is used. The offset load F is replaced by a moment and a force acting at C, as shown in Fig. 3.12. Thus

B.M. between A and $C = R_A x$

$$=\frac{F\left(d+b\right)}{L}x$$

B.M. increasing linearly from zero to $\frac{F(d+b)}{L}a$ at *C*. Similarly,

B.M. between *C* and
$$B = R_B x'$$

$$=\frac{F(a-d)}{L}x'$$

B.M. increasing linearly from zero to $\frac{F(a-d)}{L}b$ at C.



Fig. 3.12 the moment is applied on the beam *AB* by vertical loads.

Example 3.1

Draw the S.F. and B.M. diagrams for the beam loaded as shown in Fig. 3.13, and determine (**a**) the position and magnitude of the maximum B.M., and (**b**) the position of any point of contraflexure.



Fig. 3.13 the S.F. and B.M. diagrams for the beam loaded.

Solution:

(a) Taking the moments about *A*,

$$5R_B = (5 \times 1) + (7 \times 4) + (2 \times 6) + (4 \times 5) \times 2.5$$
$$R_B = \frac{5 \times 28 \times 12 \times 50}{5} = 19 \text{ kN}$$
$$R_A + R_B = 5 + 7 + 2 + (4 \times 5) = 34$$
$$R_A = 34 - 19 = 15 \text{ kN}$$

and since

Calculation of bending moments

B.M. at A and C	= 0
B.M. at <i>B</i>	$=$ $-2 \times 1 = -2 \text{ kN m}$
B.M. at <i>D</i>	= $-(2 \times 2) + (19 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13$ kN m
B.M. at <i>E</i>	$= + (15 \times 1) - (4 \times 1 \times \frac{1}{2}) = + 13 \text{ kN m}$

The B.M. diagram is shown in Fig. 3.13. The B.M. at any point between D and E at a distance x from A will be given by

$$M_{xx} = 15 \ x - 5 \ (x - 1) - \frac{4x^2}{2} = 10 \ x + 5 - 2 \ x^2$$

The maximum B.M. position is given when dM/dx = 0.

$$\frac{dM}{dx} = 10 - 4x = 0$$
 , $x = 2.5$ m

The maximum B.M. will be given by the point (or points) at which dM/dx is zero (or equal the shear force). From the S.F. diagram, the maximum B.M. occurs midway between *D* and *E* at **1.5 m** from *E*.

B.M. at this point =
$$(2.5 \times 15) - (5 \times 1.5) - (4 \times 2.5 \times \frac{2.5}{2})$$

= + 17.5 kN m

(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure between *B* and *D*. Then, B.M. at distance y from *C* will be given by

$$M_{yy} = -2y + 19 (y - 1) - 4 (y - 1) \frac{1}{2} (y - 1)$$
$$= -2y + 19y - 19 - 2y^{2} + 4y - 2 = 0$$

The point of contraflexure occurs when B.M. = 0, where Myy = 0,

$$0 = -2y^{2} + 21y - 21$$
$$2y^{2} - 21y + 21 = 0$$
$$y = \frac{21 \pm \sqrt{21^{2} - 4 \times 2 \times 21}}{4} = 1.12 \text{ m or } 9.38 \text{ m}$$

The point of contraflexure occurs at 0.12 m to the left of B.

Example 3.2

A beam *ABC* is 9 m long and supported at *B* and *C*, 6 m apart. The beam carries a triangular distribution of load over the portion *BC* together with an applied counterclockwise couple of moment 80 kN m at *B* and a u.d.1. of 10 kN/m over *AB*, as shown in Fig. 3.14. Draw the S.F. and B.M. diagrams for the beam.



Fig. 3.14 the S.F. and B.M. diagrams for the beam.

Solution:

Taking moments about *B*,

$$(R_C \times 6) + (10 \times 3 \times 1.5) + 80 = (\frac{1}{2} \times 6 \times 48) \times \frac{1}{3} \times 6$$

$$6 R_C + 45 + 80 = 288$$

$$R_C = 27.2 \text{ kN}$$

$$R_C + R_B = (10 \times 3) + (\frac{1}{2} \times 6 \times 48)$$

$$= 30 + 144 = 174$$

$$R_B = 146.8 \text{ kN}$$

and

At any distance x from C between C and B the shear force is given by

$$\mathbf{S.F.}_{XX} = -\frac{1}{2}w x + R_C$$

and by proportions

$$\frac{w}{x} = \frac{48}{6} = 8$$

$$w = 8 x \text{ kN/m}$$
S.F._{XX} = $R_C - \frac{1}{2} \times 8 x \times x$

$$= R_C - 4 x^2$$

$$= 27.2 - 4 x^2$$
B.M._{XX} = $-\left(\frac{1}{2}wx\right)\frac{x}{3} + R_C x$

$$= 27.2 x - \frac{4 x^3}{3}$$

For a maximum value,

$$\frac{d(B.M.)}{dx} = S.F. = 0$$

0 = 27.2 - 4 x²
4 x² = 27.2

x = 2.61 m from C

B.M._{max} = 27.2 (2.61)
$$-\frac{4}{3}$$
 (2.61)³
= 47.3 kN m

B.M. at A and C = 0

B.M. immediately to left of $B = -(10 \times 3 \times 1.5) = -45$ kN m

Problems

3.1 A beam *AB*, 1.2 m long, is simply supported at its ends *A* and *B* and carries two concentrated loads, one of 10 kN at *C*, the other 15 kN at *D*. Point *C* is 0.4 m from *A*, point *D* is 1 m from *A*. Draw the S.F. and B.M. diagrams for the beam inserting principal values.

[9.17, - 0.83, - 15.83 kN, 3.67, 3.17 kN m]

3.2 The beam of question 3.1 carries an additional load of 5 kN upwards at point *E*, 0.6 m from *A*. Draw the S.F. and B.M. diagrams for the modified loading. What is the maximum B.M.?

[6.67, - 3.33, 1.67, - 13.33 kN, 2.67, 2, 2.67 kN m]

3.3 A cantilever beam *AB*, 2.5 m long is rigidly built in at *A* and carries vertical concentrated loads of 8 kN at *B* and 12 kN at *C*, 1 m from *A*. Draw S.F. and B.M. diagrams for the beam inserting principal values. [-8, -20 kN; -11.2, -31.2 kN m]

3.4 A beam *AB*, 5 m long, is simply supported at the end *B* and at a point *C*, 1 m from *A*. It carries vertical loads of 5 kN at *A* and 20 kN at *D*, the center of the span *BC*. Draw S.F. and B.M. diagrams for the beam inserting principal values. [-5, 11.25, -8.75 kN; -5, 17.5 kN m]

3.5 A beam *AB*, 3 m long, is simply supported at *A* and *E*. It carries a 16 kN concentrated load at *C*, 1.2 m from *A*, and a u.d.1. of 5 kN/m over the remainder of the beam. Draw the S.F. and B.M. diagrams and determine the value of the maximum B.M. [12.3, -3.7, -12.7 kN; 14.8 kN m]

3.6 A simply supported beam has a span of 4 m and carries a uniformly distributed load of 60 kN/m together with a central concentrated load of 40 kN. Draw the S.F. and B.M. diagrams for the beam and hence determine the maximum B.M. acting on the beam.

[S.F. 140 ± 20, - 140 kN; B.M. 0, 160, 0 kN m]

3.7 A 2 m long cantilever is built-in at the right-hand end and carries a load of 40 kN at the free end. In order to restrict the deflection of the cantilever within reasonable limits an upward load of 10 kN is applied at mid-span. Construct the S.F. and B.M. diagrams for the cantilever and hence determine the values of the reaction force and moment at the support. [30 kN, 70 kN m]

3.8 A beam 4.2 m long overhangs each of two simple supports by 0.6 m. The beam carries a uniformly distributed load of 30 kN/m between supports together with concentrated loads of 20 kN and 30 kN at the two ends. Sketch the S.F. and B.M. diagrams for the beam and hence determine the position of any points of contraflexure.

[S.F. - 20, 43, - 47, 30 kN; B.M. - 12, 18.75, - 18 kN m; 0.313 and 2.553 m from l.h. support]

3.9 A beam *ABCDE*, with *A* on the left, is 7 m long and is simply supported at *B* and *E*. The lengths of the various portions are AB = 1.5 m, BC = 1.5 m, CD = 1 m and DE = 3 m. There is a uniformly distributed load of 15 kN/m between *B* and a point 2 m to the right of *B* and concentrated loads of 20 kN act at *A* and *D* with one of 50 kN at *C*.

(a) Draw the S.F. diagrams and hence determine the position from A at which the S.F. is zero.

(b) Determine the value of the B.M. at this point.

(c) Sketch the B.M. diagram approximately to scale, quoting the principal values.

[3.32 m; 69.8 kN m; 0, - 30, 69.1, 68.1, 0 kN m]

3.10 A beam *ABCDE* is simply supported at *A* and *D*. It carries the following loading: a distributed load of 30 kN/m between *A* and B; a concentrated load of 20 kN at *B*; a concentrated load of 20 kN at *C*; a concentrated load of 10 kN at *E*; a distributed load of 60 kN/m between *D* and *E*. Span AB = 1.5 m, BC = CD = DE = 1 m. Calculate the value of the reactions at *A* and *D* and hence draw the S.F. and B.M. diagrams. What are the magnitude and position of the maximum B.M. on the beam?

[41.1, 113.9 kN; 28.15 kN m; 1.37 m from A]

3.11 A beam, 12 m long, is to be simply supported at 2 m from each end and to carry a u.d.1. of 30 kN/m together with a 30 kN point load at the right-hand end. For ease transportation of the beam is to be jointed in two places, one joint being situated 5 m from the left-hand end. What load (to the nearest kN) must be applied to the left-hand end to ensure that there is no B.M. at the joint (the joint is to be a point of contraflexure)? What will be the best position on the beam for the other joint? Determine the position and magnitude of the maximum B.M. present on the beam.

[114 kN, 1.6 m from r.h. reaction; 4.7 m from l.h. reaction; 43.35 kN m]

3.12 A horizontal beam *AB* is 4 m long and of constant flexural rigidity. It is rigidly built-in at the left-hand end *A* and simply supported on a non-yielding support at the right-hand end *E*. The beam carries uniformly distributed vertical loading of 18 kN/m over its whole length, together with a vertical downward load of 10 kN at 2.5 m from the end *A*. Sketch the S.F. and B.M. diagrams for the beam, indicating all main values. [S.F. 45, - 10, - 37.6 kN; B.M. - 18.6, + 36.15 kN m]

3.14 A beam *ABCD*, 6 m long, is simply supported at the right-hand end *D* and at a point *B* l m from the left-hand end *A*. It carries a vertical load of 10 kN at *A*, a second concentrated load of 20 kN at *C*, 3 m from *D*, and a uniformly distributed load of 10 kN/m between *C* and *D*. Determine:

(a) The values of the reactions at *B* and *D*.

(b) The position and magnitude of the maximum bending moment.

[33 kN, 27 kN, 2.7 m from *D*, 36.45 k Nm]