

6.3 The Harmonic Oscillator Wave Functions

Using Equation (6.11) we see that the physically acceptable solutions of Equation (6.9), corresponding to the eigenvalues (6.19) are given by

$$\psi_n(y) = e^{-y^2/2} H_n(y) \quad \dots(6.29)$$

Returning to our original variable x , the eigenfunctions corresponding to the discrete eigenvalues E_n , given by (6.20), can be written as

$$\psi_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x) \quad n = 0, 1, 2, \dots \quad \dots(6.30)$$

$$\alpha = \sqrt{m\omega/\hbar}$$

The constant N_n can be determined by requiring that the eigenfunctions be normalized to unity. That is,

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = \frac{|N_n|^2}{\alpha} \int_{-\infty}^{\infty} e^{-y^2} H_n^2(y) dy = 1 \quad \dots(6.31)$$

Using (6.27) we obtain

$$N_n = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} \quad \dots(6.32)$$

So, the *normalized harmonic oscillator eigenfunctions* are given by

$$\boxed{\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right)^{1/2} e^{-\alpha^2 x^2/2} H_n(\alpha x), \quad n = 0, 1, 2, \dots} \quad \dots(6.33)$$

From (6.27), we have

$$\int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) dx = 0, \quad n \neq m \quad \dots(6.34)$$

showing that the eigenfunctions are orthogonal. This is in agreement with the fact that the energy eigenvalues are nondegenerate.

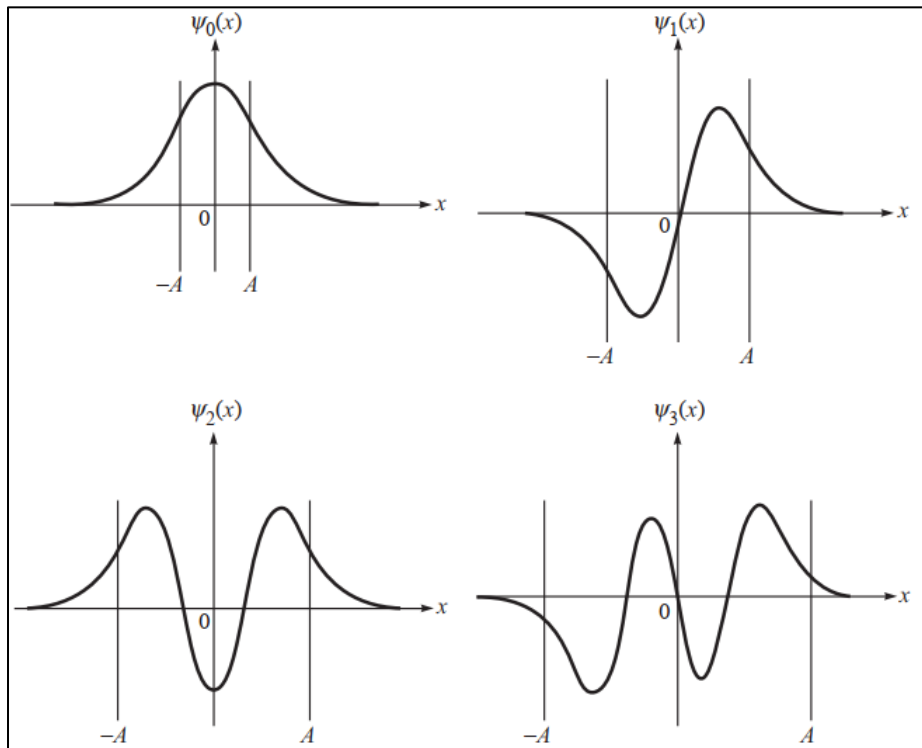
We may combine (6.31) and (6.34) by writing

$$\int_{-\infty}^{\infty} \psi_n^*(x) \psi_m(x) dx = \delta_{nm} \quad \dots(6.35)$$

showing that the eigenfunctions are **orthonormal**.

The lowest four normalized eigenfunctions are:

$$\left. \begin{aligned} \psi_0(x) &= \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \\ \psi_1(x) &= \left(\frac{2\alpha}{\sqrt{\pi}}\right)^{1/2} (\alpha x) \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \\ \psi_2(x) &= \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2} (2\alpha^2 x^2 - 1) \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \\ \psi_3(x) &= \left(\frac{\alpha}{3\sqrt{\pi}}\right)^{1/2} (2\alpha^3 x^3 - 3\alpha x) \exp\left(-\frac{1}{2}\alpha^2 x^2\right) \end{aligned} \right\} \dots(6.36)$$



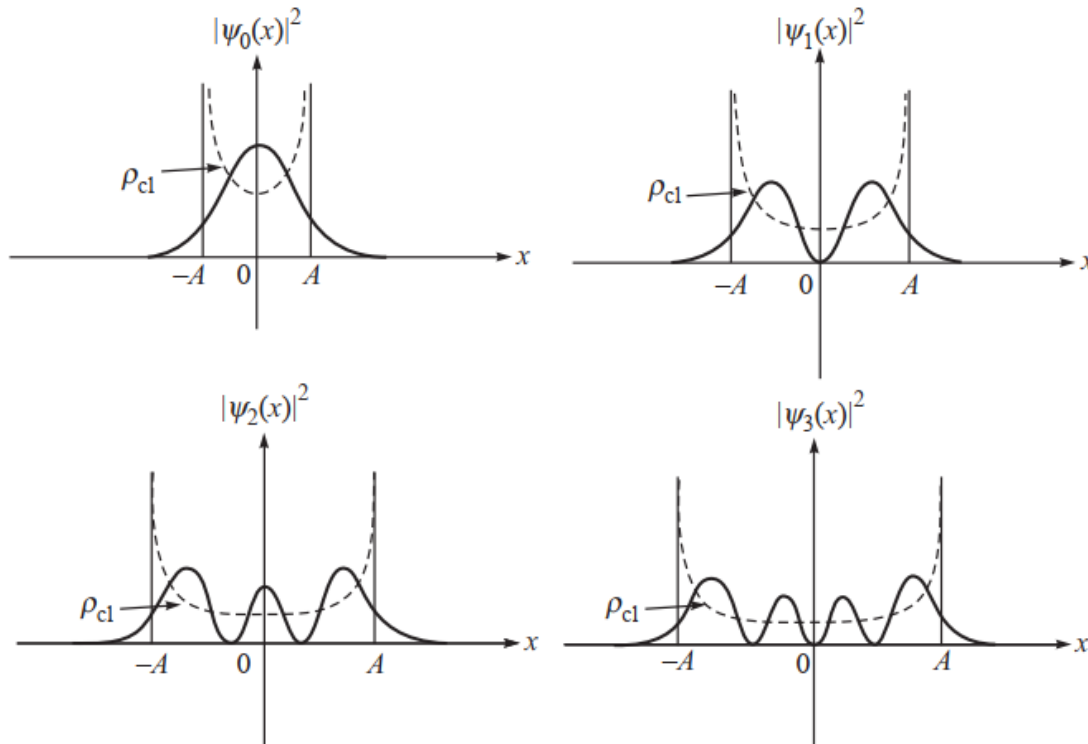
The first four harmonic oscillator wave functions, $\psi_n(x)$, $n = 0, 1, 2, 3$. The vertical lines at $x = \pm A$ show the amplitude of a classical oscillator with the same total energy.

Note

- The vertical line in each graph represents the amplitude A of a classical harmonic oscillator with the same energy.

- It is clear that there is some penetration of the wave functions into the regions $|x| > A$ that are forbidden classically. This is similar to the effect that we observed for a particle in a finite square well.
- It may be noted that the harmonic oscillator wave functions have a definite parity even when n is zero or even and odd when n is odd.

The Figure below shows the corresponding position probability densities.



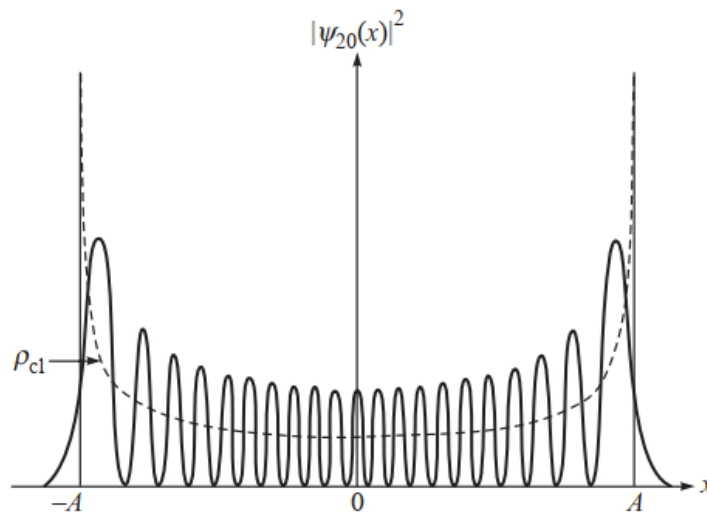
Probability densities $|\psi_n(x)|^2$ for the first four harmonic oscillator states. The dotted curves show the corresponding classical probability densities ρ_{cl} .

Note

- The (Dashed curves) are the probability densities p_{cl} for a classical harmonic oscillator.
- Classically, the probability of finding the particle at a point is inversely proportional to its speed at that point. As such, p_{cl} is maximum near the end points of the motion, where the particle moves slowly, and minimum near the equilibrium position, where it moves fast.
- It is clear from the figure that for low values of the quantum number n , the quantum mechanical probability densities $|\psi_n(x)|^2$ are quite different from the

corresponding classical probability densities p_{cl} .

- In fact, for the lowest energy state $n = 0$, the quantum behavior is exactly opposite. The probability density $|\psi_n(x)|^2$ has its maximum value at the equilibrium position $x = 0$ and decreases on either side of this position.
- However, as n increases the disagreement between the quantum and classical probability densities becomes less and less marked.



Comparison of the Quantum-mechanical and classical probability densities for a harmonic oscillator for the state $n = 20$.

From the figure note that:

- The agreement between quantum and classical behaviors as n increases is expected because, for large n , the energy interval is small compared to the total energy.
- This is also in accordance with Bohr's correspondence principle, which asserts that for large quantum numbers, quantum mechanics gives the same results as classical mechanics.

Comparison between the classical and quantum harmonic oscillator

properties	Classical harmonic oscillator	Quantum harmonic oscillator
minimum energy	$E_0 = 0$	$E_0 = \frac{1}{2} \hbar \omega$
energy spectrum	Continuous values	Discrete energy levels
probability density	inverse proportionality with its speed. ($p_{cl} \propto \frac{1}{v}$)	$p_q = \psi_n ^2$ for low n $p_q \rightarrow p_{cl}$ for large n