Problems and Exercises Ch.4

Problem 1: Evaluate (a) $\langle x \rangle$, (b) $\langle p_x^2 \rangle$, (c) $\langle p_x \rangle$ and (d) $\langle p_x^2 \rangle$ for the eigenstates of a harmonic oscillator.

Solution: (a) $\langle x \rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\; x\; \psi_n(x)\; dx$ …(6.37)

We know that the harmonic oscillator wave functions have definite parity. Thus $\psi_n(x)$ is either an even or an odd function of x. Therefore, the product $\psi_n^*(x)\;\psi_n(x)$ will always be even. Since x is odd, the integrand will always be odd and hence $\langle x \rangle = 0$.

For a complete solution from first principles, we proceed as under.

We shall use the recurrence relation (6.23b) for the Hermite polynomials which can be rewritten as

$$2\alpha x H_n(\alpha x) = H_{n+1}(\alpha x) + 2nH_{n-1}(\alpha x) \qquad \text{…(6.38)}$$

The harmonic oscillator wave functions can be written in terms of the Hermite polynomials as

$$\psi_n(x) = \left(\frac{\alpha}{2^n\, n!\sqrt{\pi}}\right)^{1/2} e^{-\alpha^2 x^2/2}\; H_n(\alpha x),\; n = 0,\, 1,\, 2,\, \ldots \qquad \text{…(6.39)}$$

Multiplying this Equation by $\left(\frac{\alpha}{2^n\, n!\sqrt{\pi}}\right)^{1/2} e^{-\alpha^2 x^2/2}$

and simplifying, we obtain

$$x\psi_n(x) = \frac{1}{\alpha\sqrt{2}}[\sqrt{n+1}\;\psi_{n+1}(x) + \sqrt{n}\;\psi_{n-1}(x)] \qquad \text{…(6.40)}$$

Substituting in (6.37),

$$\langle x \rangle = \frac{\sqrt{n+1}}{\alpha\sqrt{2}}\int_{-\infty}^{\infty} \psi_n^*(x)\;\psi_{n+1}(x)dx + \frac{\sqrt{n}}{\alpha\sqrt{2}}\int_{-\infty}^{\infty} \psi_n^*(x)\psi_{n-1}(x)dx$$

Since the oscillator wave functions are orthonormal, both the integrals on the right-hand side vanish. Therefore,

$$\langle x \rangle = 0 \qquad \text{…(6.41)}$$

(b) $\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_n^*(x)x^2\psi_n(x)dx$

$$= \frac{1}{\sqrt{2}\,\alpha}\int_{-\infty}^{\infty} \psi_n^*\; x\left[\sqrt{n+1}\;\psi_{n+1} + \sqrt{n}\;\psi_{n-1}\right]\; dx \qquad \text{Using (6.40)}$$

$$= \frac{1}{\sqrt{2}\alpha}\left[\sqrt{n+1}\int_{-\infty}^{\infty}\psi_n^*\; x\,\psi_{n+1}\,dx + \sqrt{n}\int_{-\infty}^{\infty}\psi_n^*\; x\,\psi_{n-1}\,dx\right]$$

Using (6.40) again,

$$\langle x^2 \rangle = \frac{1}{\alpha}\sqrt{\frac{n+1}{2}}\left[\int_{-\infty}^{\infty} \psi_n^* \left\{\frac{1}{\alpha}\sqrt{\frac{n+2}{2}}\,\psi_{n+2} + \frac{1}{\alpha}\sqrt{\frac{n+1}{2}}\,\psi_n\right\} dx\right]$$

$$+\frac{1}{\alpha}\sqrt{\frac{n}{2}}\left[\int_{-\infty}^{\infty} \psi_n^* \left\{\frac{1}{\alpha}\sqrt{\frac{n}{2}}\,\psi_n + \frac{1}{\alpha}\sqrt{\frac{n-1}{2}}\,\psi_{n-2}\right\} dx\right]$$

$$= \frac{\sqrt{(n+1)(n+2)}}{2\alpha^2}\int_{-\infty}^{\infty} \psi_n^*\,\psi_{n+2}\,dx + \frac{n+1}{2\alpha^2}\int_{-\infty}^{\infty} \psi_n^*\,\psi_n\,dx$$

$$+\frac{n}{2\alpha^2}\int \psi_n^*\,\psi_n\,dx + \frac{\sqrt{n(n-1)}}{2\alpha^2}\int_{-\infty}^{\infty} \psi_n^*\,\psi_{n-2}\,dx$$

Using the orthonormality of oscillator wave functions, we obtain

$$\langle x^2 \rangle = \frac{1}{2\alpha^2}\,(n+1+n)$$

or

$$\boxed{\langle x^2 \rangle = \frac{2n+1}{2\alpha^2} = \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}} \qquad \ldots(6.42)$$

(c) $\langle p \rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\,\hat{p}\,\psi_n(x)dx$

$$= \int_{-\infty}^{\infty} \psi_n^*(x)\left[-i\hbar\frac{d\psi_n(x)}{dx}\right]dx \qquad \ldots(6.43)$$

Now, if $\psi_n(x)$ is odd, then its derivative is even, and vice versa. Therefore, the integrand in the above integral is always an odd function of x. Hence $\langle p \rangle = 0$.

We can also obtain this result using recurrence relation (6.23a):

$$\frac{dH_n(\alpha x)}{d(\alpha x)} = 2n\,H_{n-1}(\alpha x) \qquad \ldots(6.44)$$

Differentiating (6.39),

$$\frac{d\psi_n(x)}{dx} = \left(\frac{\alpha}{2^n\,n!\sqrt{\pi}}\right)^{1/2} \times \left[-(\alpha^2 x)\exp\left(\frac{-\alpha^2x^2}{2}\right)H_n(\alpha x) + \exp\left(\frac{-\alpha^2x^2}{2}\right)\frac{dH_n(\alpha x)}{dx}\right]$$

$$= -\alpha^2 x\left(\frac{\alpha}{2^n\,n!\sqrt{\pi}}\right)^{1/2}\exp\left(\frac{-\alpha^2x^2}{2}\right)H_n(\alpha x) + \left(\frac{\alpha}{2^n\,n!\sqrt{\pi}}\right)^{1/2} \times \alpha\exp\left(\frac{-\alpha^2x^2}{2}\right)\frac{dH_n(\alpha x)}{d(\alpha x)}$$

Using (6.39) and (6.44) this becomes

$$\frac{d\psi_n(x)}{dx} = -\alpha^2 x\psi_n(x) + \left(\frac{\alpha}{2^n n!\sqrt{\pi}}\right)^{1/2} \alpha \exp\left(\frac{-\alpha^2 x^2}{2}\right) [2nH_{n-1}(\alpha x)]$$

$$= -\alpha^2 x\psi_n(x) + (2\alpha n)\frac{1}{\sqrt{2n}}\left(\frac{\alpha}{2^{n-1}(n-1)!\sqrt{\pi}}\right)^{1/2} \exp\left(\frac{-\alpha^2 x^2}{2}\right) H_{n-1}(\alpha x)$$

$$= -\alpha^2 x\psi_n(x) + 2\alpha\left(\frac{n}{2}\right)^{1/2} \psi_{n-1}(x)$$

Using (6.40),

$$\frac{d\psi_n(x)}{dx} = \frac{-\alpha}{\sqrt{2}}[\sqrt{n+1}\ \psi_{n+1}(x) + \sqrt{n}\ \psi_{n-1}(x)] + \frac{2\alpha}{\sqrt{2}}\sqrt{n}\ \psi_{n-1}(x)$$

Or,

$$\frac{d\psi_n(x)}{dx} = \frac{\alpha}{\sqrt{2}}[\sqrt{n}\ \psi_{n-1}(x) - \sqrt{n+1}\ \psi_{n+1}(x)] \qquad \ldots(6.45)$$

Substituting in (6.43) and using the orthonormality of eigenfunctions, we get

$$\langle p \rangle = 0 \qquad \ldots(6.46)$$

(d) $\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_n^*(x)\ \hat{p}^2\ \psi_n(x)\ dx$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi_n^*(x)\ \frac{d^2\psi_n(x)}{dx}\ dx$$

$$= \frac{-\hbar^2\alpha}{\sqrt{2}}\int_{-\infty}^{\infty} \psi_n^*\ \frac{d}{dx}\ [\sqrt{n}\ \psi_{n-1} - \sqrt{n+1}\ \psi_{n+1}]\ dx \qquad \text{Using (6.45)}$$

$$= \frac{-\hbar^2\alpha}{\sqrt{2}}\left[\sqrt{n}\int_{-\infty}^{\infty}\psi_n^*\frac{d\psi_{n-1}}{dx}dx - \sqrt{n+1}\int_{-\infty}^{\infty}\psi_n^*\frac{d\psi_{n+1}}{dx}dx\right]$$

Using (6.45) again,

$$\langle p^2 \rangle = \frac{-\hbar^2\alpha^2\sqrt{n}}{2}\int_{-\infty}^{\infty}\psi_n^*\ [\sqrt{n-1}\ \ \psi_{n-2}(x) - \sqrt{n}\ \ \psi_n]\ dx$$

$$+ \frac{\hbar^2\alpha^2\sqrt{n+1}}{2}\int_{-\infty}^{\infty}\psi_n^*\ [\sqrt{n+1}\ \ \psi_n - \sqrt{n+2}\ \ \psi_{n+2}]\ dx$$

$$= \frac{-\hbar^2\alpha^2}{2}\left[\sqrt{n(n-1)}\int_{-\infty}^{\infty}\psi_n^*\,\psi_{n-2}\,dx - n\int_{-\infty}^{\infty}\psi_n^*\,\psi_n\,dx\right.$$

$$\left. -(n+1)\int_{-\infty}^{\infty}\psi_n^*\,\psi_n\,dx + \sqrt{(n+1)(n+2)}\int_{-\infty}^{\infty}\psi_n^*\,\psi_{n+2}\,dx\right]$$

Using the orthonormality of oscillator wave function, this reduces to

$$\langle p^2 \rangle = -\frac{\hbar^2 \alpha^2}{2}(0 - n - (n+1) + 0)$$

$$= \frac{2n+1}{2} \; \hbar^2 \alpha^2$$

$$= \left(n + \frac{1}{2}\right) \hbar^2 \left(\frac{m\omega}{\hbar}\right)$$

$$\boxed{\langle p^2 \rangle = \left(n + \frac{1}{2}\right) m\omega\hbar} \qquad \ldots(6.47)$$

Problem 2: Evaluate the position-momentum uncertainty product for the *n*th state of a linear harmonic oscillator.

Solution:

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\left(n + \frac{1}{2}\right) \frac{\hbar}{m\omega} - 0$$

$$= \left(n + \frac{1}{2}\right) \frac{\hbar}{m\omega}$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$$

$$= \left(n + \frac{1}{2}\right) m\omega\hbar - 0 = \left(n + \frac{1}{2}\right) m\omega\hbar$$

Multiplying the two and taking square root we obtain the position-momentum uncertainty product

$$\Delta x \, \Delta p = \left(n + \frac{1}{2}\right) \hbar, \qquad n = 0, 1, 2, \ldots \qquad \ldots(6.48)$$

This is in accordance with the uncertainty relation

$$\Delta x \; \Delta p \geq \frac{\hbar}{2}$$

For $n = 0$, (6.48) reduces to

$$\Delta x \Delta p = \frac{\hbar}{2}$$

showing that *the uncertainty product is minimum for the ground state.*

Problem 3: Obtain the expectation values of the kinetic and potential energies for the nth state of a linear harmonic oscillator.

Solution: $\langle T \rangle = \frac{1}{2m} \langle p^2 \rangle = \frac{1}{2}\left(n + \frac{1}{2}\right) \hbar\omega = \frac{E_n}{2}$

$$\langle V \rangle = \frac{1}{2}k\langle x^2 \rangle = \frac{1}{2}(m\omega^2)\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}$$

$$= \frac{1}{2}\left(n+\frac{1}{2}\right)\hbar\omega = \frac{E_n}{2}$$

Thus, the average kinetic and potential energies for a harmonic oscillator in any eigenstate are each equal to one-half the total energy, as in the case of a classical harmonic oscillator.

H.W.: Write down the required information for the linear harmonic oscillator in the state $\psi_2(x)$;

1- Figure for $\psi_2(x)$; as a function of x.

2- The parity of $\psi_2(x)$;

3- The number of nodes.

4- The energy state E_2.

H.W.: Find the expectation value of the kinetic energy for the Linear harmonic oscillator in the eigen energy state $n = 0$.

H.W.: Prove that the following wave functions are orthonormal functions.

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{1}{2}\alpha^2 x^2\right)$$

$$\psi_1(x) = \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{1/2} (\alpha x) \exp\left(-\frac{1}{2}\alpha^2 x^2\right)$$

$$\psi_2(x) = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2} (2\alpha^2 x^2 - 1) \exp\left(-\frac{1}{2}\alpha^2 x^2\right)$$

H.W.: Use the uncertainty principle to prove that the minimum energy for the linear harmonic oscillator is $\frac{1}{2}\hbar\omega$.