Eigenvalues and Eigenfunctions of L^2 and L_z

We note that the expression for L_z is simpler than those for L_x and L_y. Therefore, it is convenient to obtain simultaneous eigenfunctions of L^2 and L_z.

Let us denote the eigenvalues of L^2 and L_z by $\lambda\hbar^2$ and $m_l\hbar$, respectively, and let the corresponding common eigenfunction be $Y(\theta, \phi)$. Then the two eigenvalue equations can be written as

$$L^2 Y(\theta, \phi) = \lambda\hbar^2 Y(\theta, \phi) \qquad \ldots(7.11)$$

and

$$L_z Y(\theta, \phi) = m_l\hbar\, Y(\theta, \phi) \qquad \ldots(7.12)$$

The subscript l is attached to m for later convenience. Substituting for L^2 from (7.10) into (7.11), we obtain

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0 \qquad \ldots(7.13)$$

This equation can be solved by using the method of separation of variables. We write

$$Y(\theta, \phi) = \Theta(\theta)\ \Phi(\phi) \qquad \ldots(7.14)$$

Substituting in (7.13), multiplying by $sin^2\theta/Y(\theta,\phi)$ and rearranging, we obtain

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{\sin^2\theta}{\Theta}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta} + \lambda\Theta\right)\right]$$

The variables have separated out, and therefore, each side must be equal to a constant. We take this constant to be m_l^2 for reason which will become clear soon and obtain the following ordinary differential equations:

$$\frac{d^2\Phi}{d\phi^2} + m_l^2\,\Phi = 0 \qquad \ldots(7.15)$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{m_l^2}{\sin^2\theta}\right)\Theta = 0 \qquad \ldots(7.16)$$

Equation (7.15) can be immediately solved to give

$$\Phi(\phi) = Ae^{im_l\phi}$$

where A is an arbitrary constant. For $\Phi(\phi)$ to be single-valued we must have

$$\Phi(\phi + 2\pi) = \Phi(\phi)$$

or

$$e^{2\pi m_l i} = 1$$

or

$$m_l = 0, \pm1, \pm2, \ldots$$

Taking $A = 1/\sqrt{2\pi}$, we obtain the normalized solutions of (7.15):

$$\Phi_{m_l}(\phi) = \frac{1}{\sqrt{2\pi}} e^{im_l\phi}; \qquad m_l = 0, \pm 1, \pm 2, \ldots \qquad \ldots(7.17)$$

It can be easily shown that these functions form an orthonormal set. That is,

$$\int_0^{2\pi} \Phi^*_{m'_l}(\phi)\ \Phi_{m_l}(\phi)\ d\phi = \delta_{m_l m'_l} \qquad \ldots(7.18)$$

We can immediately note here that the function $\Phi_{m_l}(\phi)$ is an eigenfunction of the operator L_z with the eigenvalue $m_l\hbar$. Indeed,

$$\begin{aligned} L_z\ \Phi_{m_l}(\phi) &= -i\hbar\frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{2\pi}} e^{im_l\phi}\right) \\ &= m_l\hbar\left(\frac{1}{\sqrt{2\pi}} e^{im_l\phi}\right) \\ &= m_l\hbar\ \Phi_{m_l}(\phi) \end{aligned} \qquad \ldots(7.19)$$

Spherical Harmonics

After solving equation (7.16) and using (7.14) we obtain the common eigenfunctions of the operators L^2 and L_z as

$$Y_{lm_l}(\theta, \phi) = (-1)^{m_l}\left[\frac{(2l+1)\,(l-m_l)\,!}{4\pi\,(l+m_l)\,!}\right]^{1/2} P_l^{m_l}(\cos\theta)\, e^{im_l\phi},\ m_l \geq 0 \qquad \ldots(7.20)$$

and

$$Y_{lm_l}(\theta, \phi) = (-1)^{m_l}\ Y^*_{l,-m_l}(\theta, \phi), \qquad m_l \leq 0 \qquad \ldots(7.21)$$

These functions are known as the ***spherical harmonics***.

Where,

l is called the ***orbital angular momentum quantum number.*** $l = 0, 1, 2, 3, \ldots$

m_l is called the ***magnetic quantum number*** for a given l, there are only $(2l + 1)$ possible values of m_l, $\qquad m_l = -l, -l + 1, \ldots, 0, \ldots, l - 1, l$

$P_l(\xi)$ denoted, ***Legendre polynomials***, where l is the degree of the polynomial.

Also, since $P_l(\xi)$ contains only even or odd powers of ξ, depending on whether l is even or odd, we have

…(7.22)

$$P_l(-\xi) = (-1)^l \, P_l(\xi)$$

The first few Legendre polynomials are:

$$\boxed{\begin{aligned} P_0(\xi) &= 1 \\ P_1(\xi) &= \xi \\ P_2(\xi) &= \frac{1}{2}(3\xi^2 - 1) \\ P_3(\xi) &= \frac{1}{2}(5\xi^3 - 3\xi) \\ P_4(\xi) &= \frac{1}{8}(35\xi^4 - 30\xi^2 + 3) \\ P_5(\xi) &= \frac{1}{8}(63\xi^5 - 70\xi^3 + 15\xi) \end{aligned}} \qquad \ldots(7.23)$$

The ***spherical harmonics*** satisfy the ***orthonormality condition***

$$\begin{aligned} &\int Y^*_{l'm'_l}(\theta,\ \phi)\ Y_{lm_l}(\theta,\ \phi)\ d\Omega \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} Y^*_{l'm'_l}(\theta,\ \phi)\ Y_{lm_l}(\theta,\ \phi)\ \sin\theta\ d\theta \\ &= \delta_{ll'}\ \delta_{m_l m_l'} \end{aligned} \qquad \ldots(7.24)$$

where the integration is over the full range of the angular variables $(\theta,\ \phi)$ and $d\Omega$ is the element of solid angle: $d\Omega = \sin\theta\ d\theta\ d\phi$.

We have

$$L^2\, Y_{lm_l}(\theta, \phi) = l(l+1)\hbar^2\, Y_{lm_l}(\theta, \phi), \qquad l = 0, 1, 2, \ldots \qquad \ldots(7.25)$$

and

$$L_z\, Y_{lm_l}(\theta, \phi) = m_l \hbar\, Y_{lm_l}(\theta, \phi), \qquad |m_l| \le l \qquad \ldots(7.26)$$

The first few spherical harmonics are given in Table:

l	m_l	Spherical Harmonic $Y_{lm_l}(\theta, \phi)$
0	0	$Y_{0,0} = \dfrac{1}{(4\pi)^{1/2}}$
1	0	$Y_{1,0} = \left(\dfrac{3}{4\pi}\right)^{1/2} \cos\theta$
	±1	$Y_{1,\pm 1} = \mp\left(\dfrac{3}{8\pi}\right)^{1/2} \sin\theta\ e^{\pm i\phi}$
2	0	$Y_{2,0} = \left(\dfrac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$
	±1	$Y_{2,\pm 1} = \mp\left(\dfrac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta\ e^{\pm i\phi}$
	±2	$Y_{2,\pm 2} = \left(\dfrac{15}{32\pi}\right)^{1/2} \sin^2\theta\ e^{\pm 2i\phi}$

$$P_1^1(\xi) = (1-\xi^2)^{1/2}$$
$$P_2^1(\xi) = 3(1-\xi^2)^{1/2}\,\xi$$
$$P_2^2(\xi) = 3(1-\xi^2)$$
$$P_3^1(\xi) = \frac{3}{2}(1-\xi^2)^{1/2}\,(5\xi^2-1)$$
$$P_3^2(\xi) = 15\xi(1-\xi^2)$$
$$P_3^3(\xi) = 15(1-\xi^2)^{3/2}$$