Chapter Five

The Orbital Angular Momentum in Quantum Mechanics

7.1 The Orbital Angular Momentum Operator and its Cartesian Components

Classically, the angular momentum $\boldsymbol{L}$ of a particle with respect to some fixed origin O is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \ldots(7.1)$$

where **p** is the momentum of the particle and **r** is its position vector with respect to O. Thus **L** is a vector which points in a direction at right angles to the plane containing **r** and **p**.

The Cartesian components of **L** are

$$L_x = yp_z - zp_y \qquad \ldots(7.2a)$$
$$L_y = zp_x - xp_z \qquad \ldots(7.2b)$$
$$L_z = xp_y - yp_x \qquad \ldots(7.2c)$$

The corresponding quantum mechanical operators are obtained by replacing $\mathbf{p}, p_x, p_y$ and p_z by the respective operators representing them. We have

$$\mathbf{L} = -i\hbar(\mathbf{r} \times \nabla) \qquad \ldots(7.3)$$

and

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \qquad \ldots(7.4a)$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \qquad \ldots(7.4b)$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad \ldots(7.4c)$$

(Angular momentum operators)

Commutation Relations

Let us now obtain the commutation relations between L_x, L_y and L_z. For this we use the basic commutation relations between position and momentum operators,

$$[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$$

with all other pairs (for example x and p_y) commuting. We have

$$\begin{aligned}
[L_x, L_y] &= L_x L_y - L_y L_x \\
&= (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y) \\
&= yp_z zp_x - yp_z xp_z - zp_y zp_x + zp_y xp_z - zp_x yp_z \\
&\quad + zp_x zp_y + xp_z yp_z - xp_z zp_y \\
&= yp_x(p_z z - zp_z) + p_y x(zp_z - p_z z) \\
&= (zp_z - p_z z)(xp_y - yp_x) \\
&= [z, p_z](xp_y - yp_x) \\
&= i\hbar L_z
\end{aligned}$$

In similar fashion we obtain the values of the commutators $[L_y, L_z]$ and $[L_z, L_x]$. Putting all the three together, we have

$$[L_x, L_y] = i\hbar L_z \qquad \text{...(7.5a)}$$

$$[L_y, L_z] = i\hbar L_x \qquad \text{...(7.5b)}$$

$$[L_z, L_x] = i\hbar L_y \qquad \text{...(7.5c)}$$

(Commutation relations)

We find that the operators representing any two components of the orbital angular momentum do not commute. In other words, if a system of particles is in an eigenstate of one of the components, it cannot be in an eigenstate of either of the other two components.

Let us now consider the operator representing the square of the magnitude of the orbital angular momentum:

Let us evaluate its commutator with L_x:

$$[L^2, L_x] = [L_x^2 + L_y^2 + L_z^2, L_x] \qquad \ldots(7.6)$$

Since $[L_x^2, L_x] = 0$, we get

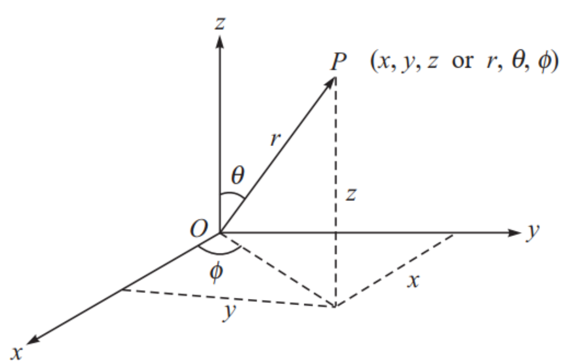
$$\begin{aligned}
[L^2, L_x] &= [L_y^2, L_x] + [L_z^2, L_x] \\
&= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z \\
&= -i\hbar(L_y L_z + L_z L_y) + i\hbar(L_z L_y + L_y L_z) \\
&= 0
\end{aligned}$$

In a similar manner, we can show that L_y and L_z also commute with L^2. Thus,

$$\boxed{[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0} \qquad \ldots(7.7)$$

This shows that the magnitude of the orbital angular momentum and any one of its Cartesian components can be simultaneously measured precisely. Therefore, it is possible to find simultaneous eigenfunctions of L^2 and any one of L_x and L_y or L_z.

Angular Momentum Operators in Spherical Polar Coordinates



The spherical and Cartesian coordinates of a point P are related as

$$\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned} \qquad \ldots(7.8)$$

with $0 \le r \le \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. After some straightforward but lengthy algebra, it can be shown that

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \quad \ldots(7.9a)$$

$$L_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \quad \ldots(7.9b)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \quad \ldots(7.9c)$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad \ldots(7.10)$$